VERTICAL MOTIONS IN AN INTENSE MAGNETIC FLUX TUBE

II: *Convective Instability*

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Abstract. The nature of convective instability in a slender magnetic flux tube is explored. A sufficient condition for stability is derived for the case of an *arbitrary* temperature profile in the external medium. The discussion allows for the possibility of a temperature difference between the interior and exterior of the tube. Special cases of our sufficiency condition reduce to Schwarzschild's criterion and its generalisation by Gough and Tayler (1966).

The distribution of stable and unstable eigenvalues, for the particular case of a *linear* temperature profile, is discussed in detail.

For a tube of *infinite* depth, with a uniform temperature gradient Λ_0' inside the tube equal to that in the ambient medium, a *necessary and sufficient condition for convective stability* to occur inside the tube is

$$\frac{4}{\gamma(\Lambda_0')^2}\left(\frac{\gamma-1}{\gamma}+\Lambda_0'\right)\left(\frac{\gamma}{2}+\frac{c_0^2}{v_A^2}\right)+\left(1+\frac{1}{2\Lambda_0'}\right)^2>0\,,$$

where c_0 and v_A are the sound and Alfvén speeds inside the tube, and γ is the ratio of specific heats. The *stable* modes form a *continuous spectrum*; the unstable modes are *discrete* and infinite in number.

In a tube of *finite* depth d, with a linear temperature profile, a necessary and sufficient condition for convective stability is derived. There exists a critical depth d^* such that tubes of depth d are stable if and only if $d < d^*$. In a finite tube, only discrete eigenvalues occur.

The critical depth d^* is determined for a wide range of conditions, and the results applied to the Sun. Under the assumptions of the present model, intense flux tubes are convectively stable if sufficiently shallow (with depths $1\text{–}2\times10^3$ km or less). Tubes that extend deeper into the solar convection zone are potentially (convectively) unstable, but may be stabilised for sufficiently strong magnetic fields (typically greater than about a kilogauss). The observed downdrafts *inside* intense flux tubes, if a manifestation of convective instability, are thus likely to be a *transient* phenomenon in which the field inside the tube is further intensified until hydrostatic equilibrium obtains. *Convective instability in a flux tube is thus a possible means of achieving kilogauss field strengths.*

1. Introduction

It is now generally agreed (see reviews by Stenflo, 1976; Harvey, 1977) that the magnetic field of the quiet regions of the photosphere is made up of intense flux tubes, located in the boundaries of supergranules. The field in these flux tubes is estimated to be in the range 1–2 kG, their diameters 100–300 km.

Why the Sun seems to arrange preferentially the general field at its surface into slender flux tubes, in opposition to the powerful expansive pressure of the confined magnetic field, is not completely clear. Of course, the simple kinematical effect of the supergranular flow, tending to concentrate the magnetic field in the corners and boundaries of supergranules, is well understood (Parker, 1963; Weiss, 1964; Clark, 1966). But some other means is required for compressing fields to the very high

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pressures of kilogauss fields. One suggestion is that the gas inside the flux tubes is cooler than the surroundings, being actively refrigerated by the generation and emission of overstable Alfvén waves (Parker, 1976; Roberts, 1976a). More recently, Parker (1978) has suggested, as an alternative explanation, that the reduced heat transport in the kinematically compressed magnetic field (of a few hundred gauss) leads to an almost adiabatic temperature gradient inside the tube, so that the temperature inside the thermally insulated tube is cooler than its surroundings. This then leads to an enhancement of the already existing downdraft in the tube and, as a result, the field is further compressed to eventually reach observed kilogauss values. A third possibility has been suggested by Galloway *et al.* (1977), who have shown that convection in a Boussinesq fluid can compress magnetic fields to strengths beyond equipartition values.

By whatever means the Sun actually achieves the compression of magnetic fields into intense flux tubes, the fact remains that such fields exist. Given, then, the existence of such a slender kilogauss flux tube, what motions are likely to occur inside the tube? In an attempt to answer this question, Roberts and Webb (1978) developed a system of non-linear *zeroth-order* equations governing vertical motions in a slender tube. Their theory allowed for a complete description of the effects of buoyancy, temperature stratification, and compressibility. The discussion in Roberts and Webb (1978) was primarily directed towards wave motions in the tube.

In this paper we consider the nature of convectively unstable motions in a magnetic flux tube. We investigate the role of the magnetic field in preventing such instability, including the effect of the splaying of the field lines with height. We should note that distinct from the convective instability is the tendency for the surface of the flux tube to be unstable to fluting (interchange instability). This has recently been considered by Meyer *et al.* (1977).

It turns out that we are able to obtain sufficient conditions for stability against the tendency towards convection for a flux tube with an arbitrary temperature profile. In the special case of a *linear* profile we are able to describe in detail both the velocity field and its instability boundaries. Finally, we argue that convective instability may be a means of *achieving* intense fields.

2. The Equilibrium State in a Slender Flux Tube

The equilibrium configuration of an intense magnetic flux tube is sketched in Figure 1. The internal pressure $p_0(z)$ and density $\rho_0(z)$ are related by the barometric relation

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g; \tag{1}$$

while the ideal gas equation determines the internal temperature $T_0(z)$:

$$p_0(z) = \frac{k}{\hat{m}} \rho_0(z) T_0(z), \tag{2}$$

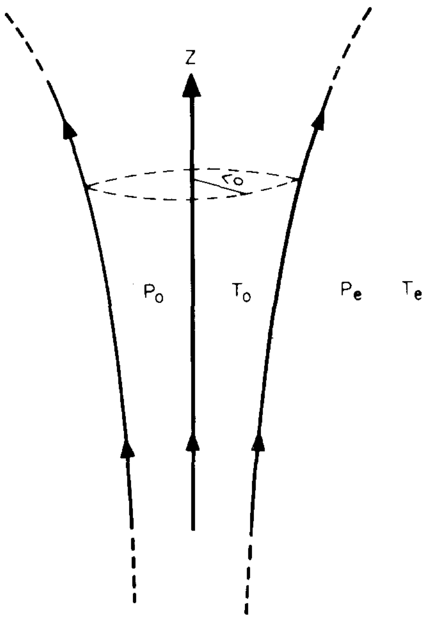

Fig. 1. The equilibrium configuration of an intense magnetic flux tube.

where k is Boltzmann's constant, $\hat{m}$ is the mean particle mass and g is the acceleration due to gravity. Lateral pressure balance demands that the external confining gas pressure $p_e(z)$ satisfy

$$p_0(z)+\frac{B_0^2(z)}{2\mu}=p_e(z)\,, \tag{3}$$

where $B_0(z)\hat{z}$ is the *zeroth-order* (induction) field of the flux tube. The external atmosphere is also in hydrostatic equilibrium, and so

$$\frac{\mathrm{d}p_e}{\mathrm{d}z}=-\rho_e g\,, \tag{4}$$

where $\rho_e(z)$ is the density in the exterior region.

From (3) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B_0^2(z)}{2\mu}\right)=g(\rho_0-\rho_e)\,. \tag{5}$$

Introduce the scale-heights $\Lambda_0\equiv kT_0(z)/\hat{m}g$ and $\Lambda_e\equiv kT_e(z)/\hat{m}g$, of the internal and external atmospheres, and write

$$\Lambda_0(z)+\lambda(z)=\Lambda_e(z)\,, \tag{6}$$

where $\lambda(z)$ is a measure of the temperature difference between the two regions. Then

(5) may be combined with (2) and the corresponding equation for the exterior region to give

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B_0^2(z)}{2\mu}\right)=-\frac{1}{\Lambda_0(z)}\left(\frac{B_0^2(z)}{2\mu}-\frac{\lambda(z)}{\Lambda_e(z)}p_e(z)\right). \tag{7}$$

Now, in the presence of a strong magnetic field, we may expect a temperature difference to exist between the interior and the exterior of the tube, despite the natural tendency for thermal conduction to smooth out such a difference. There are strong theoretical reasons (Parker, 1955, 1976, 1978; Roberts, 1976a, b) for believing this conjecture; indeed, if the flux tube is not cooler than its surroundings it is difficult to imagine how such intense fields occur. The observational evidence is presently inconclusive; in fact, the tubes appear as bright dots (Frazier, 1970) against the surrounding photosphere. They appear bright because they are less dense than their surroundings, so that one sees deeper into the Sun where the ambient temperature is higher than at the surface (Spruit, 1977).

Now the effect of a magnetic field upon the *energetics* of the basic-state is not well understood, so it is necessary to model the likely behaviour of the field in creating a temperature difference. There are clearly several possibilities. For example, it is reasonable to suppose that the stronger the field (relative to the confining pressure), the larger the temperature difference the field creates.

Introduce

$$\tau(z)=\frac{\Lambda_e(z)-\Lambda_0(z)}{\Lambda_e(z)}, \tag{8}$$

the ratio of the temperature difference between the internal and external media to the external temperature. Then, to model the behaviour of the magnetic field, we may suppose that $\tau(z)$ is proportional to the ratio of the magnetic pressure to the external gas pressure, and write

$$\tau(z)=\theta\left(\frac{B_0^2(z)}{2\mu p_e(z)}\right), \tag{9}$$

where θ is a constant of proportionality. It is assumed that θ is less than or equal to one; also, for a cool interior θ is positive.

