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On the zeros of derivatives of Bessel functions*)

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1. Introduction

We use $j_{\nu k}$ and $c_{\nu k}$ to denote the kth positive zeros of the Bessel function $J_\nu(x)$ and of the general Bessel function

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \qquad 0 \leqslant \alpha < \pi, \tag{1.1}$$

where $Y_\nu(x)$ is the Bessel function of second kind.

Similarly we denote with $j'_{\nu k}$ and $c'_{\nu k}$ the kth positive zeros of the derivative with respect to x of $J_\nu(x)$ and $C_\nu(x)$ respectively. As for $c_{\nu k}$, the dependence of $c'_{\nu k}$ on α is usually omitted.

Recently many results have been obtained on the *monotonicity*, *concavity* and *convexity* with respect to ν of $j_{\nu k}$ and more generally of $c_{\nu k}$. For example R. McCann in [9] and J. T. Lewis and M. E. Muldoon in [6] showed independently that $j_{\nu k}/\nu$ decreases as ν increases provided $\nu > 0$.

Later E. Makai [8] proved this property with an ingenious application of the Sturm comparison theorem, and in [5] the authors obtained similar more stringent properties for $c_{\nu k}$ with $k = 2, 3, \ldots$. The corresponding properties can be extended to $k = 1$, but only for $0 \leqslant \alpha \leqslant \frac{\pi}{2}$.

Further properties concerning the *concavity* and *convexity* of $j_{\nu k}$ or, more generally of $c_{\nu k}$, have been discussed in [1], [2], [3], [4], [5].

We observe that the results cited above were motivated by some physical problems concerning the explanation for the origin of the vortex lines which are produced in superfluid helium where its container is rotated [6, p. 171].

The principal tool used by the authors in several works mentioned above is the following Watson's integral formula [10, p. 508]

$$\frac{\mathrm{d}}{\mathrm{d}\nu} c_{\nu k} = 2 c_{\nu k} \int_0^\infty K_0(2 c_{\nu k} \sinh t)\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t. \tag{1.2}$$

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where $K_0(x)$ is the modified Bessel function of order zero. Since for $c'_{\nu k}$ there is an integral formula similar to (1.2), we hope that we can prove analogues of *concavity* (*convexity*) results for the zeros $c'_{\nu k}$ of $C'_\nu(x)$. However we observe that the integral formula for $\frac{\mathrm{d}}{\mathrm{d}\nu} c'_{\nu k}$ corresponding to (1.2) (see (2.1) in the next section) is more complicated, so in general it is not easy to obtain the corresponding properties for $c'_{\nu k}$.

In this work we are concerned with the *concavity* of $c'_{\nu k}$ with respect to ν in the case $c'_{\nu k} > |\nu|$.

2. Preliminaries

In this section we recall some results which will be useful in the sequel. The first one is Watson's formula for the derivative of $c'_{\nu k}$ with respect to ν [10, p. 510]

$$\frac{\mathrm{d}}{\mathrm{d}\nu} c'_{\nu k} = \frac{2\,c'_{\nu k}}{c'^2_{\nu k} - \nu^2} \int_0^\infty (c'^2_{\nu k} \cosh 2t - \nu^2)\, K_0(2\,c'_{\nu k} \sinh t)\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t \tag{2.1}$$

where $K_0(x)$ is the modified Bessel function of order zero.

Concerning the function $K_0(x)$ we need the following integrals [10, p. 388]

$$\int_0^\infty K_0(x)\, \mathrm{e}^{-ax}\, \mathrm{d}x = A(a) = \begin{cases} \dfrac{\operatorname{arc\,cos} a}{\sqrt{1-a^2}}, & |a| < 1 \\ 1, & a = 1 \\ \dfrac{\operatorname{arc\,cosh} a}{\sqrt{a^2-1}}, & a > 1 \end{cases} \tag{2.2}$$

$$\int_0^\infty K_0(x)\, x\, \mathrm{d}x = 1\,. \tag{2.3}$$

Moreover we recall the inequality [10, p. 487]

$$j'_{\nu,1} > \sqrt{\nu(\nu+2)}, \quad \nu > 0\,. \tag{2.4}$$

For our purposes we need an upper estimate for the integral (2.2). This is given by the following result.

Lemma 2.1. For $a > 1$ the inequality

$$\frac{\operatorname{arc\,cosh} a}{\sqrt{a^2-1}} < \frac{22}{15} - \frac{3}{5}a + \frac{2}{15}a^2 \tag{2.5}$$

holds.

