Z angew Math Phys 46 (1995)
(ZAMP)

0044-2275/95/050812-06 \$ 1.50 + 0.20

A generalization of Cesàro's relation for plane finite deformations

By Alexander Chiskis, Dept of Solid Mechanics, Materials and Structures, Faculty of Engineering, Tel-Aviv University, 69978 Ramat-Aviv, Israel (Fax: 972-3-640-7617)

1. Introduction

The construction of a vector field from its gradient poses a problem if only some "part" of the gradient is known.

Let $\boldsymbol{y}$ be a vector field and let the tensor

$$\boldsymbol{K} = \boldsymbol{y} \otimes \nabla \tag{1.1}$$

be its gradient. Very often, the known "part" of the gradient is a symmetric tensor $\boldsymbol{A}$ that is a combination of $\boldsymbol{K}$. Two important examples are: the linear combination,

$$\boldsymbol{A}_L = \frac{1}{2}(\boldsymbol{K} + \boldsymbol{K}^T) \tag{1.2}$$

and the nonlinear combination,

$$\boldsymbol{A}_N = (\boldsymbol{K}^T\boldsymbol{K})^{1/2}. \tag{1.3}$$

We treat here the construction of $\boldsymbol{y}$ using the nonlinear tensor (1.3).

Consider the kinematics of a deformed body, where $\boldsymbol{y}$ is the position vector of a body point which we can treat in terms of either a Lagrangian or Eulerian description. Here we let $\boldsymbol{A}_L$ and $\boldsymbol{A}_N$ be symmetric strain tensors either for small, linear relation (1.2), or for finite, nonlinear relation (1.3), deformation (see Truesdell [1]). The problem can then be formulated in mechanical terms: first, how can the displacement field $\boldsymbol{y}$ be calculated from the strain tensor $\boldsymbol{A}$ and, secondly, what are the necessary restrictions on the tensor field $\boldsymbol{A}$—the so-called "compatibility equation"—for this calculation to yield a unique vector field.

The problem with a linear tensor defined by $\boldsymbol{A}_L$ by (1.2) was solved by Cesàro [2] (see also Sokolnikoff [3]). Cesàro represented the displacement field by quadrature in terms of the strain tensor for the three-dimensional case. He also obtained the previously known compatibility conditions.

In nonlinear mechanics the compatibility conditions have been written in different forms by many authors (see, for example, the review in Truesdell and Toupin [4]). The reconstruction of the displacement field via (nonlinear) quadrature of the strain tensor for finite deformation is discussed by Shield [5], where two approaches—using $\boldsymbol{A}_N$ as well as $\boldsymbol{A}_N^2$ are discussed. It was shown that in the two-dimensional case the quadrature can be written explicitly using the tensor $\boldsymbol{A}_N^2$ and two angles of rotation of two initially orthogonal material elements. The approach of using angles of the material elements to the axes of the coordinate system is discussed also by Slepyan [6]. Slepyan uses a tensor A_N^2 and introduces 18 angles to treat the three-dimensional case. Using the resulting relations, after very complex calculations the quadrature for the two-dimensional case was constructed. However the compatibility equation was not presented.

The reconstruction of the displacement field using polar decomposition of the gradient $\boldsymbol{K}$ by the rotation tensor and by the symmetrical stretch tensor $\boldsymbol{A}_N$ is discussed by Shield [5], starting from the three-dimensional case. It was shown that in the two-dimensional case the quadrature can be written. The results were presented in a very complicated form and no simplification was performed for this case. Thus the very simple relation between the rotation angle gradient and the tensor $\boldsymbol{A}_N$ as well as the very simple form of the compatibility equation remained obscured.

Reconstruction of a displacement field in the three-dimensional case was discussed, among others, by Shamina [7]* and Lurie [8]. In Shamina's article the rotation vector is used while Lurie's [8] approach is very similar to one of differential geometry: using a representation of Christoffel symbols via A_N^2, Lurie reduces the problem to solving an auxiliary linear partial derivative tensor equation with variable space coefficients. A discussion connected to the existence of a global solution (and additional references) is given in Ciarlet [9].