With the above form of $\tau(z)$, Equation (7) gives

$$\frac{1}{B_0}\frac{\mathrm{d}B_0}{\mathrm{d}z}=-\frac{(1-\theta)}{2\Lambda_0(z)}. \tag{10}$$

Combined with (1) and (2) this gives

$$B_0^2(z)\sim(p_0(z))^{1-\theta}. \tag{11}$$

Flux conservation implies that $r_0^2(z)B_0(z)=r_0^2(0)B_0(0)$, where $r_0(z)$ is the radius of the tube at the height z above the arbitrary reference level $z=0$ (chosen to correspond to optical depth $\tau_{5000} = 1$ in the solar atmosphere). Thus

$$r_0(z) \sim (p_0(z))^{-\frac{1}{4}(1-\theta)}. \tag{12}$$

Much of the subsequent discussion will be given in terms of the function $c_0^2(z)/v_A^2(z)$, where $c_0(z) = (\gamma p_0(z)/\rho_0(z))^{1/2}$ is the sound speed and $v_A(z) = B_0(z)/(\mu\rho_0(z))^{1/2}$ is the Alfvén speed inside the flux tube. (γ is the ratio of specific heats, which may be a function of z.)

It may be noted that the parameter c_0^2/v_A^2 is related to the plasma bêta inside the tube by

$$\frac{c_0^2(z)}{v_A^2(z)} = \tfrac{1}{2}\gamma\beta_0(z),$$

where $\beta_0(z) = 2\mu p_0(z)/B_0^2(z)$.

The densities inside and outside the tube are related by

$$\frac{\rho_e(z)}{\rho_0(z)} = 1 + (1-\theta)\left(\frac{\beta_e}{1-\beta_e}\right),$$

where $\beta_e = 2\mu p_e(z)/B_0^2(z)$, $(\beta_e < 1)$.

Now the parameter c_0^2/v_A^2 satisfies (from (1), (2), and (10)) the equation

$$\frac{\gamma v_A^2}{c_0^2}\frac{d}{dz}\left(\frac{c_0^2}{\gamma v_A^2}\right) = \frac{-\theta}{\Lambda_0(z)}, \tag{13}$$

and so (for $\theta > 0$) $c_0^2/\gamma v_A^2$ increases with depth $(-z)$ in a cool tube in such a way that

$$\frac{c_0^2(z)}{\gamma v_A^2(z)} \sim (p_0(z))^{\theta}. \tag{14}$$

In particular, if $\theta = 0$ (so that the temperatures inside and outside the tube are equal), then

$$\frac{1}{\gamma}c_0^2(z) \sim v_A^2(z)$$

and

$$B_0^2(z) \sim p_0(z), \qquad r_0(z) \sim p_0^{-1/4}(z). \tag{15}$$

The relation in (15) was first given by Parker (1955) in his discussion of magnetic buoyancy and the formation of sunspots.

3. Vertical Motions in a Slender Flux Tube

The equations governing motions of a perfectly conducting, inviscid, ideal gas embedded in a magnetic field have been presented in Roberts and Webb (1978) (hereinafter to be termed Paper I). In a cylindrical coordinate (r, θ, z) system, in which there is no azimuthal dependence, each of the physical variables is expanded in its Maclaurin series about $r = 0$. Retaining the *zeroth-order* ($r \to 0$) terms leads to the following set of non-linear equations, describing vertical adiabatic motions:

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial z}-\rho g\,, \tag{16}$$

$$\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial z}=\frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial z}\right), \tag{17}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right)+\frac{\partial}{\partial z}\left(\frac{\rho v}{B}\right)=0\,, \tag{18}$$

where $\rho(z, t)$ is the density, $p(z, t)$ the pressure, $v(z, t)$ the vertical (zeroth-order) velocity inside the flux tube of vertical (zeroth-order) field $B(z, t)$. The details of the derivation of these equations are given in Paper I. Here we shall discuss the linear perturbations from the equilibrium state described in Section 2 by use of Equations (16)–(18).

The linear equations describing *perturbations* about the basic-state described in Section 2 follows immediately from (16)–(18):

$$\rho_0(z)\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial z}-\rho g\,, \tag{19}$$

$$\frac{\partial p}{\partial t}+p_0'(z)v=c_0^2(z)\left(\frac{\partial \rho}{\partial t}+\rho_0'(z)v\right), \tag{20}$$

$$B_0(z)\frac{\partial \rho}{\partial t}-\rho_0(z)\frac{\partial B}{\partial t}+(B_0(z)\rho_0'(z)-B_0'(z)\rho_0(z))v+\rho_0(z)B_0(z)\frac{\partial v}{\partial z}=0\,, \tag{21}$$

where v, p, ρ, and B now refer to the *perturbations* (so, for example, the *total* density is $\rho_0+\rho$), and a dash denotes differentiation with respect to the vertical coordinate z.

The flux tube is related to its surroundings by assuming that the pressure perturbation in the exterior region is negligible. Then the radial component of the momentum equation gives

$$p+\frac{1}{\mu}B_0(z)B=0\,. \tag{22}$$

It should be noted that the use of (22) has recently been criticised by Wilson (1979), who argues that its use leads to a degenerate solution. His criticism is discussed in detail in Roberts (1979), and is shown to be without foundation.

Assuming a time-dependence of the form $e^{i\omega t}$, (19)–(22) may be combined to give a second order differential equation for the velocity amplitude $\hat{v}(z)$, defined by

$v = \hat{v}(z)\, e^{i\omega t}$, namely

$$\frac{d^2\hat{v}}{dz^2} + \left(\frac{c_T^{2\prime}}{c_T^2} - \frac{B_0'}{B_0} - \frac{1}{H_0}\right)\frac{d\hat{v}}{dz} +$$
$$+\left[\frac{\omega^2 - N_0^2}{c_T^2} - \frac{d}{dz}\left(\frac{B_0'}{B_0} + \frac{g}{c_0^2}\right) - \left(\frac{B_0'}{B_0} + \frac{g}{c_0^2}\right)\left(\frac{c_T^{2\prime}}{c_T^2} - \frac{N_0^2}{g}\right)\right]\hat{v} = 0\,, \tag{23}$$

where

$$c_T^2 = \frac{c_0^2 v_A^2}{(c_0^2 + v_A^2)}\,.$$

The Brunt-Väisälä frequency, $N_0(z)$, and the density scale-height, $H_0(z)$, are defined by

$$\frac{1}{H_0(z)} = -\frac{\rho_0'(z)}{\rho_0(z)}\,, \qquad N_0^2(z) = \frac{g}{H_0(z)} - \frac{g^2}{c_0^2(z)}\,.$$

Note that in deriving Equation (23), we have made use of Equation (1) only in the basic-state. The temperatures inside and outside the tube may be unequal. However, for the special case $T_0(z) = T_e(z)$ the equilibrium state is completely characterized by the scale-height $\Lambda_0(z)$:

$$c_0^2(z) = \gamma g \Lambda_0(z)\,,\; v_A^2(z) = v_A^2(0)\frac{\Lambda_0(z)}{\Lambda_0(0)}\,,\; c_T^2(z) = c_T^2(0)\frac{\Lambda_0(z)}{\Lambda_0(0)}\,,$$

$$\frac{1}{H_0(z)} = \frac{1 + \Lambda_0'(z)}{\Lambda_0(z)}\,, \qquad N_0^2(z) = \frac{g}{\Lambda_0(z)}\left(\frac{\gamma - 1}{\gamma} + \Lambda_0'(z)\right). \tag{24}$$

Equation (23), for $T_0 = T_e$ and γ assumed constant, then reduces to

$$\frac{d^2\hat{v}}{dz^2} - \frac{1}{2\Lambda_0(z)}\frac{d\hat{v}}{dz} + \left(\frac{\omega^2 - N_0^2(z)}{c_T^2(z)} + \left(1 - \frac{\gamma}{2}\right)\frac{N_0^2(z)}{c_0^2(z)}\right)\hat{v} = 0\,. \tag{23$'$}$$

Equation (23)′ is Equation (31) of Paper I.

4. Sufficient Conditions for Convective Stability

In order to discuss the stability of a slender flux tube, it is convenient to write Equation (23) in the canonical Sturm-Liouville form:

$$\frac{d}{dz}\left(\sigma(z)\frac{d\hat{v}}{dz}\right) + [\omega^2 r(z) - q(z)]\hat{v} = 0\,, \tag{25}$$

where

$$\sigma(z) = \frac{\rho_0(z) c_T^2(z)}{B_0(z)}\,, \qquad r(z) = \frac{\rho_0(z)}{B_0(z)}\,,$$

and

$$q(z) = \frac{\rho_0 c_T^2}{B_0}\left[\frac{N_0^2}{c_T^2} + \frac{d}{dz}\left(\frac{B_0'}{B_0} + \frac{g}{c_0^2}\right) + \left(\frac{B_0'}{B_0} + \frac{g}{c_0^2}\right)\left(\frac{c_T^{2\prime}}{c_T^2} - \frac{N_0^2}{g}\right)\right].$$

Together with boundary conditions of the general type

$$\left.\begin{aligned} a_1\hat{v}(0)+a_2\hat{v}'(0)=0 \\ b_1\hat{v}(-d)+b_2\hat{v}'(-d)=0 \end{aligned}\right\} \tag{26}$$

(for constants a_1, a_2, b_1, and b_2), imposed at levels $z=0$ and $z=-d$, Equation (25) constitutes the standard form of the Sturm-Liouville boundary-value problem (see, for general theory, Ince (1944); and, in the context of a stratified fluid, Yih (1965)). (Particular forms of these boundary conditions will be considered in Section 6, with reference to the special case treated there.)

