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# Wave splitting of the Timoshenko beam equation in the time domain

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## 1. Introduction

During the last decade, time domain approaches to inverse scattering problems based on so called wave splitting have been highly successful. Particularly, in the field of electromagnetics, but also in the context of continuous mechanical systems, wave splitting in conjunction with invariant imbedding or Green's function technique has been used to solve a number of interesting inverse scattering problems [6]. The systems considered have in general been such that they can be modeled by hyperbolic second-order differential equations, but wave splitting has also been performed on some non-hyperbolic equations [15]. Of particular interest for us in the present context is the wave splitting of the fourth-order Euler-Bernoulli (E-B) equation given in [15].

In the analysis of stationary vibrations of beams, it is in many cases considered sufficient to model the beams by means of the Euler-Bernoulli equation. This simple model can be found in many textbooks on vibrations and waves [3]. It is even used for cases of anharmonic loading by means of a Fourier decomposition of the fields [5]. The Euler-Bernoulli equation is an equation which in many respects resembles the parabolic heat equation. While being a perfectly respectable equation for vibrations of not too short a wavelength, it shares one clearly undesirable feature of the latter equation; for transient phenomena it predicts an infinite speed of propagation. As for the heat equation, it is perhaps fair to say that few people have cared about this, since the equation is sufficiently accurate for most of the problems considered by engineers. However, in the context of inverse scattering problems, it turns out that the infinite speed of propagation causes serious problems [15]. It is in fact fatal in the sense that it apparently precludes the possibility of reconstructing the material properties from scattering data. This is due to the fact that the reflection operator for all times will depend on the material properties of the entire scattering region, which makes imbedding approaches less likely to be effective.

While the Euler-Bernoulli equation is unsuitable for inverse dynamical problems, it should be mentioned that the inverse problem for the static Euler-Bernoulli equation, with a sought non-linear deflection-dependent load, admits an, in fact entirely elementary, solution [11].

The unphysical nature of the dynamic Euler-Bernoulli equation has perhaps been of little concern to most investigators, but for other reasons more accurate equations have been developed. The standard derivation of the Euler-Bernoulli equation has at a preliminary stage a term containing the effects of rotational inertia of the beam sections (see, e.g., [3]). This term is usually discarded as being in some sense small. As pointed out in Ref. [2], the Euler-Bernoulli equation with the rotational inertia term included, commonly referred to as the Rayleigh (R) equation, has the advantage over the "plain" Euler-Bernoulli equation of having an upper bound on the phase velocity. This does, however, not save it from being unphysical in precisely the same manner as the Euler-Bernoulli equation. But what is even more surprising is that the Rayleigh equation even has an upper bound on its group velocity, and still allows an infinite speed of propagation. This somewhat confusing circumstance will be clarified in an appendix.

A considerable improvement from this point of view is offered by the Timoshenko (T) equation, derived in [13]. In this equation, the effects of both rotational inertia and shearing of the beam sections are taken into account. Incidentally, Timoshenko shows that as far as the calculation of eigenfrequencies is concerned, the shearing term in a typical case is roughly four times as important as the rotational inertia term. From the present point of view, the shearing term is all-important, as it makes the equation hyperbolic and thus removes the infinite wave speed.

There are some considerations on the accuracy of the Timoshenko equation which should be made. The Timoshenko equation (as well as the Euler-Bernoulli and Rayleigh equations) is derived under the assumption that the wave-length is greater than the extension of the cross section of the beam. If this assumption is not fulfilled the three-dimensional equations of linear elasticity should presumably have to be used, the beam acting as an elastic wave-guide. This probably means that when inverse problems for a beam is considered, the results of reconstructing various quantities, varying with the length coordinate, could only be known to be accurate to within this approximation. One would not expect the results to be accurate (compared to three-dimensional theory) at length scales less than the transverse extension of the beam.

Timoshenko has made some comparisons of the eigenfrequency predictions from the Timoshenko equation with those from three-dimensional elasticity in the case of very slender or very flat beams of rectangular cross section [14]. An analysis along similar lines, i.e., a comparison with exact solutions, could be used to assess the accuracy of an inverse solution based on the Timoshenko equation. The point to be made is that, while a procedure for reconstructing the beam properties based on the Timoshenko equation can be "exact", the results can still be inaccurate to within the limits posed by the accuracy of the Timoshenko equation itself.