Proof. Since $\operatorname{arc\,cosh} a = \log(a + \sqrt{a^2+1})$ we have to show that the function

$$f(a) = \sqrt{a^2-1}\left(\frac{22}{15} - \frac{3}{5}a + \frac{2}{15}a^2\right) - \log(a + \sqrt{a^2-1})$$

is positive. Since $f(1) = 0$, it is sufficient to prove that $f'(a) > 0$ for $a > 1$. We get $\sqrt{a^2 - 1}\, f'(a) = \frac{2}{5}(a-1)^3$ which clearly is positive for $a > 1$.

3. The function $j'_{\nu\kappa}$ and its monotonic properties

Let us denote the kth positive zero of $C'_\nu(x)$ by $c'_{\nu k}$ where $C_\nu(x)$ is the same as in (1.1). Now let $\kappa = k - \frac{\alpha}{\pi}$ and let the function $j'_{\nu\kappa}$ be defined by $j'_{\nu\kappa} \equiv c'_{\nu k}$ $(\nu > 0)$ where κ is considered to be parameter. Hence $j'_{\nu\kappa}$ is a solution of the integrodifferential equation (2.1). It is not difficult to show that the right hand side of (2.1) is Lipschitzian with respect to $c'_{\nu k}$, provided $c'_{\nu k} > 0$ and $c'_{\nu k} \neq |\nu|$, so we have the uniqueness of the solutions of the initial value problem at least on the domain $c > |\nu|$. We suspect that this uniqueness cannot be extended to the line $c = |\nu|$. We hope that we can return to the case $0 < c' < |\nu|$ in a subsequent paper.

Incidentally we observe that our notation permits to obtain easily an interesting property concerning the behaviour of the zeros $c'_{\nu k}$ with respect to α where the dependence on α is omitted. Our result complements a similar result obtained by Lorch and Newman for the positive zeros $c_{\nu k}$ of $C_\nu(x)$ [7]. Precisely we show that $j'_{\nu\kappa}$ increases as κ increases $(\alpha = (k - \kappa)\pi)$. To this end first we consider the case $\nu = 1/2$; then $C_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin(x + \alpha)$ and the function $\gamma = \gamma(\kappa) = j'_{1/2,\kappa}$ satisfies the relation

$$\tan \Gamma = 2\gamma \tag{3.1}$$

where $\Gamma = \Gamma(\kappa) = \gamma + (k - \kappa)\pi$. Let us observe that if $\gamma = \frac{1}{2}$ then $\kappa = \kappa_0 = \frac{3}{4} + \frac{1}{2\pi} = 0.90915\ldots$. Differentiating (3.1) with respect to κ we obtain

$$\gamma'(1 - 2\cos^2 \Gamma) = \pi .$$

We shall consider only the case $\gamma(\kappa) > \frac{1}{2}$. Then by (3.1) then $\Gamma > 1$, hence $\cos^2 \Gamma < \frac{1}{2}$ and $\gamma' > 0$ if $\kappa > \kappa_0$. This means that $j'_{1/2,\kappa}$ strictly increases as κ increases. Now let $\sigma = \sigma(\nu)$ be a solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\sigma = \frac{2\sigma}{\sigma^2 - \nu^2} \int_0^\infty (\sigma^2 \cosh 2t - \nu^2) K_0(2\sigma \sinh t)\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t \tag{3.2}$$

with the initial condition $\sigma\left(\frac{1}{2}\right) = \gamma(\kappa)$ for $\kappa > \kappa_0$. If $\gamma(\kappa) > 1/2$ then the uniqueness of the solutions of this initial value problem yields that $\sigma(\nu) = j'_{\nu\kappa}$ and

$$j'_{\nu\kappa_1} < j'_{\nu\kappa_2} \quad \text{if} \quad \kappa_0 < \kappa_1 < \kappa_2 \quad \text{and} \quad j'_{\nu,\kappa_1} > |\nu| .$$

Since $\alpha = (k - \kappa)\pi$ we obtain that the kth positive zero $c'_{\nu k}$ of $C'_\nu(x)$ decreases as α increases and $0 < \alpha < \pi$, as long as $c'_{\nu k} > |\nu|$.