It follows from the general Sturm-Liouville theory that eigenvalues ω^2 are *real* and, provided d is finite, these eigenvalues form an *infinite* sequence that may be ordered in increasing magnitude; thus

$$\omega_1^2<\omega_2^2<\cdots<\omega_n^2<\cdots.$$

Furthermore, $\omega_n^2\to\infty$ as $n\to\infty$ (Ince, 1944, Chapter X).

Thus, under the boundary conditions (26), with d finite, there are an infinite number of eigenmodes with eigenfrequencies $\omega_j (j=1,2,3,\ldots)$. If $\omega_j^2>0$, then the jth mode is *stable.* If $\omega_j^2<0$, then the jth mode is *unstable.* Thus, in particular, the basic-state is *unstable* if $\omega_1^2<0$ and (since $\omega_1^2<\omega_j^2$, $j>1$) this will be the mode of maximum growthrate. Unfortunately, Sturm-Liouville theory does not provide us with a direct calculation of ω_1^2. However, we may elicit further information on ω^2 (in particular the sign of ω^2) by obtaining a first integral of Equation (25).

Multiplying (25) by $\hat{v}^*$, the complex conjugate of $\hat{v}$, and integrating by parts, it follows that

$$\omega^2=\frac{\int_{-d}^{0}(q(z)|\hat{v}|^2+\sigma(z)|\hat{v}'|^2)\,\mathrm{d}z-[\sigma(z)\hat{v}^*\hat{v}']_{-d}^{0}}{\int_{-d}^{0}r(z)|v|^2\,\mathrm{d}z}. \tag{27}$$

It is clear from the form of (27) that our earlier restriction that d be finite is now no longer necessary. Thus, in the subsequent discussion d may be infinite.

Suppose, now, that the boundary conditions are such that

$$[\sigma(z)\hat{v}^*v']_{-d}^{0}\leq 0$$

(as are the special cases of (26) examined in subsequent sections). Then it follows from (27), on noting that $\sigma(z)>0$ and $r(z)>0$, that a sufficient condition for ω^2 to be positive, and thus a sufficient condition for stability, is

$$q(z)>0 \quad \text{for all } z, \quad -d\leq z\leq 0.$$

Thus, under such boundary conditions, a *sufficient condition for stability* is

$$\frac{N_0^2}{c_T^2}+\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B_0'}{B_0}+\frac{g}{c_0^2}\right)+\left(\frac{B_0'}{B_0}+\frac{g}{c_0^2}\right)\left(\frac{c_T^{2\prime}}{c_1^2}-\frac{N_0^2}{g}\right)>0 \tag{28}$$

throughout $-d\leq z\leq 0$.

It is of interest to examine the sufficiency condition (28), under the assumption that the interior of the flux tube is cooler than its surroundings, and γ constant. In terms of the function $\tau(z)$, introduced in Equation (8), (28) may be written in the form

$$\left[\left(\frac{2-\gamma}{\gamma}\right)(1-\tau(z))+\frac{\tau'(z)}{\tau(z)}\left(1+\frac{v_A^2(z)}{c_0^2(z)}\right)\Lambda_0(z)\right]\times$$
$$\times\frac{\tau(z)c_0^2(z)}{(1-\tau(z))(c_0^2(z)+v_A^2(z))}>-\Lambda_0'(z)-\left(\frac{\gamma-1}{\gamma}\right), \tag{29}$$

to be satisfied throughout $-d\leq z\leq 0$.

In terms of the parameter θ, introduced in (9), the sufficient condition for stability, inequality (29), becomes

$$\frac{(2/\gamma)-(1-\theta)}{(2/\gamma)(c_0^2/v_A^2)+(1-\theta)}\left(\frac{c_0^2}{c_0^2+v_A^2}\right)\theta>-\Lambda_0'(z)-\left(\frac{\gamma-1}{\gamma}\right), \tag{30}$$

to be satisfied throughout the atmosphere.

There are two special cases of (30) of immediate interest. In the extreme circumstances where thermal conduction is so efficient as to remove the temperature difference, so that $\tau=\theta=0$, (30) reduces to

$$0>-\Lambda_0'-\left(\frac{\gamma-1}{\gamma}\right).$$

Thus, a sufficient condition for stability in a slender tube, which is in temperature balance with its surroundings (i.e. $\Lambda_0=\Lambda_e$), is that

$$N_0^2(z)>0 \quad \text{throughout} \quad -d\leq z\leq 0. \tag{31}$$

This is the usual condition for convective stability in the absence of a magnetic field (Schwarzschild, 1906).

In the special case for which $\theta=1$, and $\tau(z)=B_0^2(z)/2\mu p_e(z)$, it follows from Equation (10) that $B_0'(z)\equiv 0$ and $\rho_0=\rho_e$. Thus, the case $\theta=1$ corresponds to a uniform (vertical) column of magnetic field; in this case, a sufficient condition for stability to convection in a *uniform* slender flux tube is

$$\frac{B_0^2/\mu}{B_0^2/\mu+\gamma p_0(z)}>-\Lambda_0'-\left(\frac{\gamma-1}{\gamma}\right) \quad \text{throughout} \quad -d\leq z\leq 0.$$

This is in fact the sufficient condition for stability in the presence of a uniform, laterally *unbounded*, magnetic field, originally derived by Gough and Tayler (1966) using the energy principle of Bernstein *et al.* (1958). Our analysis shows that the above condition also holds in a *slender* flux tube, provided the magnetic field is uniform.

Returning to the general sufficiency condition (30), we sketch in Figure 2 the condition (30), plotting c_0^2/v_A^2 against Λ_0' for various values of θ in the range $0\leq\theta\leq 1$, and for $\gamma=1.2$. It is clear from the figure that the cooler the interior of the tube (i.e., the larger the value of θ), the more likely that the sufficient condition for

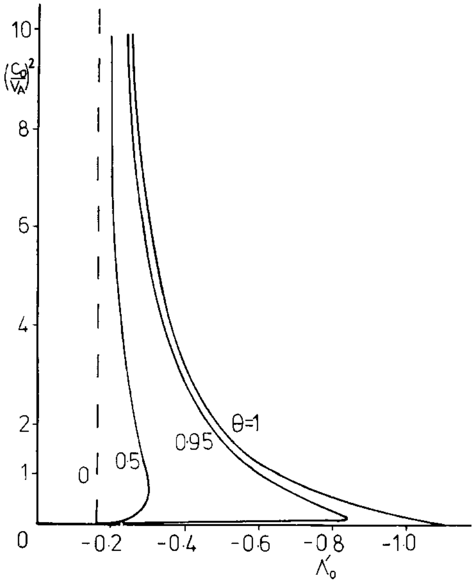

Fig. 2. Sufficient condition (Equation (30)) for stability in a flux tube with a cool interior. Stability is to the left of the curves. The larger the value of the parameter θ, the cooler the interior of the tube.

stability is satisfied, for a fixed magnetic field. It may be noted (see Equation (13)) that c_0^2/v_A^2 increases with depth for $\theta > 0$ (cool interior), and so for $\theta = 1$ the sufficient condition is simply

$$\frac{v_A^2(-d)}{c_0^2(-d)+v_A^2(-d)} > -\Lambda_0' - \left(\frac{\gamma-1}{\gamma}\right),$$

where d is the depth of the layer.

Finally, returning to (29), we may note that for the special case $\tau(z) = \tau_0$, a constant, considered in Parker (1976) and Roberts (1976b), the sufficient condition for stability reduces to

$$\left(\frac{2-\gamma}{\gamma}\right)\frac{c_0^2}{c_0^2+v_A^2}\tau_0 > -\Lambda_0' - \left(\frac{\gamma-1}{\gamma}\right),$$

to be satisfied throughout the atmosphere.

The sufficiency conditions derived above provide useful information as to the prevention of instability in the tube. However, the criteria are clearly of somewhat limited value, simply by being only sufficient conditions: the conditions may be violated at some depth or indeed for all depths, and yet the tube be stable. Thus, in order to obtain a more precise condition for stability, we shall consider, in Section 6, the special case of a *linear* temperature profile inside the flux tube. We shall further assume, for the sake of simplicity, that temperatures inside and outside the tube are equal, so that $\tau = 0$. If the temperatures are different, then the tube is likely (as we have seen from the above sufficiency conditions) to be even more stable that the analysis in Section 6 reveals.