Recent years have witnessed a renewed interest in the derivation of the various beam equations and their appropriate boundary conditions. Two examples of this are Ref. [12], wherein the beam equations are derived in a novel fashion starting from non-linear continuum mechanics, and Ref. [7], in which a careful analysis from three-dimensional elasticity reveals flaws in some commonly employed boundary conditions.

In electromagnetics, the problem of wave splitting in wave-guides has recently been solved [8]. This opens the possibility that a similar analysis could be performed for an elastic wave-guide, offering a more accurate, but of course more complicated, basis for attacking inverse problems on a beam. However, an analysis along these lines falls outside the scope of the present paper.

The aim of this paper is to present a wave splitting for the Timoshenko equation, as well as to analyze the hyperbolicity of the Timoshenko equation and its less physical relatives, the Euler-Bernoulli and Rayleigh equations, see also Ref. [2]. We emphasize again that the purpose of deriving the wave splitting is to provide a necessary tool for subsequent developments, i.e., the application of invariant imbedding and Green's function techniques to transient wave propagation problems in beam theory.

## 2. Basic equations

The T equation, which includes both rotational inertia and shear, for a uniform symmetric beam is [12], [10, Sect. 10]

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)u(z,t)+\frac{1}{r_0^2c_2^2}\frac{\partial^2 u(z,t)}{\partial t^2}=0. \tag{2.1}$$

The $z$-axis is the undeformed length axis of the beam. Perpendicular to this axis there are the $x$- and $y$-axes, respectively, which coincide with the principal axes of inertia of the cross-sectional area. The $y$-axis, denoted the vertical axis, is oriented so that every cross section is symmetric with respect to $x = 0$. The vertical displacement $u(z, t)$ is measured in the $y$-direction. All fields in this paper are assumed quiescent at time $t < t_0$, where $t_0$ is a fixed time. The two velocities $c_1$ (effective shear velocity) and $c_2$ (rod velocity) are defined by

$$\begin{cases} c_1 = \sqrt{\dfrac{\alpha G}{\varrho}} \\ c_2 = \sqrt{\dfrac{E}{\varrho}} \end{cases}$$

and the radius of gyration of the beam section is defined as

$$r_0 = \sqrt{\frac{I}{A}}.$$

In these definitions $E$ is Young's modulus, $\varrho$ the density of the beam, $G$ is the shear modulus. The factor $\alpha$ is a geometrical quantity given by

$$\alpha = \frac{bI}{SA}$$

where

$$S = \int_{y \geq 0} y \, dA = -\int_{y<0} y \, dA$$

is the static moment of the upper portion of the beam section. The coordinate $y$ is the height coordinate measured from the geometrical centerline of the beam. Furthermore, $b$ = width of the beam at $y = 0$. This is measured in the direction perpendicular to the length of the beam and perpendicular to the $y$ direction. $I$ and $A$ are the moment of inertia and the area of the beam section, respectively. For a rectangular beam section $\alpha = 2/3$, and for a circular beam section $\alpha = 3/4$.

The T equation (2.1) is hyperbolic with two families of characteristic curves, see the analysis in Appendix A

$$\begin{cases} t = \pm \dfrac{z}{c_1} + \text{constant} \\ t = \pm \dfrac{z}{c_2} + \text{constant}. \end{cases}$$

If shear can be ignored, the T equation (2.1) simplifies to the R equation

$$\frac{\partial^4 u(z, t)}{\partial z^4} + \frac{1}{r_0^2 c_2^2} \frac{\partial^2 u(z, t)}{\partial t^2} - \frac{1}{c_2^2} \frac{\partial^4 u(z, t)}{\partial z^2 \, \partial t^2} = 0.$$

If also the rotational inertia is ignored the E-B equation is obtained

$$\frac{\partial^4 u(z, t)}{\partial z^4} + \frac{1}{r_0^2 c_2^2} \frac{\partial^2 u(z, t)}{\partial t^2} = 0.$$

Neither the E–B nor the R equation are hyperbolic in the $t$-direction, see Appendix A. These equations imply infinite propagation speed, and for this reason they are not appropriate as models for transient wave propagation phenomena.