In this article we discuss only the plane problem and construct a quadrature for the displacement field. This quadrature is different from that of Shield [5] and Slepyan [6] where the tensor A_N^2 is used and it is significantly simpler than Shield's representation using the tensor $\boldsymbol{A}_N$. We use the polar decomposition of the gradient $\boldsymbol{K}$; i.e. $\boldsymbol{K}$ is represented in the well-known form of multiplication of a rotation tensor by a symmetrical stretch tensor $\boldsymbol{A}_N$. Only one angle of rotation is necessary to describe the rotation tensor in the plane problem, which is required to determine $\boldsymbol{K}$ and to find the displacement field by direct integration. The determination of the angle of rotation is significantly simplified by the commutativity of plane rotations.

Additionally, the compatibility equation, obtained by using $\boldsymbol{A}_N$ instead of the commonly used A_N^2, is written in a very brief new form.

2. Generalized formulation

If the tensor $\boldsymbol{K}$,

$$\boldsymbol{K} = \boldsymbol{y} \otimes \nabla, \tag{2.1}$$

is a plane gradient of some plane vector field $\boldsymbol{y}$, then the field $\boldsymbol{y}$ can be determined by integration

$$\boldsymbol{y}(B) = \boldsymbol{y}(A) + \int_A^B \boldsymbol{K}\, d\boldsymbol{l} \tag{2.2}$$

uniquely (in a simply-connected domain) if and only if

$$\boldsymbol{K} \times \nabla = 0. \tag{2.3}$$

* We should note that the author refers in [7] to a previously derived quadrature for the plane case and states that such quadrature was also derived independently by Novozilov. However no references are given nor is a quadrature presented in [7].

We suppose this statement to be well-known. Attention now is focused on the "integrability" condition, (2.3). Using the polar decomposition we can represent $\boldsymbol{K}$ as

$$\boldsymbol{K} = \boldsymbol{Q}\boldsymbol{A}, \tag{2.4}$$

where $\boldsymbol{Q}$ is the plane rotation tensor, determined by an angle of rotation θ, and $\boldsymbol{A}$ is a symmetric tensor, $\boldsymbol{A} = \boldsymbol{A}^T$.

We show that the integrability condition (2.3) leads to a restriction on the symmetrical tensor field $\boldsymbol{A}$—"the compatibility equation"—and leads to a representation of the angle of rotation θ via $\boldsymbol{A}$ due to a quadrature.

It is convenient to use a unit vector, $\boldsymbol{k}$, orthogonal to the plane. Then $\boldsymbol{Q}$ can be written as

$$\boldsymbol{Q}(\theta) = \cos\theta \boldsymbol{1} + \sin\theta \boldsymbol{k} \times \boldsymbol{1}, \tag{2.5}$$

where $\boldsymbol{1}$ is a plane unit tensor.

Substituting (2.4) into (2.3) we have

$$(\partial_\alpha \boldsymbol{Q})\boldsymbol{A} \times \boldsymbol{e}_\alpha + \boldsymbol{Q}(\boldsymbol{A} \times \nabla) = 0, \tag{2.6}$$

where $\nabla = \partial_\alpha \boldsymbol{e}_\alpha$ denotes the nabla-operator in plane Cartesian coordinates, and the symbol ∂_α denotes a partial derivative with respect to the corresponding Cartesian coordinate. According to (2.5) we obtain

$$\partial_\alpha \boldsymbol{Q} = (\partial_\alpha \theta)\boldsymbol{k} \times \boldsymbol{Q}. \tag{2.7}$$

Substituting (2.7) into (2.6) and attaching $\partial_\alpha \theta$ and $\boldsymbol{e}_\alpha$ we arrive at

$$\boldsymbol{k} \times \boldsymbol{Q}\boldsymbol{A} \times \nabla\theta + \boldsymbol{Q}(\boldsymbol{A} \times \nabla) = 0. \tag{2.8}$$

