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How to approximate the solutions of certain free boundary problems for the Laplace equation by using the contraction principle

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1. Introduction and main results

We derive a procedure, based on the contraction principle (Banach fixed point theorem), for numerically approximating the solutions of the following free boundary problem with hydrodynamic applications.

Problem 1 (see Fig. 1). Let be given a function $a(p) = a(x, y)$: $\mathbb{R}^2 \to \mathbb{R}$ which is continuous, strictly positive, weakly monotone increasing in y, and σ-periodic in x (for some value $\sigma > 0$). Let $\mathbb{B}$ denote the Banach space of all continuous, σ-periodic functions f: $\mathbb{R} \to \mathbb{R}$, endowed with the maximum norm $\|f\| = \max\{|f(x)|: x \in \mathbb{R}\}$. Given $F \in \mathbb{B}$, we define $\mathbb{X} = \{f \in \mathbb{B}: f > F\}$. For each $f \in \mathbb{X}$, let $\Omega(f) = \{p = (x, y) \in \mathbb{R}^2: F(x) < y < f(x)\}$ and let $U(f; p)$ be the σ-periodic (in x) solution of the boundary value problem $\nabla^2 U = 0$ in $\Omega(f)$, $U = 0$ on f, $U = 1$ on F (where f and F are viewed as their graphs in $\mathbb{R}^2$ *)). We seek a function $\tilde{f} \in \mathbb{X}$ such that

$$|\nabla U(\tilde{f}; p)| = a(p) \quad \text{on } \tilde{f}, \tag{1}$$

i.e., for each $p \in \tilde{f}$, $|\nabla U(\tilde{f}; q)| \to a(p)$ as $q \to p$, $q \in \Omega(f)$.

The existence of a function $\tilde{f} \in \mathbb{X}$ satisfying (1) follows from Beurling [7, Theorem 2]. The solution in unique (for given $\sigma > 0$ and functions $a(p)$ and $F \in \mathbb{B}$) by the Lindelöf principle (see [8, pp. 16–21] and [1, Lemma 4]). If $a(x, y)$ is a real analytic function of each coordinate variable, then according at a result of Lewy [9], the curve $\tilde{f}$ is analytic and $U(\tilde{f}; p)$ can be harmonically continued across $\tilde{f}$. Therefore, the interior normal derivative $D_n U(\tilde{f}; p)$ exists on $\tilde{f}$ and satisfies $D_n U(\tilde{f}; p) = |\nabla U(\tilde{f}; p)| = a(p)$.

Our method for approximating $\tilde{f}$ requires a certain family of operators T_ε: $\mathbb{X} \to \mathbb{X}$, $0 < \varepsilon < 1$, which we now proceed to define. For any $p, q \in \mathbb{R}^2$, we let $d_a(p, q)$ denote the infinum of $\|\gamma\|_a := \int_\gamma a(p')\,|dp'|$ among all

*) Since a function f: $\mathbb{R} \to \mathbb{R}$ is defined by its graph, we have $f = \text{graph}(f) := \{(x, f(x)): x \in \mathbb{R}\}$ Thus $p \in f$ means $p \in \text{graph}(f)$, etc.

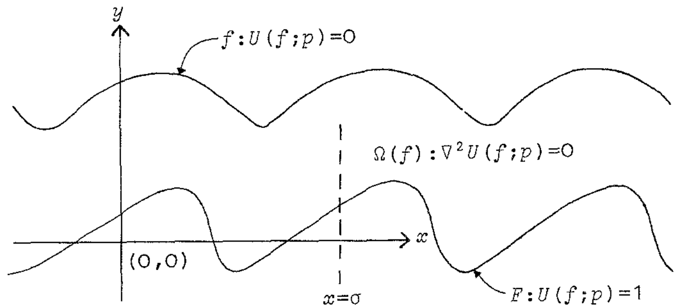

Figure 1

rectifiable curves γ joining p and q. (Thus d_a is a generalized distance function in $\mathbb{R}^2$. In the important case where $a(p) \equiv c$, we have $d_a(p, q) = c \cdot |p - q|$.) For $0 < \varepsilon < 1$, we let the auxilliary operators $\Phi_\varepsilon: \mathbb{X} \to \mathbb{X}$ and $\Psi_\varepsilon: \mathbb{X} \to \mathbb{X}$ be defined such that

$$\Phi_\varepsilon(f) = \{p \in \Omega(f): U(f; p) = \varepsilon\} \tag{2}$$

and

$$\Psi_\varepsilon(f) = \{p = (x, y) \in \mathbb{R}^2: y > f(x) \text{ and } d_a(f; p) = \varepsilon\}, \tag{3}$$

where $d_a(f; p) = \min\{d_a(q, \mathrm{p}): q \in f\}$. (The proof that these sets are indeed functions in $\mathbb{X}$ (i.e., graphs of functions in $\mathbb{X}$) is deferred to § 2.) Finally, we define

$$T_\varepsilon = \Psi_\varepsilon \circ \Phi_\varepsilon, \quad \text{i.e.,} \quad T_\varepsilon(f) = \Psi_\varepsilon(\Phi_\varepsilon(f)) \quad \text{for } f \in \mathbb{X} \text{ and } 0 < \varepsilon < 1. \tag{4}$$

Thus, for any $f \in \mathbb{X}$ and $0 < \varepsilon < 1$, $T_\varepsilon(f) \in \mathbb{X}$ is that curve whose points lie above, and at a generalized distance ε from, the level curve at height ε of the function $U(f; p)$. (See Fig. 2.) Notice that Φ_ε, Ψ_ε and T_ε are monotone operators, e.g.,

$$T_\varepsilon(f) \leqq T_\varepsilon(g) \quad \text{whenever } f \leqq g \text{ in } \mathbb{X}. \tag{5}$$