The T equation (2.1), which is of fourth order, can be written as a system of equations in $(u, \partial_z u, \partial_z^2 u, \partial_z^3 u)^t$.

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \partial_z u \\ \partial_z^2 u \\ \partial_z^3 u \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{1}{c_1^2 c_2^2}\dfrac{\partial^4}{\partial t^4} - \dfrac{1}{r_0^2 c_2^2}\dfrac{\partial^2}{\partial t^2} & 0 & \left(\dfrac{1}{c_1^2} + \dfrac{1}{c_2^2}\right)\dfrac{\partial^2}{\partial t^2} & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_z u \\ \partial_z^2 u \\ \partial_z^3 u \end{pmatrix}.$$

For the sake of future studies of the inhomogeneous beam it is appropriate to formulate the analysis in terms of $u(z, t)$ (the vertical displacement of the centerline of the beam) and $\psi(z, t)$ (the angle of rotation of the cross section of the beam) and their first $z$-derivatives, rather than in $(u, \partial_z u, \partial_z^2 u, \partial_z^3 u)^t$. With these dependent variables the T equation for an inhomogeneous beam reads

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} & \dfrac{\partial \log(\alpha GA)}{\partial z} & -\dfrac{\partial \log(\alpha GA)}{\partial z} & 1 \\ 0 & \dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2} + \dfrac{c_1^2}{r_0^2 c_2^2} & -\dfrac{c_1^2}{r_0^2 c_2^2} & -\dfrac{\partial \log(EI)}{\partial z} \end{pmatrix} \begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix}.$$

Notice that this equation only contains four independent quantities $r_0$, $c_1$, $c_2$ and $EI$, since

$$\alpha GA = \frac{EIc_1^2}{r_0^2 c_2^2}.$$

For a homogeneous beam, the T equation simplifies to

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} & 0 & 0 & 1 \\ 0 & \dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2} + \dfrac{c_1^2}{r_0^2 c_2^2} & -\dfrac{c_1^2}{r_0^2 c_2^2} & 0 \end{pmatrix} \begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix} \tag{2.2}$$

which contains only three independent quantities $r_0$, $c_1$, $c_2$.

## 3. Wave splitting

### *3.1. Wave splitting in terms of the eigenvalue operators*

The purpose of this section is to introduce a transformation which diagonalizes the homogeneous T equation in (2.2). To this aim, introduce a transformation of the dependent variables

$$\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} = \mathscr{P} \begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix} \tag{3.1}$$

with formal inverse

$$\begin{pmatrix} u \\ \psi \\ \partial_z u \\ \partial_z \psi \end{pmatrix} = \mathscr{P}^{-1} \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix}. \tag{3.2}$$

The intended result of this transformation is to bring the T equation to the following form:

$$\frac{\partial}{\partial z}\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} \tag{3.3}$$

where the operators $\lambda_i$ are the eigenvalues of the T equation. Explicit representations of these operators are found in Section 3.2. The matrix $\mathscr{P}^{-1}$ expressed in these operators is

$$\mathscr{P}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\lambda_1(1-\mathscr{U}\lambda_2^2) & -\lambda_2(1-\mathscr{U}\lambda_1^2) & \lambda_1(1-\mathscr{U}\lambda_2^2) & \lambda_2(1-\mathscr{U}\lambda_1^2) \\ -\lambda_1 & -\lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_1^2 - \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} & \lambda_2^2 - \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} & \lambda_1^2 - \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} & \lambda_2^2 - \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} \end{pmatrix}$$

where the operator $\mathscr{U}$ is defined as

$$\mathscr{U}f(t) = \frac{r_0 c_2^2}{c_1}\left[\sin\left(\frac{c_1 \,\cdot}{r_0}\right) * f(\cdot)\right](t).$$

