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Transversal homoclinic points and hyperbolic sets for non-autonomous maps I

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The aim of this paper is to generalize the concept of hyperbolic sets and transversal homoclinic points to non-autonomous systems. We treat maps rather than differential equations. The background is as follows: Consider a planar autonomous system of ODE's with a hyperbolic equilibrium and a homoclinic solution. Consider time dependent perturbation of such systems, the perturbation not necessarily being periodic in time. As in the periodic case we investigate the time-one map which, however, is no longer autonomous. Thus we consider maps $f\colon \boldsymbol{Z} \times \boldsymbol{R}^2 \to \boldsymbol{Z} \times \boldsymbol{R}^2$ with $f(n, z) = (n + 1, f_n(z))$ where the maps $f_n(z)$ depend on time n.

In Paragraph 1 we develop the concept of transversal homoclinic orbits and hyperbolic sets for these non-autonomous system. We consider the case of a map with a hyperbolic orbit which admits infinitely many homoclinic orbits. Moreover, we present a construction of a hyperbolic set for such maps. This construction does neither need the theory of invariant manifolds (Kirchgraber [2]) nor the notion of exponential dichotomies (Palmer [7, 8]).

In Paragraph 2 we prove the Shadowing Lemma for non-autonomous maps admitting a hyperbolic set.

In Paragraph 3 we introduce symbolic dynamics for time dependent maps. Then we prove that a system containing a hyperbolic set admits the "time shift" as a subsystem. This generalizes a result of Smale to non-autonomous systems.

In a second part, also to appear in this volume (see Stoffer [13]) we shall adapt the method of Melnikov to construct transversal homoclinic orbits to the non-autonomous situation. To guarantee the existence of hyperbolic sets one has to assume that the so-called Melnikov function has infinitely many simple zeroes with derivatives bounded away from zero. We shall show that the Melnikov function coincides with the well known Melnikov integral if the map is the time-one map of a differential equation. Furthermore we shall show that the Melnikov function can be interpretated as an approximation of the distance between the stable and the unstable invariant manifold of the hyperbolic orbit. Finally we shall apply the theory developed to almost periodically perturbed sytems. We shall show that under certain conditions there exists an m such that the time-m map restricted to time values $m\boldsymbol{Z}$ admits the "time shift" as a subsystem. This generalizes results of Scheurle [10] and brings some kind of order into the system.

This paper is an extract from the author's thesis [12].

1. Construction of hyperbolic sets

To motivate the assumptions we shall make we keep in mind the following example. We consider the differential equation

$$\dot{x} = g(t, x, \mu) = g^0(x) + \mu g^1(t, x, \mu) \quad x \in \boldsymbol{R}^2. \tag{1}$$

Let this differential equation have a hyperbolic equilibrium solution z_0 for $\mu = 0$. Thus for small μ there is a hyperbolic solution $z(t)$ which stays near z_0. Further we assume that for $\mu = 0$ Eq. (1) has a homoclinic solution $x(t)$, i.e. a solution $x(t) \neq z(t)$ with $|x(t) - z(t)| \to 0$ for $t \to \pm\infty$. We shall *not* assume that g is periodic in time. We shall consider maps rather than differential equations. The map we discuss, however, may be the Poincaré return map of Eq. (1). We restrict ourselves to integer values of the time t. For $n \in \boldsymbol{Z}$, $x \in \boldsymbol{R}^2$ there is a unique solution $\varphi(t; n, x, \mu)$ of Eq. (1) with $\varphi(n; n, x, \mu) = x$. The time-one map takes the point $(n, x) \in \boldsymbol{Z} \times \boldsymbol{R}^2$ to the point $(n+1, \varphi(n+1; n, x, \mu)) \in \boldsymbol{Z} \times \boldsymbol{R}^2$. We formulate our hypotheses in the language of maps. We consider the discrete dynamical system

$$\xi_{n+1} = f(\xi_n), \quad \text{with} \quad \xi_n \in \boldsymbol{Z} \times \boldsymbol{R}^2. \tag{2}$$

A set $X := \{\xi_n \mid \xi_n = (n, x_n) \in \boldsymbol{Z} \times \boldsymbol{R}^2\}$ is called an *orbit* if for each $n \in \boldsymbol{Z}$ $f(n, x_n) = (n+1, x_{n+1})$ holds. Let us make the following assumptions motivated by the preceeding differential equation example:

A1) The function $f: \boldsymbol{Z} \times \boldsymbol{R}^2 \to \boldsymbol{Z} \times \boldsymbol{R}^2$, $(n, x) \mapsto f(n, x) = (n+1, f_n(x))$ is an invertible C^1-map.