One can easily define a set $\tilde{\mathbb{X}} = \{f \in \mathbb{B}: F_1 \leqq f \leqq F_2\} \subset \mathbb{X}$, where $F_1, F_2 \in \mathbb{X} \cap C^2(\mathbb{R})$, such that

$$\tilde{f} \in \tilde{\mathbb{X}} \quad \text{and} \quad T_\varepsilon: \tilde{\mathbb{X}} \to \tilde{\mathbb{X}} \quad \text{for all } 0 < \varepsilon < 1. \tag{6}$$

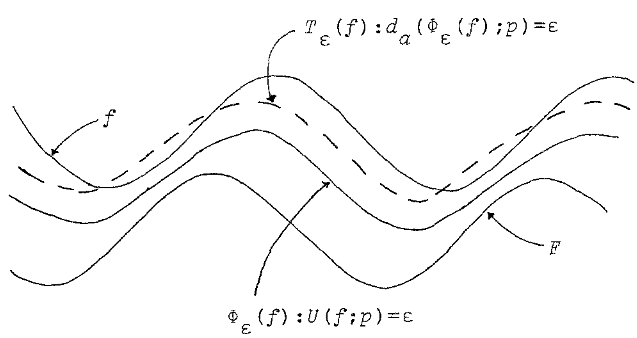

Figure 2
The operators $\Phi_\varepsilon: \mathbb{X} \to \mathbb{X}$ and $T_\varepsilon: \mathbb{X} \to \mathbb{X}$ in the case $a(p)$ = constant.

In order that $\tilde{f} \in \tilde{\mathbb{X}}$, it suffices that $|\nabla U(F_1; p)| > a(p)$ on F_1 and $|\nabla U(F_2; p)| < a(p)$ on F_2, by [7, Theorem 1]. By the monotonicity of T_ε, we have $T_\varepsilon: \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ provided that $F_1 < T_\varepsilon(F_i) < F_2$, $i = 1, 2$.

Our main result is

Theorem 1: *In Problem 1, assume the function $F(x)$ is Lipschitz continuously differentiable (L.c.d.) in $\mathbb{R}$. Then T_ε is contracting in $\tilde{\mathbb{X}}$ for any $0 < \varepsilon < 1$, i.e.,*

$$\| T_\varepsilon(f) - T_\varepsilon(g) \| \leqq \alpha \| f - g \| \quad \text{for all } f, g \in \tilde{\mathbb{X}}, \tag{7}$$

where $0 \leqq \alpha = \alpha(\varepsilon) < 1$. Thus, T_ε has a unique "fixed point" $\tilde{f}_\varepsilon \in \tilde{\mathbb{X}}$, which can be obtained by successive approximations, i.e.,

$$\| T_\varepsilon^n(f) - \tilde{f}_\varepsilon \| \leqq \frac{\alpha^n}{1 - \alpha} \| T_\varepsilon(f) - f \| \quad \text{for all } f \in \tilde{\mathbb{X}} \text{ and } n \in \mathbb{N}. \tag{8}$$

Moreover,

$$\| \tilde{f}_\varepsilon - \tilde{f} \| \to 0 \text{ as } \varepsilon \to 0+, \quad \text{where } \tilde{f} \in \tilde{\mathbb{X}} \text{ solves (1).} \tag{9}$$

The proof of Theorem 1 is given in §§ 3 and 4. The contraction principle, on which Theorem 1 is based, is discussed in, for example, [11], where the estimate (8) is also derived.

Remark 1: Basically, our method for approximating $\tilde{f}$ is to first choose ε very small, so that $\| \tilde{f}_\varepsilon - \tilde{f} \|$ is small, and then approximate $\tilde{f}_\varepsilon$ to within a small error by applying T_ε a sufficient number of times to some function $f \in \tilde{\mathbb{X}}$. It is not hard to see that $\tilde{f} \in \mathbb{X}$ satisfying (1) continues to exist and be approximable by this method even in some cases where the curve F is not the graph of a function, for example, when $a(p) \equiv 1$ and F is the periodic extension of the square-tooth curve in Fig. 5. The main difficulty of the method is: at each step in the inductive determination of the functions $T_\varepsilon^n(f)$, one must numerically approximate the function $U(T_\varepsilon^n(f); p)$ in $\Omega(T_\varepsilon^n(f))$ in order to approximately obtain the function $\Phi_\varepsilon(T_\varepsilon^n(f)) \in \mathbb{X}$ (see § 5). Some approximate solutions of Problem 1, which were obtained by this method, are graphed in Figs. 4 and 5.

Remark 2: Given $0 < \varepsilon < 1$ and functions $f, g \in \mathbb{X}$ satisfying $f \leqq \tilde{f}_\varepsilon \leqq g$, it follows from the monotonicity of T_ε (Eq. (5)) that

$$T_\varepsilon^n(f) \leqq \tilde{f}_\varepsilon \leqq T_\varepsilon^n(g) \quad \text{for all } n \in \mathbb{N}.$$

These inequalities provide a more practical means of estimating $\| T_\varepsilon^n(f) - \tilde{f}_\varepsilon \|$ than does (8), especially since $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$.

Remark 3: In the case where $a(x, y)$ is a real analytic function of each coordinate variable, the estimate (9) has the stronger form: $\| \tilde{f}_\varepsilon - \tilde{f} \| = O(\varepsilon)$ as $\varepsilon \to 0+$.