Time convolutions are defined by a star (*) throughout this paper, i.e.,

$$(f(\cdot) * g(\cdot))(t) = \int_{-\infty}^{t} f(t - t')g(t')\, dt'.$$

The operators $\lambda_1^2$ and $\lambda_2^2$ that occur in the definition of the matrix operator $\mathscr{P}^{-1}$ above can also be represented as

$$\begin{cases} \lambda_1^2 = \dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2} - V(\cdot) * \\[2ex] \lambda_2^2 = \dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2} + V(\cdot) * \end{cases}$$

where the function $V(t)$ is

$$V(t) = \frac{1}{r_0 c_2 t^2} H(t) I_2(t/\tau)$$

where

$$\tau = \frac{r_0(c_2^2 - c_1^2)}{2c_1^2 c_2}$$

and $I_2$ is the modified Bessel function of order 2, and $H(t)$ is the Heaviside step function.

In order to express the operator $\mathscr{P}$ in a simple way, it is convenient to introduce the functions $Q(t)$ and $S(t)$ ($I_0$ is the modified Bessel function of order 0, and $J_1$ is the Bessel function of order 1)

$$\begin{cases} Q(t) = \dfrac{1}{4} r_0 c_2 H(t) \displaystyle\int_0^{t/\tau} I_0(\xi')\, d\xi' \\[2ex] S(t) = \dfrac{c_1}{r_0} H(t) \displaystyle\int_0^{c_1 t/r_0} \dfrac{J_1(\xi')}{\xi'}\, d\xi' \end{cases}$$

and then define the operators $\mathscr{Q}$ and $\mathscr{S}$ as the time convolutions

$$\begin{cases} \mathscr{Q}f(t) = (Q(\cdot) * f(\cdot))(t) \\[2ex] \mathscr{S}f(t) = \dfrac{c_1}{c_2}[f(t) + (S(\cdot) * f(\cdot))(t)]. \end{cases}$$

The operator $\mathscr{Q}$ satisfies

$$2\mathscr{Q}(\lambda_1^2-\lambda_2^2)=2(\lambda_1^2-\lambda_2^2)\mathscr{Q}=1.$$

The operator $\mathscr{P}$ in (3.2) can be represented as

$$\mathscr{P}=\mathscr{Q}\begin{bmatrix} -\left(\lambda_2^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) & -\mathscr{S}\lambda_2 & \mathscr{S}\lambda_2-\lambda_1 & 1\\ \lambda_1^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} & \mathscr{S}\lambda_1 & -(\mathscr{S}\lambda_1-\lambda_2) & -1\\ -\left(\lambda_2^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) & \mathscr{S}\lambda_2 & -(\mathscr{S}\lambda_2-\lambda_1) & 1\\ \lambda_1^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} & -\mathscr{S}\lambda_1 & \mathscr{S}\lambda_1-\lambda_2 & -1 \end{bmatrix}.$$

*3.2. Representation of the eigenvalue operators*

Explicit representations of the matrix entries in (3.3) can be obtained in a systematic fashion by a simple ansatz. Specifically, the representations of $\lambda_1$ and $\lambda_2$ are

$$\begin{cases} \lambda_1 u_1^+(z,t)=\dfrac{1}{c_1}\dfrac{\partial u_1^+}{\partial t}+(F_1(\cdot)*u_1^+(z,\cdot))(t)\\[2ex] \lambda_2 u_2^+(z,t)=\dfrac{1}{c_2}\dfrac{\partial u_2^+}{\partial t}+(F_2(\cdot)*u_2^+(z,\cdot))(t). \end{cases} \tag{3.4}$$

The functions $F_i(t)$, $i=1,2$ are identically zero for negative time. In the equations below, only a positive time argument $t$ is considered, and thus a Heaviside function $H(t)$ is suppressed in some formulae. A power series expression of, e.g., $F_1(t)$ in $t$ can be obtained from the identity

$$\sqrt{\frac{1}{2}(c_2^{-2}+c_1^{-2})\frac{\partial^2}{\partial t^2}+\sqrt{\frac{1}{4}(c_2^{-2}-c_1^{-2})^2\frac{\partial^4}{\partial t^4}-r_0^{-2}c_2^{-2}\frac{\partial^2}{\partial t^2}}}\,u_1^+(z,t)$$
$$=\frac{1}{c_1}\frac{\partial u_1^+(z,t)}{\partial t}+(F_1(\cdot)*u_1^+(z,\cdot))(t).$$

