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Mechanical heating in lineal tube flow of non-Newtonian fluids

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1. Introduction

The evaluation of heat transfer by forced heat convection in slow lineal flows of non-Newtonian fluids has been discussed in [1] on the basis of the third order Rivlin-Ericksen fluid. In [1] mechanical heating in the fluid was ignored which is a valid enough approximation for sufficiently small Brinkman numbers. For polymer melts the Brinkman number is known to vary from 0 to ∞ and, as data in Pearson [2] show, it is small for slow flows of melts in small bore channels. In this note we reconsider heat transfer for a circular tube and examine the effect of mechanical heating on the heat transfer coefficient. The largest Brinkman number dealt with has the value one so that our results pertain to low heat generation.

2. The velocity and temperature fields

The radius of the tube is a, and the origin O of cylindrical polar coordinates (r, θ, x_3) is located at the thermal entrance with $O x_3$ along the axis of the tube parallel to the unit vector $\boldsymbol{e}_3$. At $x_3 = 0$ the uniform temperature of the heated fluid is T_0, and $T_1 (< T_0)$ is the constant wall temperature of the tube for $x_3 > 0$. From [1] we recall the rectilinear velocity field in the third order fluid:

$$\begin{aligned} \boldsymbol{v} &= \{\varepsilon\, v_1(r) + \varepsilon^3 v_3(r)\}\, \boldsymbol{e}_3 \\ v_1(r) &= \frac{1}{2\mu}(a^2 - r^2) \\ v_3(r) &= -\frac{\beta_2 + \beta_3}{4\mu^4}(a^4 - r^4) \end{aligned} \tag{2.1}$$

and the average velocity v_0 through any cross section of the tube

$$v_0 = \frac{a^2 \varepsilon}{4\mu}\left[1 - \frac{2(\beta_2 + \beta_3)\, a^2}{3\mu^3}\varepsilon^2\right]. \tag{2.2}$$

In the above μ, β_2, β_3 are material parameters which are assumed to be independent of temperature, and ε is the specific driving force in the fluid, powers of which beyond the third being considered negligible (vide [1]). With regard to the temperature field we have now to consider the contribution of mechanical heating to the energy of the fluid. In terms of the local energy balance equation this occurs through the term trace $(\boldsymbol{S}\,\boldsymbol{D})$ where $\boldsymbol{S}$ is the determinate stress defined in [1] and $\boldsymbol{D}$ is the symmetric part of the velocity gradient tensor. Very simply the equation that emerges for the temperature field T is, in standard notation,

$$\varrho c_p v \frac{\partial T}{\partial x_3} = k \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x_3^2} \right\} + \frac{1}{2} \left\{ \mu \left(\frac{\partial v}{\partial r} \right)^2 + (\beta_2 + \beta_3) \left(\frac{\partial v}{\partial r} \right)^4 \right\}. \tag{2.3}$$

We may note, in relation to the solution of (2.3), that we start off knowing the velocity field through (2.1). This is a consequence of our assumption that μ, β_1, β_2 are independent of temperature since then the velocity field is uncoupled from the temperature field.

In [1] we have the dimensionless quantities

$$\theta = \frac{T - T_1}{T_0 - T_1}, \quad x = \frac{r}{a}, \quad z = \frac{x_3}{2 a \,\text{Pé}}, \quad \text{Pé} = \frac{2 a \varrho v_0 c_p}{k} \tag{2.4}$$

to which, bearing in mind (2.1), we add

$$w = \frac{\varepsilon v_1}{v_0} = \frac{2}{K(\bar{\varepsilon})} (1 - x^2) \tag{2.5}$$

where, as in [1], we have defined

$$\bar{\varepsilon} = \frac{4 a^2 (\beta_2 + \beta_3)}{\mu^3} \varepsilon^2; \quad K(\bar{\varepsilon}) = 1 - \frac{\bar{\varepsilon}}{6}. \tag{2.6}$$

By inserting (2.4) and (2.5) into (2.3) we obtain the following temperature equation in which, consistent with the third order fluid approximation, powers of ε higher than the third have been neglected:

$$\frac{v}{4 v_0} \frac{\partial \theta}{\partial z} = \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{x} \frac{\partial \theta}{\partial x} + 8 \,\text{Br} \left\{ \frac{x}{K(\bar{\varepsilon})} \right\}^2 \tag{2.7}$$

subject to

$$\theta(x, 0) = 1, \; x \in (0, 1); \quad \theta(1, z) = 0, \; z > 0. \tag{2.8}$$