We note that due to (2.5)

$$\boldsymbol{Q}_{\pi/2} = \boldsymbol{k} \times \boldsymbol{1}, \tag{2.9}$$

where $\boldsymbol{Q}_{\pi/2} = \boldsymbol{Q}(\pi/2)$. Using the commutative law for plane rotations, we have

$$\boldsymbol{k} \times \boldsymbol{Q}(\theta) = (\boldsymbol{k} \times \boldsymbol{1})\boldsymbol{Q} = \boldsymbol{Q}_{\pi/2}\boldsymbol{Q}(\theta) = \boldsymbol{Q}(\theta)\boldsymbol{Q}_{\pi/2}.$$

Then (2.8) can be rewritten as

$$\boldsymbol{Q}(\theta)(\boldsymbol{Q}_{\pi/2}\boldsymbol{A} \times \nabla\theta + \boldsymbol{A} \times \nabla) = 0. \tag{2.10}$$

Since the $\boldsymbol{Q}(\theta) \neq 0$, we obtain

$$\boldsymbol{Q}_{\pi/2}\boldsymbol{A} \times \nabla\theta + \boldsymbol{A} \times \nabla = 0, \tag{2.11}$$

which does not contain θ by only $\nabla\theta$.

To simplify (2.10), we use the algebraical identity

$$\boldsymbol{a} \times \boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{Q}_{\pi/2}\boldsymbol{a})\boldsymbol{k} = (\boldsymbol{a} \cdot \boldsymbol{Q}_{\pi/2}^T\boldsymbol{c})\boldsymbol{k} \tag{2.12}$$

for any plane vectors $\boldsymbol{a}$ and $\boldsymbol{c}$. Therefore, for any plane tensor $\boldsymbol{A}$ and any plane vector $\boldsymbol{c}$ we have

$$\boldsymbol{A} \times \boldsymbol{c} = \boldsymbol{A}\boldsymbol{Q}_{\pi/2}^T \boldsymbol{c} \otimes \boldsymbol{k}. \tag{2.13}$$

It follows that (2.11) can be written as

$$(\boldsymbol{Q}_{\pi/2}\boldsymbol{A}\boldsymbol{Q}_{\pi/2}^T(\nabla\theta) + \boldsymbol{A}\boldsymbol{Q}_{\pi/2}^T\nabla) \otimes \boldsymbol{k} = 0, \tag{2.14}$$

from which we arrive at the vector equation

$$Q_{\pi/2}AQ^T_{\pi/2}(\nabla\theta) + AQ^T_{\pi/2}\nabla = 0. \tag{2.15}$$

Since $\det K \neq 0$, then due to (2.4), A^{-1} exists. Therefore, (2.15) yields the relation for $\nabla\theta$:

$$\nabla\theta = Q_{\pi/2}A^{-1}(Q_{\pi/2}AQ^T_{\pi/2}\nabla), \tag{2.16}$$

where we have used $Q^T_{\pi/2} = -Q_{\pi/2}$. Due to (2.9) we also have

$$\boxed{\nabla\theta = -k \times A^{-1}((k \times A \times k)\nabla).} \tag{2.17}$$

The condition of integrability (2.16) is $\nabla \times \nabla\theta = 0$. Due to (2.12) we can write it as $\nabla \times \nabla\theta = (\nabla \cdot Q^T_{\pi/2}\nabla\theta)k = 0$, and, therefore, using (2.15) we obtain

$$\nabla \cdot (A^{-1}(Q_{\pi/2}AQ^T_{\pi/2}\nabla)) = 0, \tag{2.18}$$

which can also be written as

$$\boxed{\nabla \cdot (A^{-1}((k \times A \times k)\nabla)) = 0.} \tag{2.19}$$

The relation (2.19) plays the role of the compatibility equation for the tensor A. Thus if the tensor A satisfies (2.19), integration of (2.17) yields the angle of rotation θ:

$$\theta(C) = \theta(A) + \int_A^C \nabla\theta \cdot dl \equiv \theta(A) + \theta(A, C). \tag{2.20}$$