Remark 4: The following free boundary problem (Problem 2) is an interesting variant of Problem 1 in the context of doubly-connected plane regions (see [10] and [1, Lemma 11]). Let $b(p)$: $\mathbb{R}^2 \to \mathbb{R}_+$ be a continuous, strictly positive function such that $\lambda \cdot b(\lambda p)$ is weakly monotone increasing in $\lambda \geqq 0$ for each $p \in \mathbb{R}^2$, and let $\mathbb{C}$ denote the class of all curves Γ having polar coordinate representations $\Gamma = \{(\theta, r(\theta)): \theta \in \mathbb{R}\}$, where the function $r(\theta)$: $\mathbb{R} \to \mathbb{R}_+$ is continuous, strictly positive and 2π-periodic. Given $\Gamma^* \in \mathbb{C}$, we seek a curve $\mathring{\Gamma} \in \mathbb{C}$, containing Γ^* in its interior complement, such that $|\nabla \mathring{U}(p)| = b(p)$ on $\mathring{\Gamma}$, where $\mathring{U}(p)$ solves the boundary value problem $\nabla^2 \mathring{U} = 0$ in $\mathring{\Omega}$ (= the doubly connected region bounded by $\mathring{\Gamma} \cup \Gamma^*$), $\mathring{U} = 0$ on $\mathring{\Gamma}$, $\mathring{U} = 1$ on Γ^*. The conformal mapping $G(z) = i \cdot \log(z)$ (i.e., the transformation $x = -\theta$, $y = \log(r)$, where (θ, r) are polar coordinates) reduces Problem 2 to Problem 1 in the case where $\sigma = 2\pi$, $a(x, y) = r \cdot b(\theta, r)$ and $F(x) = \log(r^*(\theta))$. If $\tilde{f} \in \mathbb{X}$ solves Problem 1 in this case, then the corresponding solution of Problem 2 is given in polar coordinates by the function $\mathring{r}(\theta) = \exp(\tilde{f}(x))$. Notice that if $b(p) = 1$ in Problem 2, then $a(x, y) = \exp(y)$ in the equivalent form of Problem 1.

Remark 5: Another free boundary problem related to Problem 1 is the following. In the notation of Problem 1, given $A > 0$, one seeks a function $\hat{f} \in \mathbb{X}$ such that the set $([0, \sigma] \times \mathbb{R}) \cap \Omega(\hat{f})$ has area A, and $|\nabla U(\hat{f}; p)|$ is constant on $\hat{f}$. Exactly one function $\hat{f} \in \mathbb{X}$ has these properties. In [4], the author defined a related class of operators T_ε^*: $\mathbb{X} \to \mathbb{X}$ which preserve area (of $([0, \sigma] \times \mathbb{R}) \cap \Omega(f)$), but are neither monotone nor contracting, and showed that $\hat{f}$ can be approximated in the maximum norm by essentially the method of successive approximations using the operators T_ε^*, $0 < \varepsilon < 1$.

Additional notation. For any $p = (x, y) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, and sets $P, Q \subset \mathbb{R}^2$, we define $p + \alpha = (x, y + \alpha)$, $Q + \alpha = \{q + \alpha: q \in Q\}$, $Q + p = \{q + p: q \in Q\}$, $d_a(P; Q) = \inf\{d_a(p, q): p \in P, q \in Q\}$, $d(P; Q) = \inf\{|p - q|: p \in P, q \in Q\}$ and $d(P; q) = d(P; \{q\})$. Thus for $f \in \mathbb{B}$ and $\alpha \in \mathbb{R}$, we have

$$f + \alpha = \{(x, f(x) + \alpha): x \in \mathbb{R}\} \in \mathbb{B}.$$

For any $f \in \mathbb{B}$, we define $S(f) = \{p = (x, y) \in \mathbb{R}^2: y \geqq f(x)\}$.

2. Proof that $\Phi_\varepsilon, \Psi_\varepsilon$: $\mathbb{X} \to \mathbb{X}$

For any fixed $f \in \mathbb{X}$ and $0 < \varepsilon < 1$, the sets $\Phi_\varepsilon(f)$ and $\Psi_\varepsilon(f)$ are uniquely defined by (2) and (3). Thus it suffices to show that these sets are (graphs of) functions in $\mathbb{X}$. Now the strong maximum principle implies $0 < U(f; p) < 1$ in $\Omega(f)$. Thus $U(f; p)$ is strictly monotone decreasing in y in $\Omega(f)$, since for any sufficiently small $\delta > 0$, we have

$$V_\delta(p) := U(f; p) - U(f; p - \delta) < 0$$

on $F \cup (f - \delta)$ and hence throughout $\Omega(f - \delta)$. Thus $\Phi_\varepsilon(f)$ is (the graph of) a function $[\Phi_\varepsilon(f)](x)$: $\mathbb{R} \to \mathbb{R}$, since for each $x \in \mathbb{R}$ the equation $U(f; x, y) = \varepsilon$

is solved by exactly one value $y \in (F(x), f(x))$. Clearly $\Phi_\varepsilon(f) > F$, and the continuity and σ-periodicity of $\Phi_\varepsilon(f)$ (viewed as a function of x, not f) follows from the strict monotonicity and σ-periodicity (in x) of $U(f;p)$. Thus $\Phi_\varepsilon(f) \in \mathbb{X}$.

Given $f \in \mathbb{X}$, $d_a(f;p)$ is clearly a continuous, σ-periodic (in x) function in $\mathbb{R}^2$ such that $d_a(f;p) = 0$ in f, $d_a(f;p) > 0$ in $\mathbb{R}^2 \setminus f$, and $d_a(f;p) \to +\infty$ uniformly in x as $y \to +\infty$. Moreover, the function $d_a(f;p)$ is strictly monotone increasing in y in $S(f) := \{p = (x,y) \in \mathbb{R}^2 \colon y \geqq f(x)\}$, as is seen by the following argument. Given $\delta > 0$ and points $p, p-\delta \in S(f)$, let $\varrho = d_a(f-\delta; f) > 0$, and let γ be an arc joining p to f such that $\| \gamma \|_a < d_a(f;p) + \varrho/2$. Using the assumed monotonicity of the function $a(q)$, one sees that

$$d_a(f;p-\delta) \leqq \|(\gamma-\delta) \cap S(f)\|_a \leqq \|\gamma-\delta\|_a - \varrho \leqq \|\gamma\|_a - \varrho \leqq d_a(f;p) - \varrho/2.$$

Using these properties of $d_a(f;p)$, one can show $\Psi_\varepsilon(f) := \{p \in S(f)\colon d_a(f;p) = \varepsilon\}$ is a function in $\mathbb{X}$ by the arguments already sketched for $\Phi_\varepsilon(f)$.