Formally squaring the operators, rearranging and balancing terms lead to a system of expressions from which coefficients of arbitrarily high order can be obtained. The explicit expressions up to sixth order in $t$ are

$$\begin{cases} F_1(t) = -\dfrac{c_1^3(1+q)}{r_0^2 c_2^2}\left\{\dfrac{1}{4} + \dfrac{3+2q}{64}\dfrac{c_1^4(1+q)t^2}{r_0^2 c_2^2} + \dfrac{7+10q+4q^2}{3072}\right. \\ \qquad \times \left(\dfrac{c_1^4(1+q)t^2}{r_0^2 c_2^2}\right)^2 + \dfrac{77+172q+140q^2+40q^3}{1474560} \\ \qquad \left. \times \left(\dfrac{c_1^4(1+q)t^2}{r_0^2 c_2^2}\right)^3 + O(t^8)\right\} \\ F_2(t) = \dfrac{c_1^2(1+q)}{r_0^2 c_2}\left\{\dfrac{1}{4} - \dfrac{3-2q}{64}\dfrac{c_1^2(1+q)t^2}{r_0^2} + \dfrac{7-10q+4q^2}{3072}\right. \\ \qquad \times \left(\dfrac{c_1^2(1+q)t^2}{r_0^2}\right)^2 - \dfrac{77-172q+140q^2-40q^3}{1474560} \\ \qquad \left. \times \left(\dfrac{c_1^2(1+q)t^2}{r_0^2}\right)^3 + O(t^8)\right\} \end{cases} \tag{3.5}$$

where the constant $q$ is

$$q = \frac{c_2^2 + c_1^2}{c_2^2 - c_1^2}.$$

Based upon physical considerations, the constant $q$ is taken to be larger than 1 or

$$q > 1 \Leftrightarrow \frac{c_2}{c_1} > 1.$$

This condition is met if the following reasonable assumptions are valid:

$$\alpha < 2 \quad \text{and} \quad \nu \geq 0.$$

The first assumption is motivated by the fact that $\alpha$ is the ratio between the average shear stress and the shear stress at $y = 0$. The latter is approximately equal to the maximum shear stress, so in most reasonable cases $\alpha$ does not exceed 1. The second assumption is related to the Poisson ratio $\nu$

$$\nu = \frac{E - 2G}{2G}$$

which for "ordinary" media is larger than 0. The exceptions are few, e.g., cork and some composites, and for the beam applications addressed in this paper this is not a strong limitation.

The asymptotic behavior of the functions $F_i(t)$, $i = 1, 2$ for large time $t$ can readily be obtained by Laplace transform techniques. The result is

$$\begin{cases} F_1(t) = \dfrac{e^{t/\tau}}{\sqrt{\pi c_1 \tau^2}} \Big\{ -(8t^3 q(q+1)/\tau^3)^{-1/2} \\ \qquad + \dfrac{3(1+q^2)}{8q^2} (8t^5 q(q+1)/\tau^5)^{-1/2} + O((t/\tau)^{-7/2}) \Big\} \\ F_2(t) = \dfrac{e^{t/\tau}}{\sqrt{\pi c_2 \tau^2}} \Big\{ (8t^3 q(q-1)/\tau^3)^{-1/2} \\ \qquad - \dfrac{3(1+q^2)}{8q^2} (8t^5 q(q-1)/\tau^5)^{-1/2} + O((t/\tau)^{-7/2}) \Big\} \end{cases} \tag{3.6}$$

where

$$\tau = \frac{r_0 c_2}{c_1^2(q+1)} = \frac{r_0(c_2^2 - c_1^2)}{2c_1^2 c_2}.$$

This asymptotic behavior seems to suggest that the functions $F_i(t)$, $i = 1, 2$ are related to the modified Bessel functions. This surmise is confirmed by another series representation of the functions $F_i(t)$, $i = 1, 2$.