The rotation tensor is then given by

$$Q(\theta) = Q(\theta(A) + \theta(A, C)) = Q(\theta(A))Q(\theta(A, C)). \tag{2.21}$$

Here we have used the property that Q represents plane rotations. Finally, due to (2.2) we obtain

$$y(B) = y(A) + Q(\theta(A)) \int_A^B Q(\theta(A, C))A(C)\, dl_C. \tag{2.22}$$

The resulting field y, reconstructed by means of A, will represent both a deformation field and plane rigid body motion (where the latter is given by three unknown constants of integration: $y(A)$ and the angle $\theta(A)$).

The relation (2.19) can be written in brief form

$$\operatorname{div}(A^{-1} \operatorname{div} A^{\times}) = 0, \tag{2.23}$$

where the tensor

$$A^{\times} \equiv -k \times A \times k. \tag{2.24}$$

It is of interest to note that the tensor $A^{\times}$ has a very simple representation in the principal axes e_1 and e_2 of the symmetric tensor $A = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2$, namely

$$A^{\times} = \lambda_2 e_1 \otimes e_1 + \lambda_1 e_2 \otimes e_2. \tag{2.25}$$

We observe that the principal values of A and $A^{\times}$ are permuted.

3. The Lagrange and Eulerian description

To apply the results obtained in Section 2 to continuum mechanics, we first relate the notation introduced above to the standard notations which describes the kinematics of a deformed body.

Let $\boldsymbol{x}$ and $\boldsymbol{X}$ be the position vectors of a body-point in the current and in the reference configurations respectively. If we use the Lagrange description, then we assume $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$ and the deformation gradient $\boldsymbol{F}$ is introduced as the derivative of $\boldsymbol{x}$ with respect to $\boldsymbol{X}$

$$\boldsymbol{F} = \boldsymbol{x} \otimes \nabla, \qquad F_{\alpha\beta} = \frac{\partial x_\alpha}{\partial X_\beta}. \tag{3.1}$$

Using polar decomposition, $\boldsymbol{F}$ is represented (see, for example, Truesdell [4]) as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}, \tag{3.2}$$

where $\boldsymbol{R}$ is an orthogonal tensor called the rotation tensor and $\boldsymbol{U}$ is the symmetric positive definite tensor called right stretch tensor. Clearly, the relation between this notation and that introduced in Section 2 is

$$\boldsymbol{x} \to \boldsymbol{y}, \qquad \boldsymbol{F} \to \boldsymbol{K}, \qquad \boldsymbol{U} \to \boldsymbol{A}, \qquad \boldsymbol{R}(\theta) \to \boldsymbol{Q}(\theta), \tag{3.3}$$

and ∇ is the nabla-operator in the reference configuration space. Therefore, the right stretch tensor $\boldsymbol{U}$ must satisfy the compatibility equation (2.19)

$$\nabla \cdot (\boldsymbol{U}^{-1}((\boldsymbol{k} \times \boldsymbol{U} \times \boldsymbol{k})\nabla)) = 0, \tag{3.4}$$

which is, in fact, the integrability condition for the angle of rotation θ. The relation (2.17) has form

$$\nabla\theta = -\boldsymbol{k} \times \boldsymbol{U}^{-1}((\boldsymbol{k} \times \boldsymbol{U} \times \boldsymbol{k})\nabla) \tag{3.5}$$

and upon carrying out the integration in the reference configuration, we determine θ; therefore, we can reconstruct the deformation gradient $\boldsymbol{F}$.