3. The proof that $T_\varepsilon\colon \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ is a contraction for any $0 < \varepsilon < 1$

To begin with, one sees using the assumed monotonicity of the function $a(q)$ that

$$d_a(f;p) \leqq d_a(f+\delta; p+\delta) \leqq d_a(g; p+\delta), \qquad p \in S(f),$$

and hence

$$\Psi_\varepsilon(g) \leqq \Psi_\varepsilon(f) + \delta,$$

both for any $\delta \geqq 0$ and $f, g \in \mathbb{B}$ satisfying $g \leqq f + \delta$, and it follows (by interchanging f and g) that

$$\|\Psi_\varepsilon(f) - \Psi_\varepsilon(g)\| \leqq \|f-g\| \quad \text{for all } f, g \in \mathbb{B}.$$

Therefore (7) follows (in view of the definition $T_\varepsilon = \Psi_\varepsilon \circ \Phi_\varepsilon$) if, for any $0 < \varepsilon < 1$, we can determine a value $0 \leqq \alpha = \alpha(\varepsilon) < 1$ such that

$$\|\Phi_\varepsilon(f) - \Phi_\varepsilon(g)\| \leqq \alpha \|f-g\| \quad \text{for all } f, g \in \tilde{\mathbb{X}}.$$

Due to the monotonicity of the operators Φ_ε, it actually suffices to show that (for some value $0 \leqq \alpha = \alpha(\varepsilon) < 1$)

$$\Phi_\varepsilon(f+\delta) \leqq \Phi_\varepsilon(f) + \alpha\delta \quad \text{for all } f \in \tilde{\mathbb{X}} \text{ and } 0 \leqq \delta \leqq \tilde{\delta}, \tag{11}$$

where $\tilde{\delta} = \|F_2 - F_1\|$. It is convenient to divide our somewhat involved proof of (11) into four lemmas, which we now state. Clearly (11) follows by combining Lemmas 2 and 4. The value $0 < \varepsilon < 1$ is assumed fixed for the remainder of this section.

Lemma 1: *If F is L.c.d., then there exists a constant $\Theta > 0$ such that*

$$U(f+\delta; p+\delta) \leqq (1 - \Theta\delta)\,\varepsilon \quad \textit{for } f \in \tilde{\mathbb{X}}, p \in \Phi_\varepsilon(f) \textit{ and } 0 \leqq \delta \leqq \tilde{\delta}.$$

Lemma 2: *If F is L.c.d. and $\Theta > 0$ is the constant in Lemma 1, then*

$$\Phi_\varepsilon(f+\delta) \leqq \Phi_\varepsilon(f) + (1 - [\Theta\,\varepsilon/M(f;\delta)])\,\delta \quad \textit{for all } f \in \tilde{\mathbb{X}} \textit{ and } 0 \leqq \delta \leqq \tilde{\delta},$$

where

$$M(f;\delta) = \max\{|D_y U(f+\delta;p)| : p \in E(f;\delta)\} \quad \textit{and}$$
$$E(f;\delta) = \text{closure}(\Omega(\Phi_\varepsilon(f)+\delta) \cap S(\Phi_\varepsilon(f+\delta))).$$

Lemma 3: $\inf\{d(f;\Phi_\varepsilon(f)) : f \in \tilde{\mathbb{X}}\} > 0.$

Lemma 4: $\sup\{M(f;\delta) : f \in \tilde{\mathbb{X}}, 0 \leqq \delta \leqq \tilde{\delta}\} < \infty.$

Proof of Lemma 1: We have

$$U(f+\delta;p+\delta) \leqq U(f;p) \cdot \max\{U(f+\delta;q) : q \in F+\delta\} \tag{12}$$

for any $p \in \Omega(f)$, $f \in \tilde{\mathbb{X}}$ and $\delta \geqq 0$ by the maximum principle, since (12) obviously holds for all $p \in f \cup F$. Again using the maximum principle, we find that $U(f+\delta;p) \leqq U(F_3;p)$ on $F \cup (f+\delta)$ and hence throughout $\Omega(f+\delta)$ for $0 \leqq \delta \leqq \tilde{\delta}$, where $F_3 = F_2 + \tilde{\delta}$. Therefore, it follows from (12) that

$$U(f+\delta;p+\delta) \leqq \zeta(\delta) \cdot U(f;p) \quad \text{for } p \in \Omega(f),\ f \in \tilde{\mathbb{X}} \text{ and } 0 \leqq \delta \leqq \tilde{\delta}, \tag{13}$$

where $\zeta(\delta) = \max\{U(F_3;q) : q \in F+\delta\} \leqq 1$. Due to our assumptions that $F(x)$ and $F_2(x)$ are L.c.d. in $\mathbb{R}$, the function $\nabla U(F_3;p) : \Omega(F_3) \to \mathbb{R}^2$ has a continuous extension to closure $(\Omega(F_3))$ satisfying $\eta := \min\{|\nabla U(F_3;p)| : p \in F \cup F_3\} > 0$. Thus if we set $L = \max\{|D_x F(x)|, |D_x F_2(x)| : x \in \mathbb{R}\}$ and $\Theta = \eta/\sqrt{1+L^2}$, then $D_y U(F_3;p) \leqq -\Theta < 0$ on $F \cup F_3$ and hence throughout $\Omega(F_3)$, from which it follows that

$$\zeta(\delta) \leqq 1 - \Theta\,\delta, \quad 0 \leqq \delta \leqq \tilde{\delta}. \tag{14}$$

Our assertion follows by combining (13) and (14).