4. Basic results

First we have to compute the second derivative of $j'_{\nu\kappa}$ with respect to ν. The formula (3.2) can be rewritten in the form

$$\frac{d\sigma}{d\nu} = 2\int_0^\infty f(t,\sigma,\nu)\,K_0(2\sigma\sinh t)\,dt$$

where

$$f(t,\sigma,\nu) = \frac{\sigma}{\sigma^2-\nu^2}(\sigma^2\cosh 2t - \nu^2)\,e^{-2\nu t}.$$

The differentiation with respect to ν gives

$$\begin{aligned}\frac{d^2\sigma}{d\nu^2} &= 2\int_0^\infty [f_\sigma\sigma' + f_\nu]\,K_0(2\sigma\sinh t)\,dt\\ &\quad + 2\int_0^\infty f K'_0(2\sigma\sinh t)\,2\sigma'\sinh t\,dt\end{aligned} \tag{4.1}$$

where $f_\sigma = \frac{\partial}{\partial\sigma}f$, $f_\nu = \frac{\partial f}{\partial\nu}$ and $\sigma' = \frac{d\sigma}{d\nu}$. An integration by parts of the second integral in (4.1) shows that this is equivalent to

$$\begin{aligned}&\int_0^\infty f\frac{\sigma'}{\sigma}\frac{\sinh t}{\cosh t}K'_0(2\sigma\sinh t)\,2\sigma\cosh t\,dt\\ &= \left[f\frac{\sigma'}{\sigma}\tanh t\,K_0(2\sigma\sinh t)\right]_0^\infty - \int_0^\infty K_0(2\sigma\sinh t)\frac{d}{dt}\left\{f\frac{\sigma'}{\sigma}\tanh t\right\}dt,\end{aligned} \tag{4.2}$$

where the term in the brackets is equal to zero due to the asymptotic relations

$$K_0(x) = \begin{cases} O\left(\log\frac{1}{x}\right), & x > 0, \quad x \to 0\\ O(e^{-x}), & x \to \infty.\end{cases}$$

Concerning the integral on the right hand side in (4.2) we have

$$\frac{d}{dt}\left\{f\tanh t\right\} = f_t\tanh t + f\frac{1}{\cosh^2 t},$$

therefore we get

$$\frac{d^2\sigma}{d\nu^2} = 2\int_0^\infty\left(f_\sigma\sigma' + f_\nu - \frac{\sigma'}{\sigma}f_t\tanh t - \frac{\sigma'}{\sigma}f\frac{1}{\cosh^2 t}\right)K_0(2\sigma\sinh t)\,dt$$

By straightforward calculations we obtain

$$\frac{d^2\sigma}{dv^2} = 2\int_0^\infty \left[\frac{\sigma'}{\sigma}\tanh^2 t + 2v\frac{\sigma'}{\sigma}\tanh t - 2t + \frac{4\sigma^2\sinh^2 t}{(\sigma^2 - v^2)(\sigma^2\cosh 2t - v^2)}(v - \sigma\sigma')\right] \cdot f K_0(2\sigma\sinh t)\,dt, \qquad \sigma \neq |v|. \tag{4.3}$$

Now we prove the following two Lemmas.

Lemma 4.1. Let $\sigma(v)$ be a solution of the differential equation (3.2) in a neighborhood of $v = v_0$ and $\sigma(v_0) > |v_0|$. Then

$$\left.\frac{d}{dv}\sigma(v)\right|_{v=v_0} > 1. \tag{4.4}$$

Proof. We make use of the integral formula (2.2) for $a = 1$

$$\int_0^\infty K_0(x)e^{-x}\,dx = 1.$$

By the substitution $x = 2\sigma\sinh t$ in this integral we obtain

$$\int_0^\infty K_0(2\sigma\sinh t)e^{-2\sigma\sinh t}2\sigma\cosh t\,dt = 1.$$

In order to prove the inequality (4.4) by (3.2) it is sufficient to show that

$$(\sigma^2\cosh 2t - v^2)e^{2\sigma\sinh t} > (\sigma^2 - v^2)e^{2vt}$$

which is clearly true because $\sigma > |v|$.

Corollary 4.1. Let $j'_{v_0\kappa} > |v_0|$, then

$$j'_{v\kappa} > j'_{v_0\kappa} + v - v_0, \quad v > v_0.$$

Proof. The result follows from the fact that the function $j'_{vk} - v$ increases as v increases, because by Lemma 4.1 with $\sigma(v) = j'_{v\kappa}$, we obtain $\frac{d}{dv}j'_{v\kappa} - 1 > 0$.