A2) For all $n \in \boldsymbol{Z}$, $x \in \boldsymbol{R}^2$ the matrix $Df_n(x)$ is invertible. For technical reasons we assume that

$$f_n(x + y) = f_n(x) + Df_n(x)\, y + \hat{f}_n(x, y)$$

$$Df_n(x + y) = Df_n(x) + D_2 \hat{f}_n(x, y)$$

holds (D_2 denotes the derivative with respect to the second argument) with the following estimates

$$|Df_n(x)| \leqq c, \quad |\hat{f}_n(x, y)| \leqq c\,|y|^2, \quad |D_2 \hat{f}_n(x, y)| \leqq c\,|y|$$

for some constant c. Let analogous estimates hold for the inverse maps $f_n^{-1}(x)$.

A3) *Eq. (2) has a hyperbolic orbit $Z = \{\zeta_n\}$, i.e. the set

$$Z = \{\zeta_n \mid \zeta_n = (n, z_n) \text{ for } n \in \boldsymbol{Z}\}$$

is an orbit and there are unit vectors E_n^+ and $E_n^- \in \boldsymbol{R}^2$ and numbers Λ_n^+ and Λ_n^- such that the equations

$$Df_n(z_n)\, E_n^{\pm} = \Lambda_n^{\pm}\, E_{n+1}^{\pm}$$

and the inequalities

$$|\Lambda_n^+| < \Lambda^+ < 1 < \Lambda^- < |\Lambda_n^-|$$

hold for $n \in \boldsymbol{Z}$.

A4) There exists a homoclinic orbit

$$X = \{\xi_n \mid \xi_n = (n, x_n) \text{ for } n \in \boldsymbol{Z}\} \text{ to } Z = \{\zeta_n\},$$

i.e. X is an orbit different from Z and $|\xi_n - \zeta_n| \to 0$ holds for $n \to \pm\infty$. Every point of X is called a homoclinic point.

The aim of this paragraph is to show how the hyperbolic structure of Z carries over to homoclinic orbits. In Proposition 1.1 we treat the case where we consider one single homoclinic orbit of Z. The result leads us to the definition of transversal homoclinic points and hyperbolic sets. Finally Theorem 1.10 treats the general case where there are infinitely many transversal homoclinic orbits of Z.

Proposition 1.1. Let Assumptions A1)–A4) be satisfied. Then for all $n \in \boldsymbol{Z}$ there exist unit vectors e_n^+, e_n^- and numbers t_n^+, t_n^- all uniquely determined up to a factor ± 1 satisfying

i) $e_{n+1}^{\pm} = \dfrac{1}{t_n^{\pm}} Df_n(x_n)\, e_n^{\pm}$ for $n \in \boldsymbol{Z}$

ii) $\lim_{n \to \infty} |e_n^+ - E_n^+| = 0$ and $\lim_{n \to -\infty} |e_n^- - E_n^-| = 0$.

Moreover, if in addition the vectors e_0^+ and e_0^- are linearly independent then the factors ± 1 can be chosen such that

iii) $\lim_{n \to -\infty} |e_n^+ - E_n^+| = 0$ and $\lim_{n \to \infty} |e_n^- - E_n^-| = 0$

iv) there are constants $N \in \boldsymbol{N}$, $\theta \in (0, 1)$ and $\tau > 1$ such that for all $|n| > N$

$$\frac{1}{\tau} < |t_n^+| < \theta < 1$$

$$\tau > |t_n^-| > \frac{1}{\theta} > 1 .$$

Before we prove this proposition we develop a tool to describe directions of vectors. We define the *direction of the vector* $v \neq 0$ as the set

$$R(v) := \{x \mid x = \alpha v, \alpha \in \boldsymbol{R} - \{0\}\}.$$

The direction of a vector v can be described by any vector $v' \in R(v)$. We represent the direction of each vector $v = a E_n^+ + b E_n^-$ with $b \neq 0$ by the point $v' = \frac{a}{b} E_n^+ + E_n^-$ of the straight line