$$\begin{cases} F_1(t) = \dfrac{1}{c_1 \tau^2} \displaystyle\sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-1)^k (q+1)^{-k} W_k(t/\tau) \\ F_2(t) = \dfrac{1}{c_2 \tau^2} \displaystyle\sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (q-1)^{-k} W_k(t/\tau) \end{cases}$$

where

$$\binom{\frac{1}{2}}{k} = \frac{\Gamma\left(\frac{3}{2}\right)}{k!\,\Gamma\left(\frac{3}{2} - k\right)}$$

are binomial coefficients, and the functions $W_k(\xi)$ are integrals over the modified Bessel functions of order, $k$, i.e.,

$$W_k(\xi) = \partial_\xi^{-k+1} \frac{k I_k(\xi)}{\xi}, \qquad k = 1, 2, 3, \ldots$$

where the anti-derivative $\partial_\xi^{-1}$ is

$$\partial_\xi^{-1} f(\xi) = \int_0^\xi f(\xi')\, d\xi'.$$

The series expansions of the functions $W_k(\xi)$ are

$$W_k(\xi) = \sum_{j=0}^{\infty} \frac{k(k+2j-1)!}{2^{k+2j}j!(k+j)!(2k+2j-2)!} \xi^{2k+2j-2}.$$

This series expansion provides an independent derivation of the power series expansions of $F_i(t)$, $i = 1, 2$ given in (3.5).

For numerical purposes, and to present an alternative derivation of the asymptotic behavior of the functions $F_i(t)$, $i = 1, 2$, the recurrence relations of the functions $W_k(\xi)$ are derived. This is most conveniently done by rewriting the functions $W_k(\xi)$ as

$$\begin{cases} W_1(\xi) = \dfrac{I_1(\xi)}{\xi} \\ W_k(\xi) = \dfrac{k}{(k-2)!} \displaystyle\int_0^{\xi} (\xi - \xi')^{k-2} \frac{I_k(\xi')}{\xi'} \, d\xi', \qquad k = 2, 3, 4, \ldots . \end{cases}$$

For large arguments, the functions $W_k(\xi)$ behave as $\xi^{-3/2}e^{\xi}$. To see this, use the recurrence relations for modified Bessel functions to derive

$$W_k(\xi) = i_{k-2,k}(\xi) - i_{k-1,k+1}(\xi) \tag{3.7}$$

where

$$i_{m,n}(\xi) = \partial_{\xi}^{-m} I_n(\xi), \qquad m = -1, 0, 1, 2, 3, \ldots, n = 1, 2, 3, \ldots .$$

The functions $i_{m,n}(\xi)$ can be found recursively, up and down, by combining the following two formulae [9]:

$$\begin{cases} m(1-m)i_{m+1,n}(\xi) = 2\xi(1-m)i_{m,n}(\xi) - [(m-1)^2 - n^2 - \xi^2]i_{m-1,n}(\xi) \\ \qquad + \xi(2m-3)i_{m-2,n}(\xi) - \xi^2 i_{m-3,n}(\xi) \\ 2i_{m,n}(\xi) - i_{m+1,n+1}(\xi) - i_{m+1,n-1}(\xi) = 0. \end{cases}$$

The asymptotic behavior of the functions $i_{m,n}(\xi)$ is calculated in Ref. [9, page 215]. This provides an independent way of computing the asymptotic behavior of the functions $F_i(t)$, $i = 1, 2$ given in (3.6). The dominant terms on the right hand side of (3.7), which are proportional to $\xi^{-1/2}e^{\xi}$, cancel, and the leading term is therefore proportional to $\xi^{-3/2}e^{\xi}$, which agrees with the result presented above in (3.6).