Lemma 4.2. If $\sigma(v)$ statisfies the same conditions of Lemma 4.1 and additionally

$$\left.v_0\frac{d}{dv}\sigma(v)\right|_{v=v_0} < \sigma(v_0),$$

then

$$\frac{d^2}{dv^2}\sigma(v)\bigg|_{v=v_0} < 0.$$

Proof. By our conditions we have

$$2v_0\frac{\sigma'}{\sigma}\tanh t - 2t < 2|v_0|\frac{\sigma'}{\sigma}t - 2t < 0.$$

Thus by (4.3) it is sufficient to show that

$$g(t) \equiv \frac{\sigma'}{\sigma}\tanh^2 t + \frac{4\sigma^2\sinh^2 t}{(\sigma^2 - v^2)(\sigma^2\cosh 2t - v^2)}(v - \sigma\sigma') < 0.$$

Since $\sigma' > 1$ and $v - \sigma\sigma' < 0$ moreover

$$\frac{\sigma^2\cosh^2 t}{\sigma^2\cosh 2t - v^2} > 1/2, \quad t \geqslant 0$$

we obtain

$$\frac{g(t)}{\tanh^2 t} < \frac{\sigma'}{\sigma} + \frac{4}{\sigma^2 - v^2}\frac{1}{2}(v - \sigma\sigma') = \frac{2v\sigma - \sigma'(\sigma^2 + v^2)}{\sigma(\sigma^2 - v^2)}$$

$$< \frac{2v\sigma - (\sigma^2 + v^2)}{\sigma(\sigma^2 - v^2)} < 0.$$

This gives the conclusion of the Lemma 4.2.

5. The main result

The aim of this section is to study the concavity of $j'_{v\kappa}$ for $j'_{v\kappa} > |v|$ and $\kappa \geqslant 1$. According to Lemmas 4.1, 4.2 we have the concavity of $j'_{v\kappa}$ if $v \leqslant 0$. In the case $v > 0$ we need additional restrictions for $j'_{v\kappa}$ to ensure the condition $v\sigma' < \sigma$ in Lemma 4.2. These restrictions are formulated in the following result.

Theorem 5.1. If

$$\sigma(v) > \begin{cases} \sqrt{2v}, & \text{for} \quad 0 < v \leqslant 1/2 \\ v + 1/2 & \phantom{\text{for}} \quad v \geqslant 1/2 \end{cases}$$

then

$$\frac{d^2\sigma}{dv^2} < 0.$$

Proof. For the proof it is convenient to consider two cases:

a) $0 < \nu \leqslant 1/2$

b) $\nu > 1/2$.

Case a). By Lemma 4.2 it is sufficient to prove that $\sigma' < \frac{\sigma}{\nu}$. The substitution $u = 2\sigma \sinh t$ in the integral (3.2) for σ' gives

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\sigma = \int_0^\infty \frac{1 + \dfrac{2}{1 - \left(\dfrac{\nu}{\sigma}\right)^2} s^2}{(s + \sqrt{s^2 + 1})^{2\nu}\sqrt{s^2 + 1}} K_0(u)\,\mathrm{d}u \tag{5.1}$$

where $s = \frac{u}{2\sigma}$. From the inequality $\sigma \geqslant \sqrt{2\nu}$ it follows $\frac{\nu^2}{\sigma^2} \leqslant \frac{\nu}{2}$ and we have to consider the inequality

$$\frac{\nu}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\nu}\sigma < \frac{\nu}{\sigma}\int_0^\infty \frac{1 + \dfrac{2}{1 - \dfrac{\nu}{2}} s^2}{(s + \sqrt{s^2 + 1})^{2\nu}\sqrt{s^2 + 1}} K_0(u)\,\mathrm{d}u.$$

Now we are going to show that

$$\frac{1 + \dfrac{2}{1 - \dfrac{\nu}{2}} s^2}{(s + \sqrt{s^2 + 1})^{2\nu}\sqrt{s^2 + 1}} < 1 + 2\,\frac{1 - \nu - \nu^2}{1 - \dfrac{\nu}{2}}\, s, \tag{5.2}$$

for $s > 0$ and $0 < \nu \leqslant 1/2$, or equivalently

$$\varphi(\nu) = \log\left(1 + 2\,\frac{1 - \nu - \nu^2}{1 - \dfrac{\nu}{2}}\, s\right) + 2\nu\log(s + \sqrt{s^2 + 1}) + \frac{1}{2}\log(s^2 + 1)$$

$$- \log\left(1 + \frac{2}{1 - \dfrac{\nu}{2}} s^2\right) > 0.$$

First we show

$$F(s) = \varphi(0) = \log(2s + 1) + \frac{1}{2}\log(s^2 + 1) - \log(2s^2 + 1) > 0 \quad \text{for} \quad s > 0.$$

This inequality can be proved directly. We should prove that

$$(2s + 1)\sqrt{s^2 + 1} > 2s^2 + 1$$

or equivalently

$$(2s+1)^2(s^2+1)-(2s^2+1)^2 = 4s^3+s^2+4s>0$$

which is clearly true.