$$G_n := \{x \mid x = \mu E_n^+ + E_n^-, \mu \in \boldsymbol{R}\}.$$

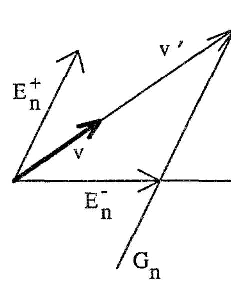

Figure 1.1

We define the map

$$\varphi_n : \boldsymbol{R}^2 - \{x \mid x = \mu E_n^+, \mu \in \boldsymbol{R}\} \to G_n$$

by

$$v = a E_n^+ + b E_n^- \mapsto v' = \varphi_n(v) = \frac{1}{b} v = \frac{a}{b} E_n^+ + E_n^-$$

which associates to a vector v point of G_n representing its direction. We want to study the change of the direction of a vector under the map $Df_n(z_n)$. We first remark that A3) implies that $Df_n(z_n)$ takes the direction of $E_n^\pm$ to the direction of $E_{n+1}^\pm$, i.e. $R(Df_n(z_n) E_n^\pm) = R(E_{n+1}^\pm)$. For $n \in \boldsymbol{Z}$ we define the map $F_n : G_n \to G_{n+1}$ by

$$F_n := \varphi_{n+1} \circ Df_n(z_n)$$

$$x = \mu E_n^+ + E_n^- \mapsto F_n(x) = \varphi_{n+1}(\mu \Lambda_n^+ E_{n+1}^+ + \Lambda_n^- E_{n+1}^-) = \mu \frac{\Lambda_n^+}{\Lambda_n^-} E_{n+1}^+ + E_{n+1}^-$$

which describes the change of direction under the map $Df_n(z_n)$. F_n maps every interval $G_n(s) := \{x \mid x = \mu E_n^+ + E_n^-, |\mu| \leqq s\}$ of the straight line G_n into the interval $G_{n+1}(s)$. The map F_n is a contraction with contraction factor $q_n = \left|\frac{\Lambda_n^+}{\Lambda_n^-}\right| < 1$. Setting $P_\mu^{(n)} = \mu E_n^+ + E_n^- \in G_n$ one has

$$F_n(P_\mu^{(n)}) = P_{\hat{\mu}}^{(n+1)} \quad \text{where} \quad \hat{\mu} = \frac{\Lambda_n^+}{\Lambda_n^-} \mu.$$

To make the proof of Proposition 1.1 more comprehensive we present the main idea of the proof. If for given m the vectors e_m^+ and e_m^- are prescribed then by property i) the vectors e_n^+ and e_n^- and the numbers t_n^+ and t_n^- are uniquely determined (up to a factor ± 1) for all $n \in \boldsymbol{Z}$. Hence one has to choose the vectors e_m^+ and e_m^- such that property ii) holds. It suffices to determine the *directions* of e_m^+ and e_m^-. We show how to construct e_m^-. The construction of e_m^+ is done analogously. The preliminary idea is the following. Consider the backward iterates $x_{m-1}, x_{m-2}, \ldots, x_{m-k}$ of x_m. Since we want $\lim_{n \to -\infty} |e_n^- - E_n^-| = 0$ to hold e_{m-k}^- is approximatively equal to E_{m-k}^- for large k. Thus one can take E_{m-k}^- as an approximation of e_{m-k}^-. Iterating the direction of E_{m-k}^- forward under the maps $Df_{m-j}(x_{m-j})\, j = k, k-1, \ldots, 1$ one gets a direction R_m^k for each k. One would expect that these directions converge for $k \to \infty$ and therefore one could determine e_m^- by putting $R(e_m^-) = \lim_{k \to \infty} R_m^k$.

Our construction is a slight modification of this idea. Instead of associating to each point x_{m-k} the direction E_{m-k} we associate to x_{m-k} a whole sector of directions which contains the direction of e_{m-k}^- we are looking for. We represent this sector of directions by an interval on the straight line G_{m-k}. The map $Df_n(x_n)$ induces a map $F_n(x_n)$ which takes the direction of y to the direction of $Df_n(x_n)\, y$. This induced map takes intervals of G_n to intervals of G_{n+1}. It turns out that this map is contracting if x_n is close enough to z_n. The intersection of all iterates of such intervals consist of a single point which represents the direction of e_n^-.