In (2.7)

$$\frac{v}{v_0} = \frac{2}{K(\bar{\varepsilon})} \left[(1 - x^2) - \frac{\bar{\varepsilon}}{8} (1 - x^4) \right]; \quad \text{Br} = \frac{\mu v_0^2}{k (T_0 - T_1)} \tag{2.9}$$

the latter quantity being the Brinkman number of the fluid. Also longitudinal heat conduction has been ignored for large Péclet numbers as discussed in [1] and [2]. In the event that Br is zero (2.7) reverts to the equation considered in [1] and this fact may be used to generate the solution of (2.7) by the principle of superposition. Thus we put

$$\theta(x, z) = \theta_1(x, z) + \theta^*(x) \tag{2.10}$$

where

$$\frac{v}{4 v_0} \frac{\partial \theta_1}{\partial z} = \frac{\partial^2 \theta_1}{\partial x^2} + \frac{1}{x} \frac{\partial \theta_1}{\partial x}, \quad \theta_1(1, z) = 0 \tag{2.11}$$

and

$$\frac{d^2 \theta^*}{dx^2} + \frac{1}{x} \frac{d\theta^*}{dx} + 8 \,\text{Br} \left\{ \frac{x}{K(\bar{\varepsilon})} \right\}^2 = 0, \quad \theta^*(1) = 0. \tag{2.12}$$

The solutions of (2.12) and (2.11) are

$$\theta^*(x) = \frac{\mathrm{Br}}{2\{K(\bar{\varepsilon})\}^2}(1-x^4) \tag{2.13}$$

$$\theta_1(x,z) = \sum_{m=1}^{\infty} A_m \Phi_m(x)\,\mathrm{e}^{-\lambda_m z} \tag{2.14}$$

where, in the latter solution, the eigenfunctions $\Phi_m(x)$, $m = 1, 2, 3, \ldots$ have the expanded forms

$$\Phi_m(x) = \sum_{p=1}^{\infty} a_p^{(m)} J_0(\Omega_p x) \tag{2.15}$$

and Ω_p, $p = 1, 2, 3, \ldots$ are the positive zeros of $J_0(\Omega)$. The eigensolutions $\{\lambda_m, a_p^m\}$ have been evaluated in [1]. It remains for the composite solution (2.10) to satisfy the first of the conditions (2.8).

3. Thermal results

The mean transport temperature as defined in [1] is

$$\bar{\theta}(z) = \int_0^1 v\,\theta\,x\,\mathrm{d}x \Big/ \int_0^1 v\,x\,\mathrm{d}x \tag{3.1}$$

so that through use of (2.9), (2.10), (2.13), (2.14), (2.15) and the first of (2.8) it is found that

$$\bar{\theta}(z) = \sum_{m=1}^{\infty} A_m^2\,\mathrm{e}^{-\lambda_m z} + \frac{5\,\mathrm{Br}}{12\{K(\bar{\varepsilon})\}^3}\left(1-\frac{4\bar{\varepsilon}}{25}\right) \tag{3.2}$$

where

$$A_m^2 = \frac{\dfrac{64}{K(\bar{\varepsilon})}\left[\sum_{p=1}^{\infty} a_p^{(m)}\left\{L_p - \dfrac{\mathrm{Br}}{2\{K(\bar{\varepsilon})\}^2}\right\}(L_p - R_p)\right]^2}{\left[\sum_{p=1}^{\infty} N_p (a_p^{(m)})^2 - \sum_{p,q} Q_p^q a_p^{(m)} a_q^{(m)} \;(p \neq q)\right]} \tag{3.3}$$

and, apart from

$$R_p = \frac{J_1(\Omega_p)}{\Omega_p^3}\left[5 - \frac{128}{\Omega_p^2} + \frac{576}{\Omega_p^4} - \frac{\bar{\varepsilon}}{2}\left(3 - \frac{140}{\Omega_p^2} + \frac{2304}{\Omega_p^4} - \frac{9216}{\Omega_p^6}\right)\right]$$

the coefficients L_p, N_p, Q_p^q are as defined in [1].

The rate of heat transferred to a channel wall is commonly described by means of a dimensionless coefficient, namely the Nusselt number

$$\mathrm{Nu} = \frac{\mathrm{d}Q_w}{k A (\Delta T)_{\mathrm{ref}}} \tag{3.4}$$

where k is thermal conductivity, Q_w the rate of flow of heat to the wall area A, d a characteristic length and $(\Delta T)_{\mathrm{ref}}$ a representative temperature difference. Evidently the choice of $(\Delta T)_{\mathrm{ref}}$ has a major influence on the value of the Nusselt number. For flows in which mechanical heating is not significant it is usual to define $(\Delta T)_{\mathrm{ref}} = \bar{T} - T_1$ in which $\bar{T}$ is the mean transport temperature (vide (3.1) above) and this choice underlies the Nusselt number values calculated in [1]. However Winter [3] from

Table 1
Coefficients A_m^2 v Br.

Br	0	0.1	0.5	1.0
A_1^2	0.8176	0.7551	0.5299	0.3043
A_2^2	0.0982	0.0931	0.0739	0.0531
A_3^2	0.0328	0.0317	0.0276	0.0228

Table 2
$\bar{\theta}$ (Z) v Br.