Proof of Lemma 2: Let $f_\delta = \Phi_\varepsilon(f+\delta)$, $\Omega_\delta = \Omega(f+\delta)$ and $U_\delta(p) = U(f+\delta;p)$ in Ω_δ for all $\delta \geqq 0$, where $f \in \tilde{\mathbb{X}}$ is fixed but arbitrary. In this notation, Lemma 1 states that

$$U_\delta(p) \leqq (1 - \Theta\,\delta)\,\varepsilon < \varepsilon \quad \text{for all } p \in f_0 + \delta,\ 0 \leqq \delta \leqq \tilde{\delta}, \tag{15}$$

(where $f_0 = f_\delta|_{\delta=0}$). Therefore $f_\delta \leqq f_0 + \delta$ for $0 \leqq \delta \leqq \tilde{\delta}$, since $U_\delta(p) = \varepsilon$ on f_δ and the function $U_\delta(p)$ is monotone decreasing in y in Ω_δ. Moreover, by the theorem of the mean we have

$$|U_\delta(x, f_\delta(x)) - U_\delta(x, f_0(x)+\delta)| \leqq (f_0(x) + \delta - f_\delta(x))\, M(f;\delta) \tag{16}$$

for all $x \in \mathbb{R}$ and $0 \leqq \delta \leqq \tilde{\delta}$, where

$$M(f;\delta) = \max\{|D_y U_\delta(p)| : f_\delta(x) \leqq y \leqq f_0(x) + \delta,\ 0 \leqq x \leqq \sigma\} < \infty.$$

Since $U_\delta(x, f_0(x)+\delta) \leqq (1-\Theta\,\delta)\,\varepsilon$ (by (15)) and $U_\delta(x, f_\delta(x)) = \varepsilon$, we conclude from (16) that

$$f_0(x) + \delta - f_\delta(x) \geqq (\Theta\,\varepsilon/M(f;\delta))\,\delta\,,$$

or, in other words,

$$f_\delta(x) \leqq f_0(x) + (1 - [\Theta\,\varepsilon/M(f;\delta)])\,\delta$$

for all $x \in \mathbb{R}$ and $0 \leqq \delta \leqq \tilde{\delta}$. But this is equivalent to our assertion.

Proof of Lemma 3 (see Fig. 3). Let $K = (M + N)/L$, where L was previously defined, $M = \max\{F_2(x) : x \in \mathbb{R}\} - \min\{F_1(x) : x \in \mathbb{R}\}$ and $N = \min\{F_1(x) - F(x) : x \in \mathbb{R}\}$. Let Ω be the finite, simply connected region bounded by $\Gamma^* \cup \Gamma$, where $\Gamma = ([-K, K] \times \{M\}) \cup (\{0\} \times [0, M])$ and $\Gamma^* = \{(x, -N + L|x|) : x \in \mathbb{R},\ |x| < K\}$, and let $U(p)$ solve the boundary value problem $\nabla^2 U = 0$ in Ω, $U = 1$ on Γ^*. $U = 0$ on Γ. Clearly $d(\gamma; 0) := \inf\{|q| : q \in \gamma\} > 0$, where we define $\gamma = \{p \in \Omega : U(p) = \varepsilon\}$. To prove Lemma 3, we will show that

$$d(\Phi_\varepsilon(f); p_0) \geqq d(\gamma; 0) \quad \text{for all } f \in \tilde{\mathbb{X}} \text{ and } p_0 \in f.$$

It suffices to show, for any fixed $f \in \tilde{\mathbb{X}}$ and $p_0 \in f$, that

$$U(p) \geqq \mathring{U}(p) \quad \text{in} \quad \Omega \cap \mathring{\Omega}\,, \tag{17}$$

where $\mathring{\Omega} = \Omega(f) - p_0$ and $\mathring{U}(p) := U(f; p + p_0)$ is the σ-periodic (in x) solution of the boundary value problem $\nabla^2 \mathring{U} = 0$ in $\mathring{\Omega}$, $\mathring{U} = 0$ on $\mathring{f} := f - p_0$, $\mathring{U} = 1$ on $\mathring{F} := F - p_0$. Let $\bar{\Omega}$ be the simply-connected region bounded by $\mathring{F}$ and $\bar{\Gamma} := (\mathbb{R} \times \{M\}) \cup (\{0\} \times [0, M])$, and let $\bar{U}(p)$ be the bounded harmonic function in $\bar{\Omega}$ satisfying the boundary conditions $\bar{U} = 0$ on $\bar{\Gamma}$, $\bar{U} = 1$ on $\mathring{F}$. Now $0 \leqq \bar{U}(p) \leqq 1$ in $\bar{\Omega}$ by the maximum principle, and $\mathring{\Omega} \subset \bar{\Omega}$, whereas $\mathring{\Omega}$ and $\bar{\Omega}$ share the lower boundary component $\mathring{F}$. Thus $\bar{U}(p) - \mathring{U}(p) \geqq 0$ on $\mathring{F} \cup \mathring{f}$ and hence throughout $\mathring{\Omega}$. Since $\Gamma \subset \bar{\Gamma}$ and $\Gamma^* \subset \text{Closure}(\bar{\Omega})$, we also have $U(p) - \bar{U}(p) \geqq 0$ on $\Gamma^* \cup \Gamma$ and hence throughout Ω. This completes the proof of (17), and therefore of Lemma 3.

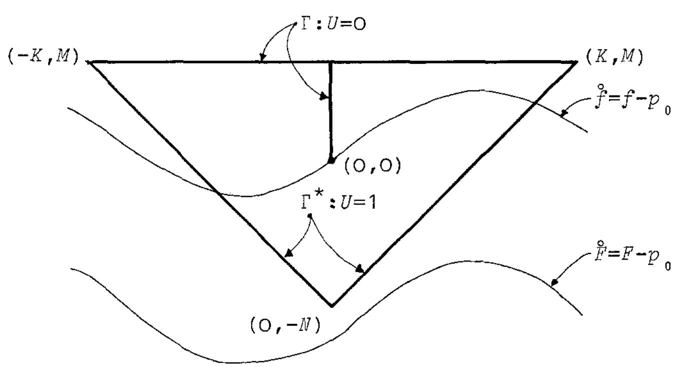

Figure 3

Proof of Lemma 4: For any $f \in \tilde{\mathbb{X}}$, $0 \leqq \delta \leqq \tilde{\delta}$ and $p \in E(f; \delta)$, we have

$$d(p; F) \geqq d(F; \Phi_\varepsilon(f+\delta)) \geqq d(F; \Phi_\varepsilon(F_1)) > 0$$