Moreover we have

$$\frac{\mathrm{d}}{\mathrm{d}v}\varphi(v) = -\frac{(1+4v-v^2)s}{(1-v/2)[1-v/2+2(1-v-v^2)s]}+2\log(s+\sqrt{s^2+1})$$
$$-\frac{s^2}{(1-v/2)(1-v/2+2s^2)}$$

and it is easy to see that $\frac{\mathrm{d}}{\mathrm{d}v}\varphi(v)$ decreases as v increases on $[0,1/2]$. Hence $\varphi(v)$ is concave, therefore, $\min\limits_{0\leqslant v\leqslant 1/2}\varphi(v)=\min\{\varphi(0),\varphi(1/2)\}$. Hence we should check the inequality $\varphi(1/2)>0$. So we have to prove that

$$\exp(\varphi(1/2) = \frac{\left(1+\frac{2}{3}s\right)(s+\sqrt{s^2+1})\sqrt{s^2+1}}{1+\frac{8}{3}s^2}>1,$$

that is

$$g(s) = 1+\frac{2}{3}s-\frac{1+\frac{8}{3}s^2}{(s+\sqrt{s^2+1})(s^2+1)}>0. \tag{5.3}$$

For $s\geqslant 0$ a simple calculation shows that

$$\frac{1+\frac{8}{3}s^2}{(s+\sqrt{s^2+1})\sqrt{s^2+1}}<\frac{4}{3},$$

and this inequality permits us to prove the (5.3) in the case $s\geqslant 1/2$. In fact we have

$$g(s)>1+\frac{2}{3}s-\frac{4}{3}=\frac{2}{3}(s-1/2)\geqslant 0,\quad s\geqslant 1/2.$$

On the interval $0\leqslant s<1/2$ we have the estimates $(s+\sqrt{s^2+1})\sqrt{s^2+1}>s+1$ and

$$g(s)>1+\frac{2}{3}s-\frac{1+\frac{8}{3}s^2}{s+1}=\frac{2s\left(\frac{5}{6}-s\right)}{s+1}>0,$$

which completes the proof of the (5.2).

Now we use (5.2) in (5.1). Then by the integral formula (2.2) at $a = 0$ and (2.3) we obtain

$$\frac{\nu}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\nu}\sigma < \frac{\nu}{\sigma}\left(\frac{\pi}{2} + \frac{1-\nu-\nu^2}{1-\frac{\nu}{2}}\frac{1}{\sigma}\right) = h(\nu,\sigma).$$

Since $\sigma \geqslant \sqrt{2\nu}$

$$h(\nu,\sigma) \leqslant h(\nu,\sqrt{2\nu}) = \frac{\pi}{2}\sqrt{\frac{\nu}{2}} + \frac{1-\nu-\nu^2}{1-\frac{\nu}{2}}\frac{1}{2}.$$

Let $u = \sqrt{\frac{\nu}{2}}$ and

$$p(u) = [1 - h(\nu,\sqrt{2\nu})](2-\nu) = 4u^4 + \pi u^3 - \pi u + 1, \quad 0 \leqslant u \leqslant 1/2.$$

Clearly the function $p(u)$ is convex for $u \geqslant 0$, hence

$$p(u) \geqslant p(1/3) + p'(1/3)(u - 1/3),$$

where $p\left(\frac{1}{3}\right) = 0.4500\ldots$ and $p'\left(\frac{1}{3}\right) = -1.5018\ldots$. This gives

$$p(u) > p(1/3) + p'(1/3)\left(\frac{1}{2} - \frac{1}{3}\right) > 0, \quad 0 \leqslant u \leqslant 1/2$$

and by the definition of $p(u)$ we have $h(\nu,\sigma) < 1$ and, consequently

$$\frac{\nu}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\nu} < 1.$$

Case b). For $\nu \geqslant 1/2$ we use the inequality $\sinh t > t$, $(t > 0)$, in (3.2) and the property that $K_0(x)$ decreases as x increases. We have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\nu} < 2\sigma\int_0^\infty \frac{\sigma^2\cosh 2t - \nu^2}{\sigma^2 - \nu^2}\mathrm{e}^{-2\nu t}K_0(2\sigma t)\,\mathrm{d}t$$

$$= \int_0^\infty \frac{\sigma^2\cosh\frac{x}{\sigma} - \nu^2}{\sigma^2 - \nu^2}\mathrm{e}^{-\frac{\nu}{\sigma}x}K_0(x)\,\mathrm{d}x.$$