5. The Local Approximation

Before discussing, in the next section, an exact solution to the velocity equation (23)′, we shall consider the so-called 'local' approximation. This serves to illustrate many of the features of the more general analysis without the attendant complexity. We suppose that the coefficients of the velocity and its derivatives in Equation (23)′ do not vary greatly with depth and, by way of illustration, we look for solutions satisfying the simple boundary conditions:

$$\hat{v} = 0 \quad \text{at} \quad z = 0\,; \tag{32}$$

and

$$\hat{v} = 0 \quad \text{at} \quad z = -d\,, \tag{33a}$$

or

$$\hat{v} \to 0 \quad \text{as} \quad z \to -\infty\,. \tag{33b}$$

Of course, such a treatment is not directly applicable to the Sun. But the results are indicative of the more general case, and so serve to give a qualitative guide to the expected behaviour in the solar convection zone.

We may write Equation (23)′ in the form

$$\hat{v}'' - \frac{1}{2\Lambda_0}\hat{v}' + \frac{1}{c_T^2}(\omega^2 - \Omega^2)\hat{v} = 0\,, \tag{34}$$

where

$$\Omega^2(z) = N_0^2 \frac{c_T^2}{c_0^2}\left(\frac{\gamma}{2} + \frac{c_0^2}{v_A^2}\right). \tag{35}$$

Note that Ω^2 has the same sign as N_0^2, and so may be positive or negative.

Assuming a locally constant atmosphere, the general solution of (34), satisfying the boundary condition (32), is

$$\hat{v} = A_1(e^{\lambda_1 z} - e^{\lambda_2 z})\,, \tag{36}$$

where λ_1, λ_2 are the roots of the quadratic

$$\lambda^2 - \frac{1}{2\Lambda_0}\lambda + \left(\frac{\omega^2 - \Omega^2}{c_T^2}\right) = 0\,, \tag{37}$$

and A_1 is an arbitrary constant. To discuss (36) further, it is convenient to consider the cases of finite and infinite depth separately.

Consider first the case of *finite depth*, for which the solution $\hat{v}$ of (36) satisfies the lower boundary condition (33a). This condition implies that

$$(\lambda_2-\lambda_1)d=2n\pi i\,, \tag{38}$$

for integer n.

Now, since (36) may be rewritten in the form

$$\hat{v}=A_1\,e^{\frac{1}{2}(\lambda_1+\lambda_2)z}\{e^{\frac{1}{2}(\lambda_1-\lambda_2)z}-e^{-\frac{1}{2}(\lambda_1-\lambda_2)z}\},$$

and from (37) we have

$$\lambda_1+\lambda_2=\frac{1}{2\Lambda_0},$$

the solution for the velocity is

$$\hat{v}=A\,e^{z/4\Lambda_0}\sin\left(\frac{n\pi z}{d}\right),$$

where A is an arbitrary constant.

Now, solving (37) gives

$$(\lambda_1-\lambda_2)^2=4\left\{\frac{1}{16\Lambda_0^2}-\frac{(\omega^2-\Omega^2)}{c_T^2}\right\},$$

which, when combined with (38), determines the eigenvalues ω^2:

$$\omega^2=\Omega^2+\left(\frac{n^2\pi^2}{d^2}+\frac{1}{16\Lambda_0^2}\right)c_T^2. \tag{39}$$

Note that if

$$\Omega^2+\frac{c_T^2}{16\Lambda_0^2}>0\,,$$

i.e. if

$$\frac{\gamma/16}{(\gamma/2)+(c_0^2/v_A^2)}>-\Lambda_0'-\left(\frac{\gamma-1}{\gamma}\right),$$

then all modes are *stable*, and we have an *infinite* sequence of positive eigenvalues ω_n^2.

However, if $-\Omega^2<c_T^2/16\Lambda_0^2$, then *unstable* modes are permissible; but as $d\to 0$ Equation (39) implies that

$$\omega^2\sim\frac{n^2\pi^2c_T^2}{d^2},$$

and so there exists a critical value d_n^* of the depth d, below which the basic-state is always stable. From (39), this critical depth d_n^* is given by

$$d_n^{*2} = \frac{-n^2\pi^2}{(\Omega^2/c_T^2)+(1/16\Lambda_0^2)}. \tag{40}$$

Thus, if $d < d_n^*$, there are no *unstable* modes. Note that the depth d_n^* is a function of n, the minimum value of which occurs for $n = 1$. Then $d_n^* = d_1^* \equiv d^*$, say, where

$$d^{*2} = \frac{-\pi^2}{(\Omega^2/c_T^2)+(1/16\Lambda_0^2)}. \tag{41}$$

Finally, we should note that there are no modes for which

$$\omega^2 < \Omega^2 + \frac{c_T^2}{16\Lambda_0^2}.$$

For then λ_1 and λ_2 are real, and so cannot satisfy condition (38), i.e. $\hat{v}$ is non-zero everywhere, except at $z = 0$.

Consider now the case of *infinite depth*, for which the solutions of the velocity Equation (34) are subject to the boundary conditions (32) and (33b). The solution (36) may now be written in the form

$$\hat{v} = A\, e^{z/4\Lambda_0} \sinh\left[\left(\frac{1}{16\Lambda_0^2}+\frac{\Omega^2-\omega^2}{c_T^2}\right)^{1/2} z\right],$$

satisfying the condition $\hat{v} = 0$ at $z = 0$.

If $(c_T^2/16\Lambda_0^2)+\Omega^2-\omega^2<0$, then condition (33b) is satisfied for all ω^2, i.e. for $\omega^2>\Omega^2+(c_T^2/16\Lambda_0^2)$ the spectrum of eigenvalues ω^2 is *continuous* (i.e. no discrete eigenvalues exist).

For $(c_T^2/16\Lambda_0^2)+\Omega^2-\omega^2>0$, $\hat{v}\to 0$ as $z\to-\infty$ provided $\omega^2>\Omega^2$. Therefore, if $\omega^2>\Omega^2$, we again have a continuous spectrum; while if $\omega^2<\Omega^2$, then there is no solution satisfying the boundary conditions.

The above illustration shows that for a finite depth, there is an infinite number of positive eigenvalues. There *may* exist a finite number of unstable modes provided Ω^2 is sufficiently negative; however, if the depth d is sufficiently small, no unstable modes exist. Qualitatively, these results agree with those to be presented in the following sections. However, we should note that differences do arise for the case of a flux tube of infinite depth, in that, in the 'local' approximation, we find a continuous spectrum for $\omega^2>\Omega^2$; whereas for the case to be presented in the next section, we find a continuous spectrum for $\omega^2>0$ and an infinite number of discrete eigenvalues for $\omega^2<0$.

6. The Linear Temperature Profile

Here we shall consider the special case of the linear profile

$$\Lambda_0(z)=\Lambda_0(0)+z\Lambda_0'(0), \tag{42}$$

for which Λ_0' is a constant. In order that the temperature and density increase monotonically with depth (see Equation (24)) it is necessary that the scale-height gradient, Λ_0', satisfy

$$-1 < \Lambda_0' < 0 .$$

Further, we shall suppose, for simplicity, that $\tau(z) \equiv 0$(i.e. the temperatures inside and outside the tube are equal) and that γ is a constant. (Note that for $\tau = 0$ the ratio $c_0^2(z)/v_A^2(z)$ is a constant.) Thus, as we have seen in Section 3, a sufficient condition for stability to convective motions is $N_0^2 > 0$. In particular, the isothermal atmosphere ($\Lambda_0' \equiv 0$) is always stable.

Under these assumptions, the governing equation for the velocity perturbation is Equation (23)′, for which an exact solution is known (Equation (45) of Paper I).

We use this solution here in order to investigate in detail the nature of motions in a linear temperature profile. To find the eigenvalues ω^2 using this solution we must, of course, specify appropriate boundary conditions on the velocity.

6.1. Boundary conditions

We have considered two alternative forms of the lower boundary condition, namely

$$\hat{v} \to 0 \quad \text{as} \quad z \to -\infty ; \tag{43a}$$

$$\hat{v} = 0 \quad \text{at} \quad z = -d . \tag{43b}$$

Thus, we are requiring that the velocity $\hat{v}$ tend to zero at a depth d, which may be infinite. It is necessary to consider the two cases separately, as the discussion in Section 5 indicates. We shall concentrate on the former condition, leaving the discussion of the case of finite depth to the end of this section.

In addition to the lower boundary condition (43a), we must specify the flow at (say) $z = 0$. We shall write this boundary condition in the general form

$$a_1 \hat{v}(0) + a_2 \hat{v}'(0) = 0 , \tag{44}$$

where $a_1 \geq 0$ and $a_2 \geq 0$. Thus, with $a_1 = 0$ and $a_2 = 1$, we allow for the possibility that the vertical velocity is a maximum (or minimum) at the observed level $z = 0$; with $a_1 = 1$ and $a_2 = 0$, condition (44) imposes a top on the flux tube at $z = 0$, beyond which no flow penetrates. In fact, our results are not sensitive to the precise form of the constants a_1 and a_2. However, it should be noted that our results *are* sensitive to the alternative cases of the tube's depth being finite or infinite.

6.2. The exact solution of the velocity equation

Consider the case of equal temperatures ($T_0 = T_e$), for which Equation (23)′ is applicable. We shall employ the boundary conditions (43a) and (44).