In an analogous way, if we use the Eulerian description, then we assume $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x})$ and the deformation gradient is given by

$$\boldsymbol{G} = \boldsymbol{X} \otimes \nabla, \qquad G_{\alpha\beta} = \frac{\partial X_\alpha}{\partial x_b} \tag{3.6}$$

with a polar decomposition

$$\boldsymbol{G} = \boldsymbol{P}\boldsymbol{W}, \tag{3.7}$$

where $\boldsymbol{P}$ is a proper orthogonal tensor and $\boldsymbol{W}$ is a symmetric positive definite tensor. Comparing (2.1), (2.4) with (3.6), (3.7) we obtain the following relations between the notations

$$\boldsymbol{X} \to \boldsymbol{y}, \qquad \boldsymbol{G} \to \boldsymbol{K}, \qquad \boldsymbol{W} \to \boldsymbol{A}, \qquad \boldsymbol{P}(\theta) \to \boldsymbol{Q}(\theta), \tag{3.8}$$

and ∇ is nabla-operator in the current configuration space.

Commonly, the tensor $\boldsymbol{W}$ is not used but instead the left stretch tensor $\boldsymbol{V}$, which is introduced by (see, for example, Truesdell [1])

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}. \tag{3.9}$$

Using the property that $\boldsymbol{F}$ and $\boldsymbol{G}$ are the inverse of each other, we obtain

$$\boldsymbol{P} = \boldsymbol{R}^T, \qquad \boldsymbol{W} = \boldsymbol{V}^{-1}, \tag{3.10}$$

and therefore we can write $\boldsymbol{G}$ as

$$\boldsymbol{G} = \boldsymbol{P}\boldsymbol{V}^{-1}. \tag{3.11}$$

We thus obtain the compatibility equation

$$\nabla \cdot (\boldsymbol{V}((\boldsymbol{k} \times \boldsymbol{V}^{-1} \times \boldsymbol{k})\nabla)) = 0, \tag{3.4}$$

and the relation for $\nabla\theta$ of the form

$$\nabla\theta = -\boldsymbol{k} \times \boldsymbol{V}((\boldsymbol{k} \times \boldsymbol{V}^{-1} \times \boldsymbol{k})\nabla). \tag{3.5}$$

References

[1] C. Truesdell, *A First Course in Rational Continuum Mechanics*, The Johns Hopkins University, Baltimore, MD 1972.

[2] E. Cesàro, *Sulle formole del Volterra, fondamentali nella teoria delle distorsioni elastiche*, Rend. Napoli, *12*, n. 3a, 311–321 (1906).

[3] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill Book Co., New-York 1951.

[4] C. Truesdell and R. Toupin, *The classical field theories*. In *Handbuch der Physik*, Vol. 3/1, Springer, Berlin-Göttingen-Heidelberg 1960.

[5] R. T. Shield, *The rotation associated with large strain*, SIAM J. Appl. Math., *25*, 3, 483–491 (1973).

[6] I. L. Slepyan, *On linearization of the compatibility equations*. In *Mechanics of Deformable Body, Series Strength and Visco-Plasticity*, Nauka, Moskow 1986, pp. 139–146 (in Russian).

[7] V. A. Shamina, *Determination of displacement vector from the components of strain tensor in nonlinear continuum mechanics*, (in Russian) Izv. Akad. Nauk SSSR Mech. Tverd. Tela, *1*, 14–22 (1974) translated as Mechanics of Solids, *9*, no. 1, 9–16 (1974).

[8] A. I. Lurie, *Nonlinear Theory of Elasticity*, North-Holland Series in Applied Mathematics and Mechanics, Vol. *36*, Amsterdam-New York 1990.

[9] P. Ciarlet, *Mathematical Elasticity Vol. 1, Three-dimensional elasticity*. Studies in Mathematics and its Applications, Vol. *20*. North-Holland Pub. Co., Amsterdam-New York 1988.

Abstract

Using a new representation for the gradient of the rotation angle, the construction of a displacement field via the stretch tensor is reduced to quadrature for plane finite deformations. The compatibility equation is written in a very brief new form.

(Received: December 6, 1993; revised: April 26, 1995)