If α is defined by $\cos\alpha = \frac{\nu}{\sigma}$ with $0 < \alpha < \frac{\pi}{2}$ then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\nu} < \frac{1}{2\sin^2\alpha}\left[A\left(\cos\alpha + \frac{1}{\sigma}\right) + A\left(\cos\alpha - \frac{1}{\sigma}\right)\right] - \operatorname{cotan}^2\alpha\,\frac{\alpha}{\sin\alpha},$$

where we have used the integral formula $A(a)$ in (2.2). Since $A(a)$ is defined only for $a > -1$, we must have $\cos\alpha - \frac{1}{\sigma} = \frac{\nu}{\sigma} - \frac{1}{\sigma} > -1$ or $\nu + \sigma > 1$ which is satisfied because in the present case $\sigma \geqslant \nu + 1/2 \geqslant 1$ and $\nu + \sigma \geqslant 3/2$. In order to apply Lemma 2.1 we should prove the inequality $\frac{d\sigma}{d\nu} < \frac{\sigma}{\nu} = \frac{1}{\cos\alpha}$, that is the inequality

$$A\left(\cos\alpha + \frac{1}{\sigma}\right) + A\left(\cos\alpha - \frac{1}{\sigma}\right) < \psi(\alpha), \quad 0 < \alpha < \frac{\pi}{2}$$

where

$$\psi(\alpha) = \frac{2\sin^2\alpha}{\cos\alpha} + 2\cos^2\alpha\,\frac{\alpha}{\sin\alpha}. \tag{5.4}$$

By (2.2) it is clear that the function $A(a)$ is convex, hence the sum $A\left(\cos\alpha + \frac{1}{\sigma}\right) + A\left(\cos\alpha - \frac{1}{\sigma}\right)$ increases when σ decreases.

Now σ and $\nu = \sigma\cos\alpha$ satisfy the restrictions $\sigma \geqslant \nu + 1/2$ and $\nu \geqslant 1/2$, hence

$$\sigma \geqslant \max_{0<\alpha<\pi/2} \left\{\frac{1}{2(1-\cos\alpha)}, \quad \frac{1}{2\cos\alpha}\right\},$$

consequently

$$\sigma \geqslant \begin{cases} \dfrac{1}{2(1-\cos\alpha)} & \text{if} \quad 0 < \alpha \leqslant \pi/3 \\[2mm] \dfrac{1}{2\cos\alpha}, & \pi/3 \leqslant \alpha < \pi/2 \end{cases}$$

so we need to show the inequalities

$$A(3\cos\alpha - 2) + A(2 - \cos\alpha) < \psi(\alpha) \quad \text{if} \quad 0 < \alpha \leqslant \pi/3 \tag{5.5}$$

and

$$A(-\cos\alpha) + A(3\cos\alpha) < \psi(\alpha) \quad \text{if} \quad \pi/3 \leqslant \alpha < \pi/2. \tag{5.6}$$

First consider the case $0 < \alpha \leqslant \pi/3$. Since $2 - \cos\alpha > 1$, by (2.2) and (2.5) we have the estimate

$$A(2 - \cos\alpha) < \frac{22}{15} - \frac{3}{5}(2 - \cos\alpha) + \frac{2}{15}(2 - \cos\alpha)^2 = B(\alpha), \tag{5.7}$$

where

$$B(\alpha) = \frac{4}{5} + \frac{1}{15}\cos\alpha + \frac{2}{15}\cos^2\alpha.$$

Let the function $\beta = \beta(\alpha)$ be defined by $\cos\beta = 3\cos\alpha - 2$ with $\beta(0) = 0$. By this definition we have $0 < \beta(\alpha) < \frac{\pi}{2}$ for $0 < \alpha < \frac{\pi}{3}$ and $\beta\left(\frac{\pi}{3}\right) = \frac{\pi}{2}$. By (5.5) and (5.7) it is sufficient to show the relation

$$\Psi(\alpha) = [\psi(\alpha) - B(\alpha)]\sin\beta - \beta > 0, \quad 0 < \alpha \leqslant \pi/3. \tag{5.8}$$

Since $\Psi(0) = 0$ and

$$\Psi\left(\frac{\pi}{3}\right) = \left[\psi\left(\frac{\pi}{3}\right) - B\left(\frac{\pi}{3}\right)\right] - \frac{\pi}{2} > 0 \tag{5.9}$$

therefore it is sufficient to show that the function $\Psi(\alpha)$ is increasing on the interval $(0, \alpha_0)$ and decreasing on $(\alpha_0, \pi/3)$ with some $\alpha_0 \in (0, \pi/3)$.