(since $f + \delta \geqq F_1$, implying $\Phi_\varepsilon(f+\delta) \geqq \Phi_\varepsilon(F_1) > F$) and

$$d(p; f+\delta) \geqq d(f+\delta; \Phi_\varepsilon(f) + \delta) = d(f; \Phi_\varepsilon(f)).$$

Applying Lemma 3, we conclude that there exists a constant $\varkappa > 0$ such that $d(p; F \cup (f+\delta)) \geqq \varkappa$ and hence

$$B_\varkappa(p) \subset \Omega(f+\delta) \tag{18}$$

for all $f \in \tilde{\mathbb{X}}$, $0 \leqq \delta \leqq \tilde{\delta}$ and $p \in E(f; \delta)$, where $B_\varkappa(p) = \{q \in \mathbb{R}^2 : |q - p| < \varkappa\}$. In view of (18), our assertion follows directly from a well known inequality obtained by differentiating the Poisson integral formula. Namely, if $W(q)$ is a harmonic function in $B_\varkappa(p)$, then

$$|\nabla W(p)| \leqq (2/\varkappa) \cdot \sup \{|W(q)| : q \in B_\varkappa(p)\}. \tag{19}$$

In fact one sees using (18) and (19) that $M(f; \delta) \leqq 2/\varkappa$ for all $f \in \tilde{\mathbb{X}}$ and $0 \leqq \delta \leqq \tilde{\delta}$.

Remark 6: The proof that T_ε is a contraction in $\tilde{\mathbb{X}}$ is simpler in the case where there exists a constant $\mu > 0$ such that $a(p+\delta) \geqq (1 + \mu\,\delta)\,a(p)$ for all $\delta \geqq 0$ and $p = (x, y) \in \mathbb{R}^2$ satisfying $y \geqq F(x)$. In this case, one can show $\Psi_\varepsilon(f+\delta) \leqq \Psi_\varepsilon(f) + (1 - (\mu\,\varepsilon/\bar{a}))\,\delta$ for all $f \in \tilde{\mathbb{X}}$ and $0 \leqq \delta \leqq \tilde{\delta}$, where $\bar{a} = \max\{a(x, F_2(x) + \tilde{\delta})) : x \in \mathbb{R}\}$. It follows by the discussion at the beginning of this section that (7) holds with $\alpha = (1 - (\mu\,\varepsilon/\bar{a}))$.

4. Proof that $\|\tilde{f}_\varepsilon - \tilde{f}\| \to 0$ as $\varepsilon \to 0+$

Let $\lambda_\pm(\varepsilon) = \max\{\pm(\tilde{f}_\varepsilon(x) - \tilde{f}(x)) : x \in \mathbb{R}\}$, $0 < \varepsilon < 1$, and let $E_\pm = \{\varepsilon \in (0, 1) : \lambda_\pm(\varepsilon) \geqq 0\}$. Since $\|\tilde{f}_\varepsilon - \tilde{f}\| = \max\{\lambda_+(\varepsilon), \lambda_-(\varepsilon)\}$, $0 < \varepsilon < 1$, it suffices to show

$$\limsup_{\varepsilon \to 0+} \lambda_\pm(\varepsilon) \leqq 0. \tag{20}$$

In order to prove (20) in the "+" case, choose (for each $\varepsilon \in E_+$) points $p_0(\varepsilon) \in \tilde{f}_\varepsilon \cap (\tilde{f} + \lambda_+(\varepsilon))$ and $p_1(\varepsilon) \in \Phi_\varepsilon(\tilde{f}_\varepsilon)$ satisfying $d_a(p_0(\varepsilon), p_1(\varepsilon)) = d_a(p_0(\varepsilon), \Phi_\varepsilon(\tilde{f}_\varepsilon)) = \varepsilon$. Since $\tilde{f}_\varepsilon \leqq \tilde{f} + \lambda_+(\varepsilon)$, $0 < \varepsilon < 1$, we conclude using the maximum principle and Lemma 1 that

$$\begin{aligned} \varepsilon = U(\tilde{f}_\varepsilon; p_1(\varepsilon)) &\leqq U(\tilde{f} + \lambda_+(\varepsilon); p_1(\varepsilon)) \\ &\leqq (1 - \Theta\,\lambda_+(\varepsilon))\,U(\tilde{f}; p_1(\varepsilon) - \lambda_+(\varepsilon)), \quad \varepsilon \in E_+. \end{aligned} \tag{21}$$

On the other hand

$$\begin{aligned} d_a(\tilde{f}; p_1(\varepsilon) - \lambda_+(\varepsilon)) &\leqq d_a(p_0(\varepsilon) - \lambda_+(\varepsilon), p_1(\varepsilon) - \lambda_+(\varepsilon)) \\ &\leqq d_a(p_0(\varepsilon), p_1(\varepsilon)) = \varepsilon \end{aligned}$$

by the monotone property of the function $a(q)$. Also, one can show using (1) that

$$||\nabla U(\tilde{f}; p)| - a(p)| \leqq \varphi(d_a(\tilde{f}; p)) \quad \text{in } \Omega(\tilde{f}),$$

where we use φ to denote an arbitrary function such that $\varphi(\alpha) \to 0$ as $\alpha \to 0+$. Therefore, if $\gamma_\varepsilon \subset \Omega(\tilde{f})$, $\varepsilon \in E_+$, denotes a smooth curve joining the point $p_1(\varepsilon) - \lambda_+(\varepsilon)$ to $\tilde{f}$, with generalized length $\|\gamma_\varepsilon\|_a < \varepsilon(1 + \varphi(\varepsilon))$, then

$$\begin{aligned} U(\tilde{f}; p_1(\varepsilon) - \lambda_+(\varepsilon)) &\leqq \int_{\gamma_\varepsilon} |\nabla U(\tilde{f}; q)|\,|dq| \leqq \int_{\gamma_\varepsilon} (a(q) + \varphi(d_a(\tilde{f}; q)))\,|dq| \\ &\leqq \|\gamma_\varepsilon\|_a (1 + \varphi(\varepsilon)) \leqq \varepsilon(1 + \varphi(\varepsilon)), \quad \varepsilon \in E_+. \end{aligned} \tag{22}$$

By combining (21) and (22), we obtain

$$(1 - \Theta\,\lambda_+(\varepsilon))(1 + \varphi(\varepsilon)) \geqq 1, \quad \varepsilon \in E_+,$$

from which (20) immediately follows in the "+" case.