To find the solutions of (23)′ it is convenient to introduce in place of z the new independent variable x, where

$$x=\frac{2\Lambda_0^{1/2}(0)}{c_T(0)}\left|\frac{\omega}{\Lambda_0'}\right|\Lambda_0^{1/2}(z)\,,\qquad x>0\,. \tag{45}$$

Then (23)′ becomes

$$\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}x^2}-\left(1+\frac{1}{\Lambda_0'}\right)\frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x}+\left(\pm 1-\frac{[s^2-(1+1/2\Lambda_0')^2]}{x^2}\right)\hat{v}=0\,, \tag{46}$$

where the + sign applies to *stable* solutions ($\omega^2>0$), and the − sign to *unstable* solutions ($\omega^2<0$). The constant s^2 is defined by

$$s^2=\frac{4}{\gamma(\Lambda_0')^2}\left(\frac{\gamma-1}{\gamma}+\Lambda_0'\right)\left(\frac{\gamma}{2}+\frac{c_0^2}{v_\mathrm{A}^2}\right)+\left(1+\frac{1}{2\Lambda_0'}\right)^2. \tag{47}$$

Note that s^2 may be positive or negative, but is positive if $N_0^2>0$.

In terms of the x variable, the boundary condition (43a) becomes

$$\hat{v}\to 0\quad\text{as}\quad x\to\infty\,; \tag*{(43a)′}$$

whilst condition (44) becomes

$$a_1\hat{v}+\tfrac{1}{2}\Lambda_0'a_2x\frac{\mathrm{d}\hat{v}}{\mathrm{d}x}=0\quad\text{at}\quad x=x_0\,, \tag*{(44)′}$$

where

$$x_0(\omega)=\frac{2\Lambda_0(0)}{c_T(0)}\left|\frac{\omega}{\Lambda_0'}\right|.$$

It is convenient to discuss the stable and unstable cases of (46) separately.

6.2.1. *Stable Solutions*

Consider the velocity equation (46) under the assumption that ω^2 is positive (so that the plus sign applies). The solutions of (46) are (see Section 5.4 of Paper I)

$$\hat{v}\sim x^{1+1/2\Lambda_0'}J_s(x)\quad\text{and}\quad x^{1+1/2\Lambda_0'}Y_s(x)\,, \tag{48}$$

where J_s and Y_s are Bessel functions of the first and second kind (Abramowitz and Stegun, 1967). Note that s may be real or imaginary.

Now *both* of these solutions tend to zero as $x\to\infty$, provided $-1<\Lambda_0'<0$. Thus, since this condition on Λ_0' is satisfied, both solutions (48) satisfy the lower boundary condition (43a)′. Furthermore, the upper boundary condition (44)′ can be satisfied for any value of x_0 simply by choosing a suitable linear combination of the two independent solutions (48). Thus, the boundary conditions no longer determine a *discrete* set of eigenvalues $\omega^2>0$ (in contrast to the results found in Section 5 for a *finite* domain, $-d\le z\le 0$). Hence, for the *infinite* domain $z\le 0$, there exists a *continuous spectrum* of eigenvalues $\omega^2>0$, whether s^2 is positive or negative.

6.2.2. *Unstable Solutions*

Suppose, now, that ω^2 is negative, so that the minus sign applies in the velocity Equation (46). The solutions of (46) are now

$$\hat{v} \sim x^{1+1/2\Lambda_0'} I_s(x) \quad \text{and} \quad x^{1+1/2\Lambda_0'} K_s(x), \tag{49}$$

where $I_s(x)$ and $K_s(x)$ are modified Bessel functions (of the first and second kind) of order s (which may be real or imaginary).

In order to satisfy the lower boundary condition (43a) we must reject the solution involving the modified Bessel function of the first kind, since it is unbounded as $x \to \infty$. So consider the second solution $x^{1+1/2\Lambda_0'} K_s(x)$. It is convenient to discuss the two cases $s^2 \geq 0$ and $s^2 < 0$ separately.

(a) $s^2 \geq 0$.

For this case $K_s(x)$ is a real positive, monotonically decreasing function of x. Also, we may show from (47) that (for $s^2 \geq 0$) we have

$$0 > \Lambda_0' > -\tfrac{3}{2} + (2/\gamma)^{1/2}.$$

Therefore, for γ in the range $1 < \gamma \leq 5/3$, $1 + 1/2\Lambda_0' < 0$ and $x^{1+1/2\Lambda_0'} K_s(x)$ is monotonically decreasing; so the boundary condition (44)′ is *not* satisfied for any value of $\omega^2 < 0$, i.e. no unstable solutions exist for $s^2 \geq 0$. Hence, subject to the boundary conditions (43a)′ and (44)′, the fluid is *stable* if $s^2 \geq 0$. Note that this is consistent with the sufficiency condition for stability obtained in Section 4.

(b) $s^2 < 0$.

Writing $s^2 = -\nu^2$, where ν is real and positive, the solution for the velocity is

$$\hat{v} \sim x^{1+1/2\Lambda_0'} K_{i\nu}(x). \tag{50}$$

The velocity $\hat{v}$ as a function of x is sketched for various Λ_0' in Figure 3. For purposes of illustration, we have taken $c_0 = v_A$ and $\gamma = 1.2$.

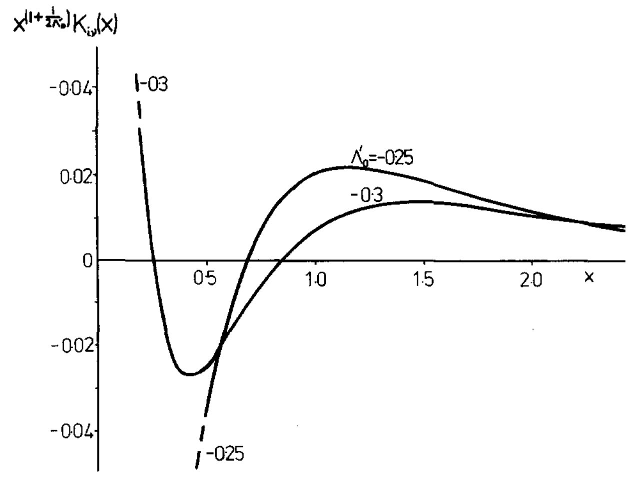

Fig. 3. The function $x^{1+1/2\Lambda_0'} K_{i\nu}(x)$ for $\gamma = 1.2$, $c_0 = v_A$ and two values of Λ_0'. For $0 > \Lambda_0' > -\frac{1}{2}$, ($\Lambda_0' < -\frac{1}{2}$) the function oscillates infinitely with increasing (decreasing) amplitude as $x \to 0$.

It may be noted that $\hat{v}$ may be expressed in terms of the Whittaker function $W_{0,i\nu}$ by observing that

$$K_{i\nu}(x)=\left(\frac{\pi}{2x}\right)^{1/2} W_{0,i\nu}(2x)\,. \tag{51}$$

(Gradshteyn and Ryzhik, 1965, p. 1063).

Now the zeros of the Whittaker function $W_{l,m}(x)$ have been investigated by Dyson (1960), in relation to the stability of an idealized atmosphere with constant shear flow and exponentially decreasing density (see also Case, 1960). Dyson found that for l real and m purely imaginary there are no complex zeros, and that there are an infinite number of simple zeros, on the positive real axis, which are bounded and have a point of accumulation at $x=0$. In our problem, $l\equiv 0$, and $m=i\nu$ is purely imaginary.

Therefore, with boundary conditions (43a) and (44), there are an infinite number of (negative) eigenvalues ω_n^2, with the property that

$$\omega_1^2<\omega_2^2<\cdots<\omega_n^2<\cdots<0\,; \tag{52}$$

furthermore,

$$\omega_n^2\to 0 \quad \text{as} \quad n\to\infty\,,$$

and ω_1^2 is finite.

Thus there are unstable ($\omega^2<0$) solutions, satisfying the boundary conditions (43a)′ and (44)′, if and only if $s^2<0$, i.e. if and only if

$$\frac{4}{\gamma(\Lambda_0')^2}\left(\frac{\gamma-1}{\gamma}+\Lambda_0'\right)\left(\frac{\gamma}{2}+\frac{c_0^2}{v_A^2}\right)+\left(1+\frac{1}{2\Lambda_0'}\right)^2<0\,.$$

Hence, a *necessary and sufficient condition* for the *tube* to be (convectively) *stable* is

$$\frac{4}{\gamma(\Lambda_0')^2}\left(\frac{\gamma-1}{\gamma}+\Lambda_0'\right)\left(\tfrac{1}{2}\gamma+\frac{c_0^2}{v_A^2}\right)+\left(1+\frac{1}{2\Lambda_0'}\right)^2>0\,. \tag{53}$$

The line $s^2=0$ is sketched in Figure 4 as a function of the parameters c_0^2/v_A^2 and Λ_0'. From (53) we see that a *sufficient* condition for $s^2>0$, and hence a sufficient condition for stability, is $N_0^2>0$, in agreement with the results obtained earlier (Equation (31)).