Z \ Br	0	0.1	0.5	1.0
0.01	0.7487	0.7279	0.6670	0.6404
0.05	0.3918	0.3972	0.4310	0.4999
0.10	0.1867	0.2078	0.2980	0.4234
0.15	0.0892	0.1178	0.2348	0.3871
0.20	0.0426	0.0748	0.2046	0.3698
0.30	0.0097	0.0444	0.1833	0.3576
0.40	0.0022	0.0374	0.1784	0.3548
0.50	0.0005	0.0359	0.1773	0.3541
0.60	0.0001	0.0355	0.1770	0.3540
∞	0	0.0354	0.1770	0.3539

Table 3
Nu (Z) v Br.

Z \ Br	0	0.1	0.5	1.0
0.01	3.693	3.781	4.228	4.996
0.05	1.468	1.697	2.658	3.958
0.10	0.690	0.978	2.151	3.665
0.15	0.329	0.645	1.918	3.531
0.20	0.157	0.487	1.806	3.467
0.30	0.036	0.374	1.728	3.422
0.40	0.008	0.348	1.709	3.412
0.50	0.002	0.343	1.705	3.409
0.60	0.000	0.341	1.704	3.409
∞	0.000	0.341	1.704	3.408

a consideration of developing temperature profiles in pipe flow has adduced persuasive reasons for the inadequacy of the usual definition of the Nusselt number where viscous dissipation is significant. One main disadvantage cited is that in this definition an attempt is made to describe two fairly unrelated quantities Q_w and $(\Delta T)_{\text{ref}} = \bar{T} - T_1$ as functions of each other.

In the current note we take $(\Delta T)_{\text{ref}} = T_0 - T_1$, whence on the basis of Fourier's law of heat conduction together with the dimensionless forms (2.4), it is easy to show that the Nusselt number assumes the local form

$$\text{Nu}\,(z) = -\frac{d}{P} \oint_{\partial R} \frac{\partial \theta}{\partial \nu}\, ds \tag{3.5}$$

where $P = 2\pi a$ is the perimeter of the tube cross-section ∂R to which ν is normal and, for the sake of uniformity with [1], the characteristic length d is take to be $2a$. An applicaton of the two dimensional Gauss theorem followed by routine manipulations similar to those used in the evaluation of $\bar{\theta}(z)$ above then leads to

$$\mathrm{Nu}(z) = \frac{1}{4}\left[\sum_{p=1}^{\infty} A_p^2 \lambda_p e^{-\lambda_p z} + \frac{16\,\mathrm{Br}}{\{K(\bar{\varepsilon})\}^2}\right]. \tag{3.6}$$

The asymptotic values of (3.2) and (3.6) are

$$\bar{\theta}(\infty) = \frac{5\,\mathrm{Br}}{12\{K(\bar{\varepsilon})\}^3}\left(1 - \frac{4\bar{\varepsilon}}{25}\right); \quad \mathrm{Nu}(\infty) = \frac{4\,\mathrm{Br}}{\{K(\bar{\varepsilon})\}^2}$$

and arise solely from mechanical heating in the fluid.

The parameter $\bar{\varepsilon}$ has been discussed in [1]. Here we choose $\bar{\varepsilon} = -0.5$ as being typical of the general behaviour for negative $\bar{\varepsilon}$ values. Numerical values of A_m^2, $m = 1, 2$ and 3, have been computed on an ICL 1903 computer for values of the Brinkman number $\mathrm{Br} = 0.1$, 0.5 and 1.0. These are shown in Table 1 and have been used to calculate the functions $\bar{\theta}(z)$ and $\mathrm{Nu}(z)$ which are displayed in Tables 2 and 3. The case $\mathrm{Br} = 0$ derives from [1] and is included for comparison although the values quoted here for $\mathrm{Nu}(z)$ have been revised in accord with (3.6). Table 2 indicates significant enlargement of the transport temperature with increasing Brinkman number outside of a small region at the thermal inlet. This is accompanied by a very marked enhancement of the local heat transfer coefficient $\mathrm{Nu}(z)$ as is evidenced by Table 3. Finally it is noteworthy that the asymptotic values $\bar{\theta}(\infty)$ and $\mathrm{Nu}(\infty)$ for a Newtonian fluid ($\bar{\varepsilon} = 0$) exceed those for the slow flow of a non-Newtonian fluid as represented by the third order Rivlin-Ericksen fluid.

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Abstract

The effect of mechanical heating on the heat transfer characteristics of a heated non-Newtonian fluid in lineal flow through a circular tube is examined. Marked enhancement in the local rate of heat transferred to the tube wall is found.

Zusammenfassung

Die Wirkung der mechanischen Wärmeerzeugung auf die Wärmeübertragung einer erwärmten nicht-Newtonschen Flüssigkeit, in geradliniger Strömung in einem kreisrunden Rohr, wird untersucht. Es wird eine merkbare Vergrößerung des lokalen Wärmeüberganges an die Rohrwand gefunden.

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