For the proof of (20) in the "−" case, for each $\varepsilon \in E_-$ let γ_ε denote a curve of steepest ascent of the function $U(\tilde{f}; q)$ whose endpoints are $p_0(\varepsilon) \in \tilde{f} \cap (\tilde{f}_\varepsilon + \lambda_-(\varepsilon))$ and $p_1(\varepsilon) \in \Phi_\varepsilon(\tilde{f}_\varepsilon) + \lambda_-(\varepsilon)$. Since $\tilde{f} \leqq \tilde{f}_\varepsilon + \lambda_-(\varepsilon)$, $0 < \varepsilon < 1$, we conclude, using the maximum principle and Lemma 1, that

$$\begin{aligned} U(\tilde{f}; p_1(\varepsilon)) &\leqq U(\tilde{f}_\varepsilon + \lambda_-(\varepsilon); p_1(\varepsilon)) \leqq (1 - \Theta\,\lambda_-(\varepsilon))\, U(\tilde{f}_\varepsilon; p_1(\varepsilon) - \lambda_-(\varepsilon)) \\ &= (1 - \Theta\,\lambda_-(\varepsilon))\,\varepsilon, \quad \varepsilon \in E_-. \end{aligned} \tag{23}$$

Due to (1), we have

$$||\nabla U(\tilde{f}; p)| - a(p)| \leqq \varphi(U(\tilde{f}; p)) \quad \text{in } \Omega(f).$$

Since $\gamma_\varepsilon \subset \Omega(\tilde{f}) \backslash \Omega(\Phi_\varepsilon(\tilde{f}))$, we conclude that

$$\begin{aligned} U(\tilde{f}; p_1(\varepsilon)) &= \int_{\gamma_\varepsilon} |\nabla U(f; q)|\,|dq| \geqq \int_{\gamma_\varepsilon} (a(q) - \varphi(\varepsilon))\,|dq| \\ &\geqq \|\gamma_\varepsilon\|_a (1 - \varphi(\varepsilon)) \geqq d_a(p_0(\varepsilon), p_1(\varepsilon))(1 - \varphi(\varepsilon)) \\ &\geqq \varepsilon(1 - \varphi(\varepsilon)), \quad \varepsilon \in E_-. \end{aligned} \tag{24}$$

By combining (23) and (24), we obtain

$$\Theta \cdot \lambda_-(\varepsilon) \leqq \varphi(\varepsilon), \quad \varepsilon \in E_-,$$

from which (20) immediately follows in the "−" case.

5. Numerical results

Our basic procedure is to choose $\varepsilon > 0$ small, so that $\|\tilde{f}_\varepsilon - \tilde{f}\|$ is small, and then inductively compute the functions $f_n = T_\varepsilon^n(f)$ (for some $f \in \mathbb{X}$) until $\|f_{n+1} - f_n\|$ is very small. We have tried out this procedure in several cases where $a(p) \equiv C$ (C a constant) (see Figs. 4 and 5).