For $-\frac{3}{2}+(2/\gamma)^{1/2}>\Lambda_0'>-1$, unstable solutions exist for all values of c_0^2/v_A^2. Note that $c_0^2/v_A^2\to 0$ does not imply $B_0\to\infty$, if $p_e(z)$ is fixed, because the magnetic pressure in the tube cannot exceed the confining external pressure. Thus, the limit $c_0^2/v_A^2\to 0$ is achieved by allowing (see (3)) $B_0(z)\to(2\mu p_e(z))^{1/2}$, for which $p_0(z)$ and $\rho_0(z)$ both tend to zero (and thus the tube is evacuated by the magnetic field).

For

$$-\left(\frac{\gamma-1}{\gamma}\right)>\Lambda_0'>-\tfrac{3}{2}+\left(\frac{2}{\gamma}\right)^{1/2},$$

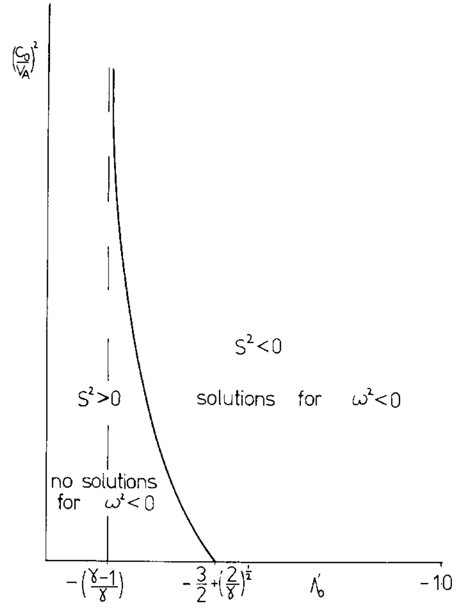

Fig. 4. The line $s^2 = 0$ (see Equation (47)), shown schematically in the $(c_0^2/v_A^2) - \Lambda_0'$ plane, dividing the region with unstable modes ($s^2 < 0$) from that with only stable modes ($s^2 > 0$). The dashed line, $N_0^2 = 0$, is the asymptote for $s^2 = 0$ as $c_0^2/v_A^2 \to \infty$.

there is a critical value of c_0^2/v_A^2, given by

$$\left(\frac{c_0^2}{v_A^2}\right)_{\text{crit}} = \frac{-\gamma}{4(\Lambda_0' + (\gamma - 1)/\gamma)}\left(\Lambda_0' + \tfrac{3}{2} + \left(\frac{2}{\gamma}\right)^{1/2}\right)\left(\Lambda_0' + \tfrac{3}{2} - \left(\frac{2}{\gamma}\right)^{1/2}\right), \quad (54)$$

below which the solution is *stable*. So, provided the field is sufficiently strong, the gas in the flux tube is stable to convection.

Also, it may be noted from Figure 4, that the effect of increasing the ratio of specific heats, γ, is that the line $s^2 = 0$ is moved to the right, giving a greater region for stability. Thus, for certain values of c_0^2/v_A^2 and Λ_0', increasing γ may stabilise the perturbation.

Consider now the fastest growing eigenvalue ω_1^2, the existence of which for $s^2 < 0$ is guaranteed by (52). The values of ω_1^2 have been determined numerically for a range of values of the parameters c_0^2/v_A^2 and Λ_0'. For convenience, it was found simpler to use the form (50) for the velocity, and to solve numerically Bessel's equation for $K_{i\nu}(x)$, matching the numerical solution to the known asymptotic forms as $x \to 0$ and $x \to \infty$.

Figure 5 shows a plot of ω_1^2 (suitably non-dimensionalised) against c_0^2/v_A^2 for various Λ_0'. For the sake of illustration, we have taken the upper boundary condition as $\hat{v}' = 0$ at $z = 0$. (The results are similar for the general boundary condition (44).)

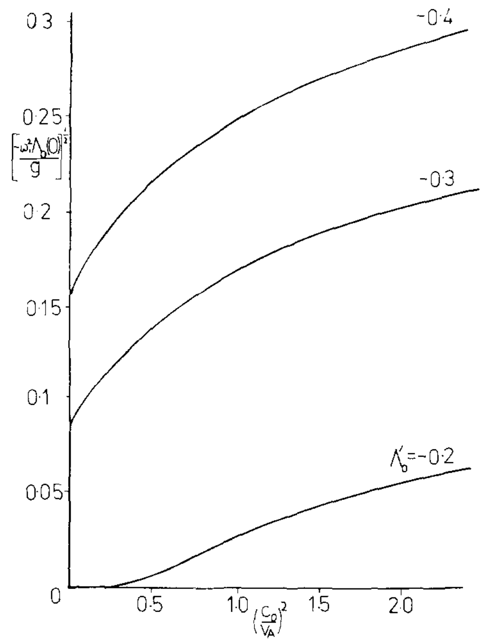

Fig. 5. The non-dimensional growthrate, $(-\omega_1^2 \Lambda_0(0)/g)^{1/2}$, of the fastest growing mode as a function of c_0^2/v_A^2 for several values of Λ_0'. We have taken $\gamma = 1.2$ and the upper boundary condition as $\hat{v}' = 0$ at $z = 0$.

Now, as $c_0^2/v_A^2 \to \infty$, $\nu \to \infty$; and we may show (by considering the asymptotic form of $K_{i\nu}(x)$ as $\nu \to \infty$) that

$$\omega_1^2 \to N_0^2(0) \quad \text{as} \quad \frac{c_0^2}{v_A^2} \to \infty .$$

Thus the behaviour of ω_1^2 at large c_0^2/v_A^2 indicated in Figure 5.

For $\Lambda_0' < -\frac{3}{2} + (2/\gamma)^{1/2}$, the mode is unstable for all values of c_0^2/v_A^2, and

$$\omega_1^2 \to \frac{\gamma g \Lambda_0'^2}{4\Lambda_0(0)} x_{\max}^2 \quad \text{as} \quad \frac{c_0^2}{v_A^2} \to 0 ,$$

where $x_{\max}$ is the largest value of x, for which the boundary condition (44)' is satisfied, with

$$\nu^2 = -\frac{1}{\Lambda_0'^2}\left(\Lambda_0' + \frac{3}{2} - \left(\frac{2}{\gamma}\right)^{1/2}\right)\left(\Lambda_0' + \frac{3}{2} + \left(\frac{2}{\gamma}\right)^{1/2}\right).$$

For

$$-\left(\frac{\gamma-1}{\gamma}\right) > \Lambda_0' > -\frac{3}{2} + \left(\frac{2}{\gamma}\right)^{1/2},$$

there is no unstable solution for $(c_0^2/v_A^2) < (c_0^2/v_A^2)_{\text{crit}}$, where $(c_0^2/v_A^2)_{\text{crit}}$ is given by (54). Figure 5 also shows that ω_1^2 is a monotonically increasing function of c_0^2/v_A^2; thus, the weaker the magnetic field, the faster the growthrate of the instability.

In Figure 6 we have plotted ω_1^2 against Λ_0' for various values of c_0^2/v_A^2. We see that there is no solution for any value of c_0^2/v_A^2 if

$$0 > \Lambda_0' > -\left(\frac{\gamma-1}{\gamma}\right),$$

i.e. if $N_0^2 > 0$; and that (for a given c_0^2/v_A^2) there is a minimum value of Λ_0', given by

$$\Lambda'_{0_{\rm crit}} = -\left(\frac{3}{2} + \frac{2}{\gamma}\frac{c_0^2}{v_A^2}\right) + \frac{2}{\gamma}\left[\left(1 + \frac{c_0^2}{v_A^2}\right)\left(\frac{\gamma}{2} + \frac{c_0^2}{v_A^2}\right)\right]^{1/2}, \tag{55}$$

for which the solution is stable if $0 > \Lambda_0' > \Lambda'_{0_{\rm crit}}$.

We also find that, for given c_0^2/v_A^2 and Λ_0', the growth-rate of the fastest-growing mode is a decreasing function of γ.

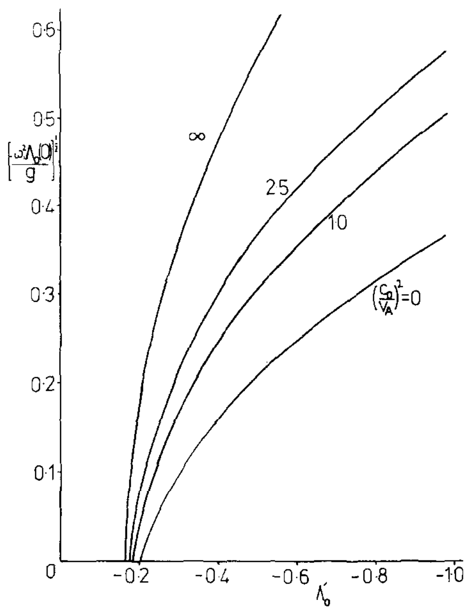

Fig. 6. The non-dimensional growthrate, $(-\omega_1^2\Lambda_0(0)/g)^{1/2}$, of the fastest-growing mode as a function of Λ_0' for various c_0^2/v_A^2. Again, for purposes of illustration, we have taken $\gamma = 1.2$ and the upper boundary condition as $\hat{v}' = 0$ at $z = 0$.