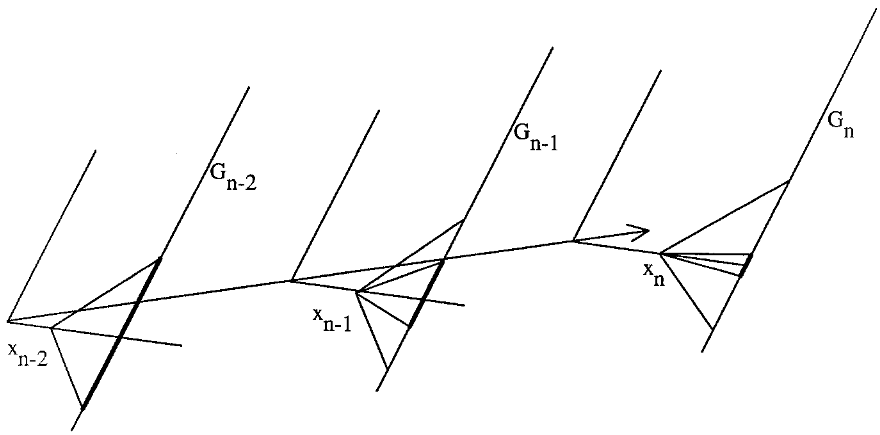

Figure 1.2

Proof of Proposition 1.1. We prove the proposition for e_n^- and t_n^-; for e_n^+ and t_n^+ it can be done analogously.

a) $Z = \{\zeta_n\} = \{(n, z_n)\}$ is the hyperbolic orbit and $X = \{\xi_n\} = \{(n, x_n)\}$ is the given homoclinic orbit of Z. We investigate the change of directions of vectors under the map $Df_n(x_n)$ for large negative n, i.e. for x_n close to z_n. To this end we define the map $\tilde{F}_n: G_n \to G_{n+1}$ by

$$\tilde{F}_n := \varphi_{n+1} \circ Df_n(x_n).$$

According to Assumption A2) one gets

$$\tilde{F}_n := \varphi_{n+1} \circ (Df_n(z_n) + D_2 \hat{f}_n(x_n, x_n - z_n)).$$

Lemma 1.2. For any $s \in (0, 1]$ there exist positive numbers $N(s)$ and $q \in (0, 1)$ such that for all $n \leqq -N(s)$ the following holds:

i) $\tilde{F}_n(G_n(s)) \subset G_{n+1}(s)$
ii) $\tilde{F}_n|_{G_n(s)}$ is contracting with contraction factor q.

Proof. Since E_{n+1}^+ and E_{n+1}^- are linearly independent there are coefficients $a_i^{(n)}$ and $b_i^{(n)}$ $i = 1, 2$ such that

$$\begin{aligned} D_2 \hat{f}_n(x_n, x_n - z_n) E_n^+ &= a_1^{(n)} E_{n+1}^+ + a_2^{(n)} E_{n+1}^- \\ D_2 \hat{f}_n(x_n, x_n - z_n) E_n^- &= b_1^{(n)} E_{n+1}^+ + b_2^{(n)} E_{n+1}^- . \end{aligned} \tag{3}$$

Let T_{n+1} be the matrix consisting of the colomn vectors E_{n+1}^+ and E_{n+1}^-. According to Assumptions A3) we have (see Fig. 1.3)

$$\begin{aligned} \Lambda^- - \Lambda^+ &\leqq |\Lambda_{n+1}^- E_{n+2}^- \pm \Lambda_{n+1}^+ E_{n+2}^+| \leqq |Df_{n+1}(z_{n+1})| \, |E_{n+1}^- \pm E_{n+1}^+| \\ &\leqq c\, |E_{n+1}^- \pm E_{n+1}^+| \end{aligned}$$

and thus

$$|E_{n+1}^- \pm E_{n+1}^+| \geqq \frac{\Lambda^- - \Lambda^+}{c}.$$

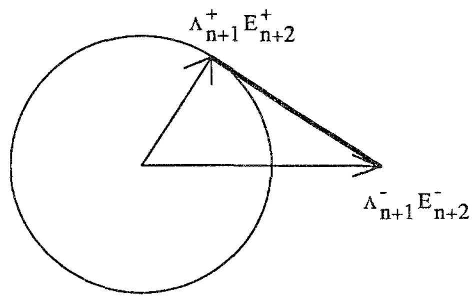

Figure 1.3

Claim. The following estimate holds

$$|T_{n+1}^{-1}| \leqq \frac{2c}{\Lambda^- - \Lambda^+}.$$

(This estimate holds for the Euclidean vectornorm and the induced matrix norm; for other norms one might have to multiply the r.h.s. by a norm factor).