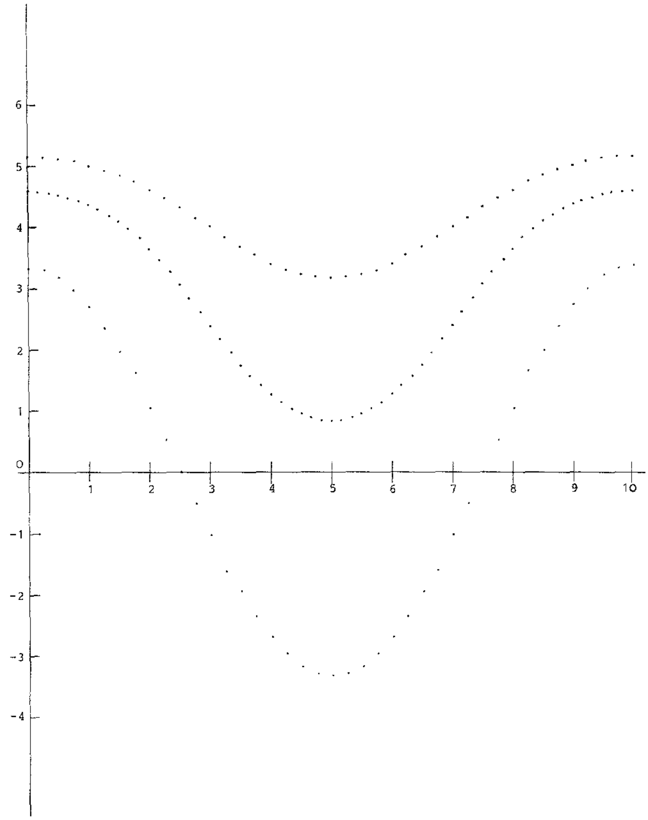

Figure 4

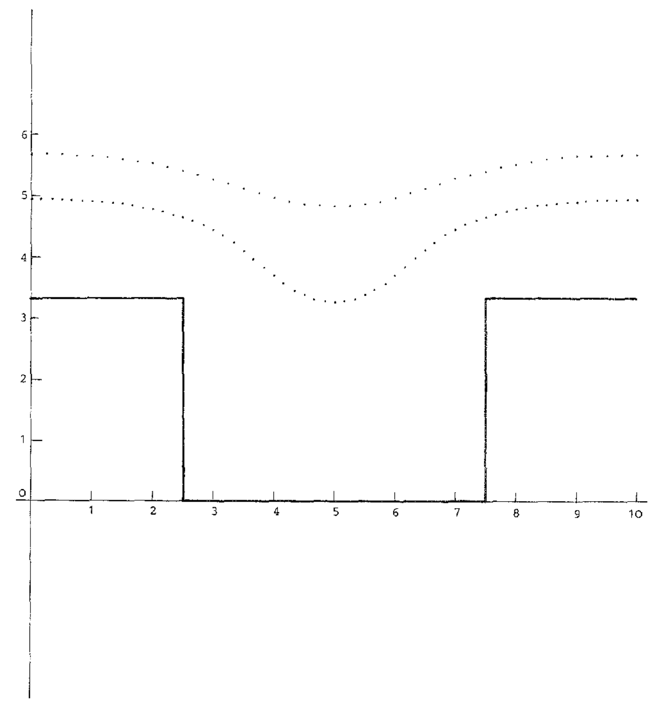

Figure 5

Our main difficulty was in making a "reasonable" choice of ε. It is important that ε be not too small, because the rate at which the curves f_n progress toward $\tilde{f}_\varepsilon$ is roughly proportional to ε. Also, the distance between neighbouring points in the rectangular grid used for the discrete computation of the functions $U(f_n; p)$, $n = 1, 2, 3, \ldots$, must be substantially smaller than (ε/C) (= the approximate distance between $\tilde{f}_\varepsilon$ and $\Phi_\varepsilon(\tilde{f}_\varepsilon)$). Thus, the cost of numerically approximating $\tilde{f}_\varepsilon$ increases rapidly as ε decreases toward 0. On the other hand, ε should be small enough so that $\|\tilde{f}_\varepsilon - \tilde{f}\|$ is "sufficiently small". One potential tool for deciding when ε fullfills this requirement would be an *a priori* bound M such that $\|\tilde{f}_\varepsilon - \tilde{f}\| \leqq M\varepsilon$, $0 < \varepsilon < 1$. In fact the author obtained such a bound (assuming $a(p) \equiv C$), but did not succeed in finding one which, in practical cases, was small enough to serve the indicated purpose. Therefore, in order to try out our method, we chose a modestly small value of ε (namely $\varepsilon = 1/5$) and computed $\tilde{f}_\varepsilon$ in this case, leaving unanswered the question of how small $\|\tilde{f}_\varepsilon - \tilde{f}\|$ is. The results thus obtained are graphed in Figs. 4 and 5.

In Fig. 4, the lowest curve represents the function $F(x) = \frac{10}{3}\cos(\pi x/5)$. The middle and upper curves approximate the functions $\tilde{f}_\varepsilon(x)$, $\varepsilon = 1/5$, which correspond to F in the cases $a(p) \equiv 3/5$ and $a(p) \equiv 2/5$, respectively. The relative improvement $\|f_{n+1} - f_n\|$ at the last completed iteration was .005 in the case $a(p) = 3/5$ and .0012 in the case $a(p) = 2/5$.

For our results in Fig. 5, we let F be a (10-periodic) square-tooth curve of height 10/3, as shown. The lower and upper computed curves approximate the function $\tilde{f}_\varepsilon(x)$ $(\varepsilon = 1/5)$ in the cases $a(p) \equiv 3/5$ and $a(p) \equiv 2/5$, respectively. The relative improvement at the final iteration was .0013 in the case $a(p) \equiv 3/5$ and .000022 in the case $a(p) \equiv 2/5$.

In all computed curves in Figs. 4 and 5, the horizontal separation of points indicates the size of the grid used for computing the functions $U(f_n; p)$.

Remark 7: Although we have not attempted this, the efficiency of our procedure for approximating $\tilde{f}$ could no doubt be improved by defining $f_{n+1} = T_{\varepsilon_n}(f_n)$, $n = 1, 2, 3, \ldots$, where the values $0 < \varepsilon_n < 1$ are initially large, but gradually decrease as $\|f_n - f_{n-1}\|$ decreases.

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Note added in proof. Due to the monotonicity of T_ε, if $T_\varepsilon(f) \leqq f$ for some $f \in \mathbb{X}$, then $f \geqq \tilde{f}_\varepsilon$ and in fact the functions $f_n = T_\varepsilon^n(f)$ decrease monotonically to their limit $\tilde{f}_\varepsilon$. Similarly, if $T_\varepsilon(f) \geqq f$, then $f \leqq \tilde{f}_\varepsilon$ and the f_n increase to $\tilde{f}_\varepsilon$ (see [12, p. 6]). This provides a simple test for determining whether a given function $f \in \mathbb{X}$ is an upper or lower bound for $\tilde{f}_\varepsilon$.

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Zusammenfassung

Wir zeigen, wie der freie Rand einer idealen Flüssigkeit, welcher einer verallgemeinerten Bernoulli-Bedingung genügt, unter geeigneten Umständen approximiert werden kann. Unsere Methode stützt sich auf eine Klasse freier Randperturbationsoperatoren T_ε, $0 < \varepsilon < 1$, welche relativ zu einer geeigneten Norm und Ränderklasse kontrahierend sind und deren Fixpunkte gegen die gewünschte Lösung der freien Randaufgabe mit $\varepsilon \to 0+$ konvergieren.

Abstract

We show how the free boundary of an ideal fluid, subject to a generalized Bernoulli condition, can (under appropriate circumstances) be approximated. Our method is based on a class of free-boundary perturbation operators T_ε, $0 < \varepsilon < 1$, which are all contracting relative to a suitable norm and class of boundaries, and whose fixed points converge to the desired free boundary solution as $\varepsilon \to 0+$.

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