6.3. Finite depth

Finally, in this subsection, we consider the effect of applying the lower boundary condition (43b), so that the tube has a *finite* depth d.

The first important (though perhaps not suprising) point to emerge is that *there no longer exists a continuous spectrum* for $\omega^2 > 0$. This is because the velocity Equation (23)$'$, together with the boundary conditions (43b) and (44), form a *regular* (Ince, 1944, Chapter X) Sturm-Liouville system (see also Section 4), which, from the general theory, has *discrete* eigenvalues only.

Again, for a given depth, we have a *necessary and sufficient condition* for the existence of *unstable* solutions. We find unstable solutions exist if and only if

$$e^{2\phi/\nu} < \frac{\Lambda_0(-d)}{\Lambda_0(0)}, \tag{56}$$

where $\nu = \sqrt{(-s^2)}$, and ϕ depends on the coefficients a_1 and a_2 occurring in the boundary conditions (44). In general ϕ satisfies

$$\pi \geq \phi \geq \tan^{-1}\left(\frac{\nu}{1 + 1/2\Lambda_0'}\right).$$

For the special case $a_1 = 0$ (for which the upper boundary condition is $\hat{v}'(0) = 0$), we find that

$$\phi = \tan^{-1}\left(\frac{\nu}{1 + 1/2\Lambda_0'}\right);$$

whilst for $a_2 = 0$ (for which $\hat{v}(0) = 0$), $\phi = \pi$.

The condition (56) is sketched for various $d/\Lambda_0(0)$ in Figure 7. In the region to the left of each curve, all the eigenvalues are positive; in the region to the right, there is at least one negative eigenvalue. Thus, for values of c_0^2/v_A^2 and Λ_0' lying to the right of the curve, there is an *unstable* mode of maximum growthrate.

Alternatively, from Figure 7, we see that for given c_0^2/v_A^2 and Λ_0' there exists a minimum depth d^*, such that for $d < d^*$, the perturbation is always *stable*. The

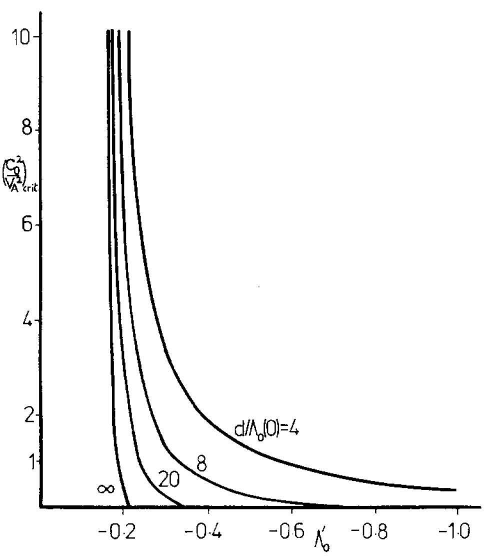

Fig. 7. Necessary and sufficient condition for instability in a tube of finite depth, d, sketched for various values of $d/\Lambda_0(0)$. For a given depth, to the left of the curve only stable modes exist and to the right there is at least one unstable mode. The upper boundary condition has been taken as $\hat{v}' = 0$ at $z = 0$, and $\gamma = 1.2$.

critical value d^* is determined (from (56)) by

$$\Lambda_0(-d^*) = \Lambda_0(0)\, e^{2\phi/\nu}. \tag{57}$$

The dependence of d^* on Λ_0' is sketched in Figure 8 for various values of the parameter c_0^2/v_A^2.

The application of these results to the solar atmosphere is considered in the following section.

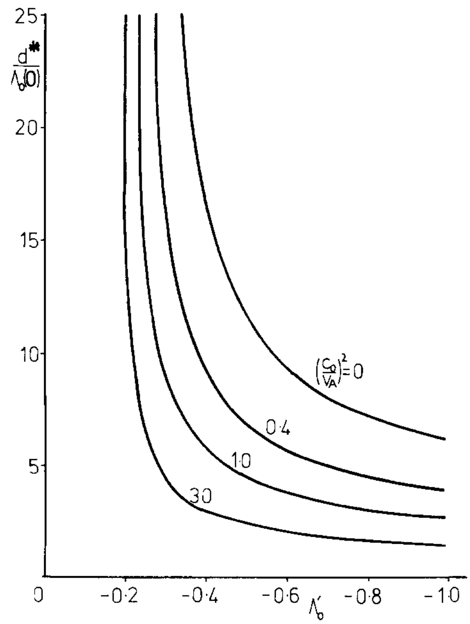

Fig. 8. The critical depth for stability as a function of Λ_0' for various c_0^2/v_A^2.

7. Intense Flux Tubes in the Sun

Before considering the application of the above analysis to the Sun, it is perhaps convenient to summarise the main results we have so far obtained. The discussion falls into two parts: (a) conditions for stability in an arbitrary temperature gradient; and (b) instability in a uniform temperature gradient.

For an *arbitrary* temperature profile (case (a)), we have shown that a sufficient condition for stability to convection is

$$\left\{\left(\frac{2-\gamma}{\gamma}\right)(1-\tau(z)) + \frac{\tau'(z)}{\tau(z)}\left(1+\frac{v_A^2(z)}{c_0^2(z)}\right)\Lambda_0(z)\right\} \times$$

$$\times \frac{\tau(z)c_0^2(z)}{(1-\tau(z)(c_0^2(z)+v_A^2(z))} > -\Lambda_0'(z) - \left(\frac{\gamma-1}{\gamma}\right),$$

to be satisfied throughout the depth of the tube (see Equation (29)). If the internal and external temperatures are equal, so that $\tau = 0$, then the above sufficiency condition is simply

$$N_0^2(z) > 0, \tag{31}$$

to be satisfied throughout the depth of the tube.

For a *linear* temperature profile (case (b)), with internal and external temperatures equal, condition (31), of course, guarantees stability to convection.

In a tube of *infinite* depth, with a linear temperature profile, a necessary and sufficient condition for convective stability is that $s^2 > 0$, i.e.

$$\frac{4}{\gamma \Lambda_0'^2}\left(\frac{\gamma-1}{\gamma}+\Lambda_0'\right)\left(\frac{\gamma}{2}+\frac{c_0^2}{v_A^2}\right)+\left(1+\frac{1}{2\Lambda_0'}\right)^2 > 0.$$

(See (53).) The *stable* ($\omega^2 > 0$) modes in a tube of infinite depth form a *continuous* spectrum; the *unstable* ($\omega^2 < 0$) modes, which occur if $s^2 < 0$, are *discrete* and infinite in number (with a point of accumulation at $\omega^2 = 0$).

In a tube of *finite* depth d, with a linear temperature profile, a necessary and sufficient condition for stability to convection is

$$e^{2\phi/\nu} > \Lambda_0(-d)/\Lambda_0(0).$$

(See inequality (56).) Thus, there exists a minimum depth d^* below which (i.e. for a tube of depth d less than d^*) only stable modes exist. The critical depth d^* is given by Equation (57), and is sketched in Figure 8. The *stable* ($\omega^2 > 0$) modes, in a tube of finite depth, are *discrete* and infinite in number; there is no continuous spectrum. If $d > d^*$, then there exists at least one *unstable* ($\omega^2 < 0$) mode; again, there is no continuous spectrum of eigenvalues.

Consider now the application of these results to the solar atmosphere. In applying our analysis to the Sun, we are, of course, supposing that the basic (static) state of a flux tube is as that described in Section 2, and that any motions in the tube are adequately described by the linear analysis of Section 3. We shall also assume that the flux tube is in temperature balance with its surroundings. Further, we shall approximate the convection zone by a linear temperature profile (or, more precisely, a linear profile for the scale height $\Lambda_e(z)$), and take a mean value of the ratio of specific heats, γ. We use Spruit's model of the solar convection zone (Spruit, 1974), and take $z = 0$ to correspond to optical depth $\tau_{5000} = 1$ (in the *surrounding* photosphere). In fact, Spruit's model gives $-\Lambda_0'$ sharply peaked at about 40 km below $\tau_{5000} = 1$. Also γ varies rapidly over the first 100 km or so below $\tau_{5000} = 1$, (Spruit, 1977). However, below a depth of about 100 km, and down to about 4000 km Λ_0' and γ are effectively constant. Thus, our assumption of a linear temperature profile is a reasonable one over this range.

The precise choice of values for Λ_0' and γ in our model is uncertain. However, if we take $\gamma = 1.15$ and Λ_0' in the range $-0.2 \geq \Lambda_0' \geq -0.3$, then the perturbation is unstable in a medium of infinite depth, for all values of c_0^2/v_A^2. Now the ratio of the sound speed to the Alfvén speed is determined by its value at $z = 0$ (i.e. at observed levels), where it is roughly unity. Thus, taking $c_0^2/v_A^2 = 1$, and the above values of Λ_0' and γ, we find that a tube of infinite depth is unstable. However, a tube of finite depth is stable provided its depth does not exceed about $1\text{–}2 \times 10^3$ km.