Differentiating $\psi(\alpha)$ we obtain

$$\psi'(\alpha) = 2\frac{\sin^3\alpha}{\cos^2\alpha} + 2\left(\sin\alpha + \frac{1}{\sin\alpha}\right)\left(1 - \frac{\alpha\cos\alpha}{\sin\alpha}\right),$$

and clearly $\psi'(\alpha) > 0$ for $0 < \alpha < \pi/2$.

Recalling that the function $\beta(\alpha)$ is defined by $\cos\beta = 3\cos\alpha - 2$, we have $\sin\beta\,\beta' = 3\sin\alpha$, hence by (5.8)

$$\sin\beta\frac{\mathrm{d}}{\mathrm{d}\alpha}\psi(\alpha) = 3\sin\alpha(1-\cos\alpha)^2\left(-\frac{6}{5}\cos\alpha + \frac{68}{15} - \frac{2}{\cos^2\alpha}\right)$$
$$+ 6\sin\alpha(1-\cos\alpha)\frac{6\cos^2\alpha + 6\cos\alpha - 2}{1+\cos\alpha}\left(1 - \frac{\alpha\cos\alpha}{\sin\alpha}\right),$$

or

$$\tilde{\Psi}(\alpha) = \frac{\cos^2\alpha(1+\cos\alpha)}{3\sin\alpha(1-\cos\alpha)^2}\sin\beta\,\Psi'(\alpha) = C_1(\alpha) + C_2(\alpha)\frac{\cos\alpha}{1-\cos\alpha}\left(1 - \frac{\alpha\cos\alpha}{\sin\alpha}\right),$$

where

$$C_1(\alpha) = (1+\cos\alpha)\left(-\frac{6}{5}\cos^3\alpha + \frac{68}{15}\cos^2\alpha - 2\right),$$

$$C_2(\alpha) = 2\cos\alpha(6\cos^2\alpha + 6\cos\alpha - 2).$$

Since $\lim_{\alpha\to+0}\tilde{\Psi}(\alpha) = 16 > 0$ and $\tilde{\Psi}\left(\frac{\pi}{3}\right) < 0$ it is sufficient to show that $\tilde{\Psi}(\alpha)$ strictly decreases. Since the functions $C_1(\alpha)$, $C_2(\alpha)$ are clearly decreasing and $C_2(\alpha) > 0$, we need only to show that

$$C_3(\alpha) = \frac{\cos\alpha}{1-\cos\alpha}\left(1 - \frac{\alpha\cos\alpha}{\sin\alpha}\right)$$

is also decreasing. To prove this first consider the derivative of $\log C_3(\alpha)$

$$\frac{C_3'(\alpha)}{C_3(\alpha)} = \frac{\alpha\sin\alpha}{\sin\alpha - \alpha\cos\alpha} - \frac{1+\cos\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}$$

and prove the statement $C_4(\alpha) < 0$ for $0 < \alpha < \pi/2$, where

$$C_4(\alpha) = \frac{C_3'(\alpha)}{C_3(\alpha)}\frac{\sin\alpha(\sin\alpha - \alpha\cos\alpha)}{2+\cos\alpha} = \alpha - \tan\alpha\frac{1+\cos\alpha+\cos^2\alpha}{2+\cos\alpha}.$$

For $C_4(0) = 0$, we have to show $C_4'(\alpha) < 0$ for $0 < \alpha < \pi/2$ which is clearly true because

$$C_4'(\alpha) = -\frac{\sin^2\alpha(1-\cos\alpha)\cos^2 + 4\cos\alpha + 2)}{\cos^2\alpha(1+\cos\alpha)^2}$$

is negative. Thus we have that $\tilde{\Psi}(\alpha)$ is strictly decreasing, hence there is only one value $\alpha_0\,(0, \pi/3)$ with $\tilde{\Psi}$ such that $\tilde{\Psi}(\alpha_0) = 0$ and $\tilde{\Psi}(\alpha) > 0$ for $0 < \alpha < \alpha_0$ and $\tilde{\Psi}(\alpha) < 0$ for $\alpha_0 < \alpha \leqslant \pi/3$. Consequently the function $\Psi(\alpha)$ has a local maximum at $\alpha = \alpha_0$ and

$$\Psi(\alpha) > \min_{0 \leqslant \alpha \leqslant \pi/3} \left\{\Psi(0), \Psi\left(\frac{\pi}{3}\right)\right\} = 0$$

i.e. the relation (5.8) holds, hence the inequality (5.5) is also true.