Proof. The induced matrix norm is defined by

$$|T| = \max \lambda_i(T^T T), \quad \text{(the maximum eigen value of } T^T T).$$

For the norm of the inverse we get

$$|T^{-1}| = \max \lambda_i((T^T)^{-1} T^{-1}) = \frac{1}{\min \lambda_i(T T^T)} = \frac{1}{\min \lambda_i(T^T T)}.$$

Since T is a two dimensional matrix we have

$$\max \lambda_i(T^T T) \cdot \min \lambda_i(T^T T) = \operatorname{Det}(T^T T) = \operatorname{Det}(T)^2$$

and thus

$$|T^{-1}| = \frac{|T|}{|\operatorname{Det}(T)|} \leqq \frac{\sqrt{2}}{|\operatorname{Det}(T)|}$$

$|\operatorname{Det}(T)|$ is the area of the parallelogram spanned by E_{n+1}^+ and E_{n+1}^-. Let α be the smaller of the two angles between E_{n+1}^+ and E_{n+1}^-. Then $|\operatorname{Det}(T)| = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$ holds and by assumption

$$|E_{n+1}^+ - E_{n+1}^-| = 2 \sin \frac{\alpha}{2} \geqq \frac{\Lambda^- - \Lambda^+}{c}.$$

Since $0 \leqq \frac{\alpha}{2} \leqq \frac{\pi}{4}$ holds one has $\cos \frac{\alpha}{2} \geqq \frac{1}{\sqrt{2}}$ and therefore

$$|T^{-1}| \leqq \frac{2c}{\Lambda^- - \Lambda^+}.$$

We solve Eq. (3) for the coefficients $a_i^{(n)}$ and $b_i^{(n)}$

$$\begin{pmatrix} a_1^{(n)} \\ a_2^{(n)} \end{pmatrix} = T_{n+1}^{-1} D_2 \hat{f}_n(x_n, x_n - z_n) E_n^+$$

$$\begin{pmatrix} b_1^{(n)} \\ b_2^{(n)} \end{pmatrix} = T_{n+1}^{-1} D_2 \hat{f}_n(x_n, x_n - z_n) E_n^-$$

and by Assumption A2) we get the estimates

$$|a_i^{(n)}| \leqq \frac{2c}{\Lambda^- - \Lambda^+} c\,|x_n - z_n|$$

$$|b_i^{(n)}| \leqq \frac{2c}{\Lambda^- - \Lambda^+} c\,|x_n - z_n|.$$

Assumption A4) implies that there is an $N(s)$ such that for $n \leqq -N(s)$

$$|x_n - z_n| \leqq \frac{\Lambda^- - \Lambda^+}{2c^2} s \min\left(\frac{1-\Lambda^+}{3}, \frac{\Lambda^- - 1}{3}\right).$$

This implies

$$|a_i^{(n)}| \leqq \min\left(\frac{1-\Lambda^+}{3}, \frac{\Lambda^- - 1}{3}\right)$$

$$|b_i^{(n)}| \leqq s \min\left(\frac{1-\Lambda^+}{3}, \frac{\Lambda^- - 1}{3}\right).$$

For $P_\mu^{(n)} = \mu E_n^+ + E_n^-$ we have

$$\begin{aligned}\tilde{F}_n(P_\mu^{(n)}) &= \varphi_{n+1} \circ (Df_n(z_n) + D\hat{f}_n(x_n, x_n - z_n))(P_\mu^{(n)}) \\ &= \varphi_{n+1}((\mu(\Lambda_n^+ + a_1^{(n)}) + b_1^{(n)}) E_{n+1}^+ + (\Lambda_n^- + b_2^{(n)} + \mu a_2^{(n)}) E_{n+1}^-) \\ &= P_{\hat{\mu}}^{(n+1)}\end{aligned}$$

where

$$\hat{\mu} = \frac{b_1^{(n)} + \mu(\Lambda_n^+ + a_1^{(n)})}{\Lambda_n^- + b_2^{(n)} + \mu a_2^{(n)}} =: \frac{\alpha_n(\mu)}{\beta_n(\mu)}.$$