If, on the other hand, we choose to apply our boundary condition at a *prescribed* depth, say $z = -2000$ km (Roberts, 1976b), and take mean values of -0.25 for the scale-height gradient and 1.2 for the ratio of specific heats, over the range $z = 0$ to $z = -2000$ km, we find (using the upper boundary condition $\hat{v}' = 0$ at $z = 0$) that the perturbation is stable for $c_0^2/v_A^2 < 1.2$, i.e. the perturbation is stable for $B_0(0) > > 1040$ G. Applying the lower boundary condition at a shallower depth would give a greater value of c_0^2/v_A^2; that is, a lower value of the field strength would be required for stability to convection.

In Table I we give the critical value of c_0^2/v_A^2, where it exists, and the corresponding value of the field at $z = 0$, for various tube depths. The depths chosen correspond to the granular scale (10^3 km), the overstable cooling depth (2×10^3 km; Roberts, 1976a), the superadiabatic depth (3×10^3 km; Parker, 1978), the supergranular scale (1.5×10^4 km), and finally the depth scale of the convection zone ($\sim 10^5$ km).

Table II gives the growth-rate of the instability, when present, for the above depths and $c_0^2/v_A^2 = 1$. For example, with $\Lambda_0' = -0.3$, a tube of depth 2000 km has a growth-rate of $5.3 \times 10^{-3}\ \mathrm{s}^{-1}$ (that is, the instability grows on an e-folding time of

TABLE I

The critical field strength (in gauss) at $z = 0$ necessary for convective stability in a flux tube of given depth d and scale-height gradient Λ_0', for $\gamma = 1.2$ and $p_e(0) = 1.3 \times 10^5$ dynes cm^{-2}

	Depth d(km)				
Λ_0'	10^3	2×10^3	3×10^3	1.5×10^4	10^5
-0.15	stable	stable	stable	stable	stable
-0.2	490	670	770	1080	1280
-0.25	760	1040	1200	1750	unstable
-0.3	940	1290	1500	unstable	unstable

TABLE II

The growth-rate (in s^{-1}), $-i\omega_1$, of the most unstable mode for various depths d and scale-height gradients Λ_0', for $c_0 = v_A$, $\gamma = 1.2$ and $\Lambda_0(0) = 152$ km

	Depth d(km)				
Λ_0'	10^3	2×10^3	3×10^3	1.5×10^4	10^5
-0.15	stable	stable	stable	stable	stable
-0.2	stable	stable	stable	stable	1.2×10^{-3}
-0.25	stable	stable	3.2×10^{-3}	4.8×10^{-3}	4.8×10^{-3}
-0.3	stable	5.3×10^{-3}	6.7×10^{-3}	7.2×10^{-3}	7.2×10^{-3}

190 s); whereas, for a depth 1.5×10^4 km, the growth-rate is $7.2 \times 10^{-3}\ s^{-1}$ (with an e-folding time of 140 s). Taking a smaller value of γ, for example $\gamma = 1.1$, the growth-rate increases to $9.7 \times 10^{-3}\ s^{-1}$ for a depth of 2000 km.

Given that convective instability occurs in the tube, what are its consequences? The instability may result in either a downflow or an upflow, which, unfortunately, linear analysis cannot discriminate between. Parker (1978) has shown that a *steady* downdraft (in the basic-state of a flux tube) leads to a temperature difference between the interior of the tube and the surrounding photosphere, and subsequently to field intensification.

In a flux tube that is in unstable equilibrium, either because its field strength is low or because it is deeply rooted, instability (as we have seen in this paper) may manifest itself as a downflow, the consequence of which is to lead to an increase in field strength* and thus eventually to a possibly stable equilibrium. An upflow would presumably result in the magnetic field being dispersed* at the surface by the diverging flow and enhanced temperature of the rising gas (Parker, 1978; Spruit, 1978).

Thus we suggest that the downdraft observed in *intense* flux tubes (Stenflo, 1976; Harvey, 1977) cannot be due to convective instability if the tubes are shallow. If the depths of the flux tubes are greater than about 2000 km, then the downflow, if resulting from convective instability, is likely to be a *transient* phenomenon, *taking place as the field strength intensifies* and the flux tube moves to a state of hydrostatic equilibrium with an increased field strength. Of course, this does not exclude the possibility of other mechanisms giving rise to a downdraft in the tube.

Consider, then, in summary, a possible scenario for the life of an intense flux tube. If the flux tube is a shallow phenomenon, then motions driven by convective forces are unlikely, and an equilibrium state presumably rules. If, however, the flux tube extends somewhat deeper (several thousand kilometers, say) into the Sun, then *a flux tube of moderate field strength* (of several hundred gauss, say) *is convectively unstable*. The result of this instability is to lead either *to the dissolution of the tube*, with the field being dispersed; or *to an increase in field strength, driven by a downdraft in the tube*, until an equilibrium, with kilogauss field strengths, is once more possible. Independently, overstable Alfvén waves will tend to cool the interior of the tube, and thus further intensify the field (Parker, 1976; Roberts, 1976a). There are, of course, uncertainties in this global description of field intensification: only a more detailed, presumably non-linear, analysis can further clarify such points.

It should be noted that both Spruit (1978) and Unno and Ando (1978) have also, and independently, made similar suggestions to the above, whilst Parker (1978) has described the effect of a *steady* downdraft on achieving field intensification. The convective instability we have investigated *generates* such a downdraft (though not necessarily a steady one).

* A *downflow* brings (to a location z) *lighter* and *cooler* fluid and so leads to a *decrease* in gas pressure, and consequently to an *increase* in field strength (as the field collapses inward under the exterior pressure). An upflow brings heavier and warmer fluid to the location z, and so weakens the magnetic field.

Finally, we note that we have neglected the effect of dissipation in our analysis. When dissipation is taken into account, overstability may arise, even when our conditions for stability are satisfied (Cowling, 1976). Dissipative processes may also relate to the possibility of cooling and field intensification (Parker, 1976; Roberts, 1976a). The effect of dissipation is clearly complicated and will in fact be the subject of a further paper in this series (Webb and Roberts, 1978).

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References

Abramowitz, M. and Stegun, I. A.: 1967, *Handbook of Mathematical Functions*, Nat. Bur. of Standards.
Bernstein, I. B., Frieman, E. A., Kruskal, M. D., and Kulsrud, R. M.: 1958, *Proc. Roy. Soc.* **A244**, 17.
Case, K. M.: 1960, *Phys. Fluids* **3**, 149.
Clark, A.: 1966, *Phys. Fluids* **9**, 485.
Cowling, T. G.: 1976, *Magnetohydrodynamics*, Ch. 4, Adam Hilger.
Dyson, F. J.: 1960, *Phys. Fluids* **3**, 155.
Frazier, E. N.: 1970, *Solar Phys.* **14**, 89.
Galloway, D. J., Proctor, M. R. E., and Weiss, N. O.: 1977, *Nature* **266**, 686.
Gough, D. O. and Tayler, R. J.: 1966, *Monthly Notices Roy. Astron. Soc.* **133**, 85.
Gradshteyn, I. S. and Ryzhik, I. M.: 1965, *Table of Integrals, Series, and Products*, Academic Press, London.
Harvey, J. W.: 1977, *Highlights of Astronomy* **4**, Part II, p. 223.
Ince, E. L.: 1944, *Ordinary Differential Equations*, Dover, New York.
Meyer, F., Schmidt, H. U., and Weiss, N. O.: 1977, *Monthly Notices Roy. Astron. Soc.* **179**, 741.
Parker, E. N.: 1955, *Astrophys. J.* **121**, 491.
Parker, E. N.: 1963, *Astrophys. J.* **138**, 552.
Parker, E. N.: 1976, *Astrophys. J.* **204**, 259.
Parker, E. N.: 1977, *Astrophys, J.* **210**, 816.
Parker, E. N.: 1978, *Astrophys. J.* **221**, 368.
Roberts, B.: 1976a, *Astrophys. J.* **204**, 268.
Roberts, B.: 1976b, *Solar Phys.* **50**, 329.
Roberts, B.: 1979, *Solar Phys.* (in preparation).
Roberts, B. and Webb, A. R.: 1978, *Solar Phys.* **56**, 5 (Paper I).
Schwarzschild, K.: 1906, *Nachr. K. Ges. Wiss. Got*, p. 41; see also in M. Schwarzschild, *Structure and Evolution of the Stars*, Princeton Univ. Press, 1958.
Spruit, H. C.: 1974, *Solar Phys.* **34**, 277.
Spruit, H. C.: 1977, Ph.D. Thesis, Utrecht.
Spruit, H. C.: 1978, in preparation.
Stenflo, J. O.: 1976 in 'Energy Balance and Hydrodynamics of the Solar Chromosphere and Corona', *IAU Colloq.* **36**, 143.
Unno, W. and Ando, H.: 1978, *Geophys. Astrophys. Fluid Dynamics*, (in press).
Webb, A. R. and Roberts, B.: 1978, in preparation.
Weiss, N. O.: 1964, *Phil. Trans, Roy. Soc.* **A256**, 99.
Wilson, P. R.: 1979, in preparation.
Yih, C.-S.: 1965, *Dynamics of Nonhomogeneous Fluids*, MacMillan.