The connections between the functions $F_i(t)$, $i = 1, 2$ and the modified Bessel functions alluded to above can also be seen in another way. Specifically, it is possible to express the functions $F_i(t)$, $i = 1, 2$ in a series expansion in terms of $I_k(t/\tau)$, $k = 0, 1, 2, \ldots$, since (see Ref. [1, replace $z$ with $iz$ in eq. 11.2.4]).

$$\begin{cases} k = 1: & i_{-1,1}(\xi) = \dfrac{\partial I_1(\xi)}{\partial \xi} = \dfrac{1}{2}(I_0(\xi) + I_2(\xi)) \\ k = 2: & i_{0,2}(\xi) = I_2(\xi) \\ k \geq 3: & i_{k-2,k}(\xi) = \dfrac{2^{k-2}}{(k-3)!} \displaystyle\sum_{j=0}^{\infty} \dfrac{(k+j-3)!}{j!}(-1)^j I_{2k+2j-2}(\xi). \end{cases}$$

By expressing the functions $W_k(\xi)$ in terms of modified Bessel functions it is obvious that the functions $F_i(t)$, $i = 1, 2$ can be expressed as a series expansion of modified Bessel functions. For small arguments this choice is as good as the power series expansion given in (3.5). However, for large arguments the modified Bessel functions $I_k(t/\tau)$, contrary to the functions $W_k(t/\tau)$, have incorrect asymptotic behavior and large cancellation effects are expected.

## A. Hyperbolicity of the Timoshenko equation

### *A.1. The Timoshenko equation*

The T equation (2.1) of Section 2 is

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)u(z, t) + \frac{1}{r_0^2 c_2^2}\frac{\partial^2 u(z, t)}{\partial t^2} = 0.$$

The underlying characteristic properties of this PDE are determined by studying the polynomial $P(x, y)$ in the $xy$-plane.

$$P(x, y) = x^4 - a^2y^2 - b^2x^2y^2 + c^4y^4$$

which is obtained by replacing $z$- and $t$-derivatives with $ix$ and $iy$, respectively. The positive constants $a^2$, $b^2$ and $c^4$ are

$$\begin{cases} a^2 = \dfrac{1}{r_0^2 c_2^2} \\ b^2 = \dfrac{1}{c_2^2} + \dfrac{1}{c_1^2} \\ c^4 = \dfrac{1}{c_2^2 c_1^2}. \end{cases}$$

Notice that

$$b^4 - 4c^4 = \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)^2 = d^4 \geq 0.$$

The principal part of the PDE corresponds to the monomial $P_m(x, y)$

$$P_m(x, y) = x^4 - b^2x^2y^2 + c^4y^4 = \left(x^2 - \frac{b^2y^2}{2}\right)^2 - \frac{d^4y^4}{4}.$$

The four distinct roots of $P_m(x, y) = 0$ are

$$\begin{cases} x = \pm y\sqrt{\dfrac{b^2+d^2}{2}} = \pm\dfrac{y}{c_2} \\ x = \pm y\sqrt{\dfrac{b^2-d^2}{2}} = \pm\dfrac{y}{c_1}. \end{cases}$$

These four roots determine the characteristic curves of the PDE in the $zt$-plane. Specifically, the normal $\hat{n}$ to the characteristic curve satisfy $P_m(n_x, n_y) = 0$.

$$\begin{cases} n_x = \pm\dfrac{n_y}{c_2} \\ n_x = \pm\dfrac{n_y}{c_1}. \end{cases}$$

The PDE is hyperbolic in the $\hat{n}$-direction if [4, p. 349]

- The complex roots of $P(x + n_x\tau, y + n_y\tau) = 0$ satisfy $\Im\tau \geq \gamma$ for all real $x$ and $y$ and some constant $\gamma$.
- $P_m(n_x, n_y) \neq 0$

If the PDE is hyperbolic in the $\hat{n}$-direction, the Cauchy problem has one and only one solution in the half space $\{\boldsymbol{r} \cdot \hat{n} \geq 0\}$.

One generic case is $\hat{n} = (0, 1)$. This corresponds to specifying Cauchy data on $t =$ constant. The roots of $P(x, y + \tau) = 0$ are

$$\tau = -y \pm \sqrt{\frac{a^2 + b^2x^2 \pm \sqrt{(a^2+b^2x^2)^2 - 4c^4x^4}}{2c^4}}$$

which implies that all roots are real

$$\Im\tau = 0$$

and the PDE is hyperbolic in the direction $\hat{n} = (0, 1)$.