To prove i) we show that $|\mu| \leqq s$ implies $|\hat{\mu}| \leqq s \cdot q$ with $q = \dfrac{2+\Lambda^+}{2+\Lambda^-} < 1$:

$$|\hat{\mu}| \leqq s \frac{\dfrac{|b_1^{(n)}|}{s} + \Lambda_n^+ + |a_1^{(n)}|}{\Lambda^- - |b_2^{(n)}| - |a_2^{(n)}|} \leqq s \frac{\dfrac{1-\Lambda^+}{3} + \Lambda^+ + \dfrac{1-\Lambda^+}{3}}{\Lambda^- - 2\dfrac{\Lambda^- - 1}{3}} = s \frac{2+\Lambda^+}{2+\Lambda^-} < s.$$

To prove ii) we show that $|\hat{\mu}_2 - \hat{\mu}_1| \leqq q\,|\mu_2 - \mu_1|$:

$$\begin{aligned}|\hat{\mu}_2 - \hat{\mu}_1| &\leqq \left|\frac{\alpha_n(\mu_2)}{\beta_n(\mu_2)} - \frac{\alpha_n(\mu_1)}{\beta_n(\mu_2)}\right| + \left|\frac{\alpha_n(\mu_1)}{\beta_n(\mu_2)} - \frac{\alpha_n(\mu_1)}{\beta_n(\mu_1)}\right| \\ &\leqq \frac{|\alpha_n(\mu_2) - \alpha_n(\mu_1)|}{|\beta_n(\mu_2)|} + \left|\frac{\alpha_n(\mu_1)}{\beta_n(\mu_1)}\right| \frac{|\beta_n(\mu_1) - \beta_n(\mu_2)|}{|\beta_n(\mu_2)|}.\end{aligned}$$

According to i) $\left|\frac{\alpha_n(\mu_1)}{\beta_n(\mu_1)}\right| \leqq s \leqq 1$ holds and therefore

$$|\hat{\mu}_2 - \hat{\mu}_1| \leqq \frac{\Lambda^+ + |a_1^{(n)}| + |a_2^{(n)}|}{|\beta_n(\mu_2)|} |\mu_2 - \mu_1| \leqq \frac{2 + \Lambda^+}{2 + \Lambda^-} |\mu_2 - \mu_1|. \quad \square$$

b) To each x_n with $n \leqq -N(s)$ we associate a sequence of sectors of directions. Each sector of directions is represented by a segment $G_n^{(k)}(s)$ $k = 0, 1, 2, \ldots$ on the straight line G_n. We define by induction with respect to k

$$G_n^{(0)}(s) := G_n(s) = \{x \mid x = \mu E_n^+ + E_n^-, |\mu| \leqq s\}$$

$$G_n^{(k)}(s) := \tilde{F}_{n-1}(G_{n-1}^{(k-1)}(s)).$$

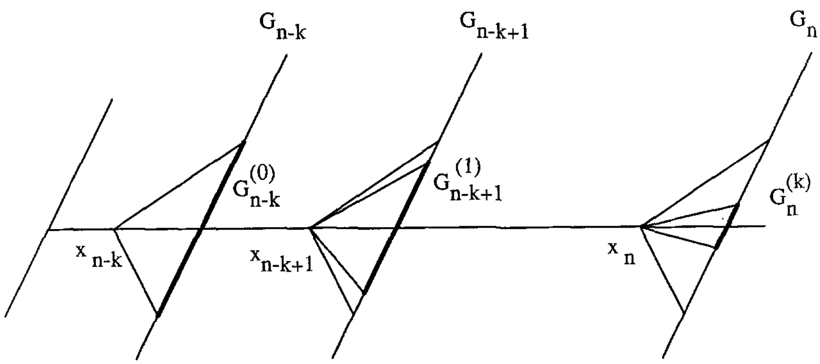

Figure 1.4

Lemma 1.3. For all $s \in (0, 1)$, $k \geqq 0$ and $n \leqq -N(s)$ the following holds:

i) $G_n^{(k)}(s)$ is compact and non-empty.
ii) $G_n^{(k+1)}(s) \subset G_n^{(k)}(s)$.
iii) For the length of $G_n^{(k)}(s)$ the estimate $|G_n^{(k)}(s)| \leqq 2q^k s$ holds with $q < 1$.
iv) If $s < s'$ then $G_n^{(k)}(s) \subset G_n^{(k)}(s')$.