To prove the inequality (5.6) we observe that the function $A_1(a) = A(-a) + A(3a)$ is well defined and convex if $-\frac{1}{3} < a < 1$ and, by (5.10) the function $\psi(\alpha)$ is strictly increasing, hence

$$\max_{\pi/3 \leqslant \alpha \leqslant \pi/2} \{A(-\cos\alpha) + A(3\cos\alpha)\} = \max\left\{A\left(-\frac{1}{2}\right) + A\left(\frac{3}{2}\right), 2A(0)\right\}$$

and

$$\min_{\pi/3 \leqslant \alpha \leqslant \pi/2} \Psi(\alpha) = \psi\left(\frac{\pi}{3}\right).$$

Therefore we should check the inequalities $A\left(-\frac{1}{2}\right) + A\left(\frac{3}{2}\right) < \psi\left(\frac{\pi}{3}\right)$ and $2A(0) < \psi\left(\frac{\pi}{3}\right)$. The first is true because it is the inequality (5.5) at $\alpha = \frac{\pi}{3}$ which has been already proved. The second inequality is also true because by (2.2) $A(0) = \frac{\pi}{2}$ and by (5.4) $\psi\left(\frac{\pi}{3}\right) = 3.45\ldots$. The proof of Theorem 5.1 is complete.

Corollary 5.1. For $\kappa \geqslant 1$ and $\nu \geqslant 0$ the function $j'_{\nu\kappa}$ is concave respect to ν. The same is true for $\kappa \geqslant 1$ and $\nu < 0$ under the additional restriction $j'_{\nu\kappa} > |\nu|$.

Proof. If $\nu \geqslant 0$ the inequality (2.4) implies that $j'_{\nu 1}$ satisfies the conditions of Theorem 5.1, hence $j'_{\nu 1}$ is concave. Clearly the same is true for $j_{\nu\kappa}$ with $\kappa > 1$, because by (3.3), $j_{\nu\kappa} > j_{\nu_1}$. Finally in the case of $\nu \leqslant 0$ the conclusion of Corollary 5.1 is a consequence of Lemma 4.1 and 4.2.

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Abstract

In this paper we are interested in the behaviour respect to ν of the kth positive zero $c'_{\nu k}$ of the derivative of the general Bessel function

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \quad 0 \leqslant \alpha < \pi,$$

where $J_\nu(x)$ and $Y_\nu(x)$ indicate the Bessel functions of first and second kind respectively. It is well known that for $c'_{\nu k} > |\nu|$, $c'_{\nu k}$ increases as ν increases. Here we prove several additional properties for $c'_{\nu k}$. Our main result is that $c'_{\nu k}$ is concave as a function of ν, when $c'_{\nu k} > |\nu| > 0$. This implies the concavity of $c'_{\nu k}$ for every $k = 2, 3, \ldots$. In the case of the zeros $j'_{\nu k}$ of $\frac{d}{dx} J_\nu(x)$ we extend this property to $k = 1$ for every $\nu \geqslant 0$.

Sommario

In questo lavoro il nostro interesse è rivolto al comportamento, rispetto a ν, del k-esimo zero $c'_{\nu k}$ della derivata della funzione cilindrica

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \quad 0 \leqslant \alpha < \pi,$$

dove $J_\nu(x)$ e $Y_\nu(x)$ indicano le funzioni di Bessel rispettivamente di prima e seconda specie. E' ben noto che nel caso $c'_{\nu k} > |\nu|$, $c'_{\nu k}$ è una funzione crescente di ν.

Qui, proviamo parecchie ulteriori proprietà per la funzione $c'_{\nu k}$. Il principale risultato è che $c'_{\nu k}$ è concava rispetto a ν, per $c'_{\nu k} > |\nu| > 0$. Questo implica la concavità di $c'_{\nu k}$ per ogni $k = 2, 3, \ldots$. Nel caso degli zeri $j'_{\nu k}$ della funzione $\frac{d}{dx} J_\nu(x)$ possiamo estendere questo proprietà anche a $k = 1$, per ogni $\nu > 0$.

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