The second case is $\hat{n} = (1, 0)$, which corresponds to specifying Cauchy data on $z =$ constant. The roots of $P(x + \tau, y) = 0$ are

$$\tau = -x \pm \sqrt{\frac{b^2y^2}{2} \pm \sqrt{\frac{d^4y^4}{4} + a^2y^2}}$$

which implies that $\Im\tau = 0$ or

$$\Im\tau = \begin{cases} y\sqrt{\sqrt{\dfrac{b^4}{4} - c^4 + \dfrac{a^2}{y^2}} - \dfrac{b^2}{2}}, & |y| \le \dfrac{a}{c^2} \\ 0, & |y| \ge \dfrac{a}{c^2} \end{cases}$$

which is bounded and the PDE is hyperbolic in the direction $\hat{n} = (1, 0)$.

The analysis presented in this section can be compared to the results in Ref. [2].

### *A.2. The Rayleigh equation*

An analogous analysis of the R equation

$$\frac{\partial^4 u(z,t)}{\partial z^4} + \frac{1}{r_0^2 c_2^2}\frac{\partial^2 u(z,t)}{\partial t^2} - \frac{1}{c_2^2}\frac{\partial^4 u(z,t)}{\partial z^2\,\partial t^2} = 0$$

implies that the polynomials are

$$P(x, y) = x^4 - a^2y^2 - b^2x^2y^2$$

and

$$P_m(x, y) = x^4 - b^2x^2y^2 = x^2(x^2 - b^2y^2).$$

The positive constants $a$ and $b$ are

$$\begin{cases} a = \dfrac{1}{r_0 c_2} \\ b = \dfrac{1}{c_2}. \end{cases}$$

These polynomials are obtained from the analysis above by letting the velocity $c_1 \to \infty$ ($c^4 \to 0$).

The four roots of $P_m(x, y) = 0$ are

$$\begin{cases} x = 0 \\ x = \pm yb = \pm\dfrac{y}{c_2}. \end{cases}$$

This equation is not hyperbolic in the direction $\hat{n} = (0, 1)$, since $P_m(0, 1) = 0$. It is, however, still hyperbolic in the $(1, 0)$, since $P_m(1, 0) = 1$ and the complex roots of $P(x + \tau, y) = 0$ are

$$\tau = -x \pm \sqrt{\frac{b^2y^2}{2} \pm \sqrt{\frac{b^4y^4}{4} + a^2y^2}}$$

which implies that $\Im\tau = 0$ or

$$\Im\tau = \frac{yb}{2}\sqrt{\sqrt{1+\frac{4a^2}{b^2y^2}}-1}$$

which is bounded for all $y$.

It is interesting to notice that in fixed frequency analysis of the Rayleigh equation the group velocity is bounded for all frequencies. This seems to contradict the infinite speed of propagation entailed by the above analysis. However, it should be recalled that the group velocity is at most a measure of the *average* energy transport velocity. It provides no upper bound on the maximum energy transport velocity.

### *A.3. The Euler-Bernoulli equation*

The E-B equation

$$\frac{\partial^4 u(z,t)}{\partial z^4} + \frac{1}{r_0^2 c_2^2}\frac{\partial^2 u(z,t)}{\partial t^2} = 0$$

implies that the polynomials are

$$P(x,y) = x^4 - a^2y^2$$

and

$$P_m(x,y) = x^4.$$

The positive constant $a$ is

$$a = \frac{1}{r_0 c_2}.$$

These polynomials are obtained from the analysis above by letting both velocities $c_1$, $c_2 \to \infty$ ($c^4, b^2 \to 0$).

The only root of $P_m(x,y) = 0$ is $x = 0$ and this equation is not hyperbolic in any direction.

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**Abstract**

In recent years, wave splitting in conjunction with invariant imbedding and Green's function techniques has been applied with great success to a number of interesting inverse and direct scattering problems. The aim of the present paper is to derive a wave splitting for the Timoshenko equation, a fourth order PDE of importance in beam theory. An analysis of the hyperbolicity of the Timoshenko equation and its, in a sense, less physical relatives—the Euler-Bernoulli and the Rayleigh equations—is also provided.

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