Proof. The proof is by induction. The assertions are true for $k = 0$:

i) By definition $G_n^{(0)}(s) = G_n(s)$ is compact and non-empty.
ii) $G_n^{(1)}(s) := \tilde{F}_{n-1}(G_{n-1}(s)) \subset G_n(s) =: G_n^{(0)}(s)$ holds by Lemma 1.2.
iii) Length of $G_n(s) = 2s$.
iv) $G_n^{(0)}(s) = G_n(s) \subset G_n(s') = G_n^{(0)}(s')$ for $s < s'$.

Let the lemma be proven up to $k - 1$. Then

i) $G_n^{(k)}(s) = \tilde{F}_{n-1}(G_{n-1}^{(k-1)}(s))$ is the image of a compact, non-empty set and thus is compact and non-empty.

ii) $G_n^{(k+1)}(s) = \tilde{F}_{n-1}(G_{n-1}^{(k)}(s) \subset \tilde{F}_{n-1}(G_{n-1}^{(k-1)}(s)) = G_n^{(k)}(s)$.

iii) By Lemma 1.2 we have:
length of $G_n^{(k)}(s) \leqq q \cdot$ length of $G_{n-1}^{(k-1)}(s) \leqq q \cdot 2q^{k-1}s$.

iv) $G_n^{(k)}(s) = \tilde{F}_{n-1}(G_{n-1}^{(k-1)}(s)) \subset \tilde{F}_{n-1}(G_{n-1}^{(k-1)}(s')) = G_n^{(k)}(s')$. □

c) By Lemma 1.3 the sets $\{P_n(s)\} := \bigcap_{k=0}^{\infty} G_n^{(k)}(s)$ are compact and consist of one single point $P_n(s)$.

Lemma 1.4. i) If $n \leqq \min(-N(s), -N(s'))$ then $P_n(s) = P_n(s')$.

ii) If $n \leqq -N(s)$ then $P_n(s) = \tilde{F}_{n-1}(P_{n-1}(s))$.

Proof. i) is a consequence of Lemma 1.3 assertion iv).

ii) By definition of $G_n^{(k)}(s)$ we have

$$\tilde{F}_{n-1}(P_{n-1}(s)) = \tilde{F}_{n-1}\left(\bigcap_{k=0}^{\infty} G_{n-1}^{(k)}(s)\right) \subset \bigcap_{k=0}^{\infty} \tilde{F}_{n-1}(G_{n-1}^{(k)}(s))$$

$$= \bigcap_{k=0}^{\infty} G_n^{(k+1)}(s) = \{P_n(s)\}. \quad \square$$

d) We prove i) and ii) of Proposition 1.1. To this end we define for $n \leqq -N(1)$

$$e_n^- := \frac{P_n(1)}{|P_n(1)|}$$

where the vectors $P_n(1)$ are defined in section c). To satisfy claim i) of Proposition 1.1 the vectors e_n^- are uniquely determined for $n > -N(1)$ (up to a factor ± 1) in an obvious way. We now check i) for $n \leqq -N(1)$. It suffices to show that $Df_{n-1}(x_{n-1})$ takes the direction of e_{n-1}^- to the direction of e_n^-. This is equivalent to

$$\varphi_n(e_n^-) = \varphi_n(Df_{n-1}(x_{n-1})\, e_{n-1}^-)$$

or

$$P_n(1) = \tilde{F}_{n-1}(P_{n-1}(1)).$$

By Lemma 1.4 this is true for $n \leqq -N(1)$ what proves i) of Proposition 1.1. To show ii) we choose a monotoniquely decreasing sequence (s_j) with $s_j \to 0$ for $j \to \infty$. According to Lemma 1.2 there is a sequence $(N(s_j))$ with $N(s_1) \leqq N(s_2) \leqq N(s_3) \leqq \cdots$ and by Lemma 1.4 for $n \leqq -N(s_j)$

$$\varphi_n(e_n^-) = P_n(1) = P_n(s_j) \in G_n(s_j)$$

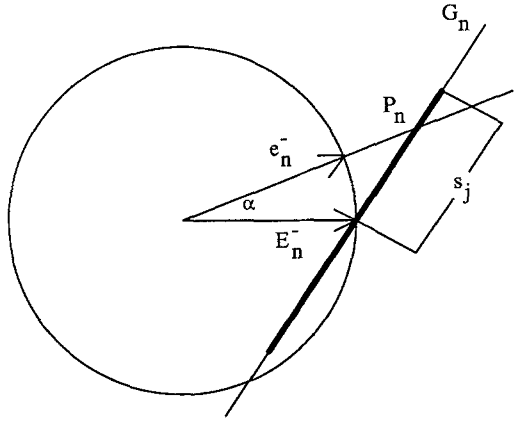

Figure 1.5

holds. Let α be the angle between e_n^- and E_n^-. Then $\sin\alpha \leqq s_j$ and (see Fig. 1.5)

$$|e_n^- - E_n^-| \leqq 2\sin\frac{\alpha}{2} = \sqrt{2(1-\cos\alpha)} = \sqrt{2\frac{\sin^2\alpha}{1+\cos\alpha}} \leqq \sqrt{2}\sin\alpha \leqq \sqrt{2}\cdot s_j$$

for $n \leqq -N(s_j)$ proving ii) of Proposition 1.1. It remains to show the uniqueness of e_n^- (up to a factor ± 1). Assume that for all $n \in \boldsymbol{Z}$ there are given vectors $\hat{e}_n$ such that the assertion i) and ii) hold and that for some m we have $\hat{e}_m \neq \pm e_m^-$. By i) we have $\hat{e}_n \neq \pm e_n^-$ for all $n \in \boldsymbol{Z}$, in particular for $n = -N(1)$. By definition of e_n^- there is a k such that

$$\varphi_{-N(1)}(\hat{e}_{-N(1)}) \notin G^{(k)}_{-N(1)}(1).$$

By definition of the sets $G_n^{(k)}(1)$ we conclude

$$\varphi_{-N(1)-i}(\hat{e}_{-N(1)-i}) \notin G^{(k-i)}_{-N(1)-i}(1) \quad \text{for} \quad i = 1, 2, \ldots, k-1$$

and finally for $i = k$

$$\varphi_{-N(1)-k}(\hat{e}_{-N(1)-k}) \notin G^{(0)}_{-N(1)-k}(1) = G_{-N(1)-k}(1).$$

Lemma 1.2 implies that $\varphi_n(\hat{e}_n) \notin G_n(1)$ holds for all $n < -N(1) - k$. This contradicts ii) which implies uniqueness.

For e_n^+ assertions i) and ii) are proven analogously.

e) We prove assertion iii) for e_n^+, i.e. we again treat the case $n \to -\infty$. We have to show that $e_0^+ \neq \pm e_0^-$ implies $\lim_{n \to -\infty} |e_n^+ - E_n^+| = 0$. We consider the segments $H_n(s) := \{x \mid x = E_n^+ + \mu E_n^-, |\mu| \leqq s\}$ of the straight lines $H_n := H_n(\infty)$ and the corresponding maps

$$\psi_n\colon \boldsymbol{R}^2 - \{x \mid x = \mu E_n^-, \ \mu \in \boldsymbol{R}\} \to H_n$$

$$v = a E_n^+ + b E_n^- \mapsto \psi_n(v) = \frac{1}{a} v = E_n^+ + \frac{b}{a} E_n^-.$$

The composed map

$$\hat{F}_n := \psi_{n-1} \circ Df_{n-1}^{-1}(x_n) \colon H_n(s) \to H_{n-1}(s)$$

has analogous properties as the map $\tilde{F}_n$.

Lemma 1.5. For any $s \in (0, 1]$ there exist numbers $\hat{N}(s)$ and $\hat{q} \in (0, 1)$ such that for all $n \leqq -\hat{N}(s)$ the following holds:

i) $\hat{F}_n(H_n(s)) \subset H_{n-1}(s)$
ii) $\hat{F}_n|_{H_n(s)}$ is contracting with contraction factor $\hat{q}$.

Proof. The proof is similar to the proof of Lemma 1.2. □

To prove iii) we show that for any given $s > 0$ the inclusion $\psi_n(e_n^+) \in H_n(s)$ holds provided n is a sufficiently large negative number. By assumption $e_n^+ \neq \pm e_n^-$ holds for all $n \in \mathbf{Z}$, in particular for $n = -N(1)$. In section d) we have shown that there is a k such that $\varphi_n(e_n^+) \notin G_n(1)$ holds for $n \leqq -N(1) - k$. This implies $\psi_n(e_n^+) \in H_n(1)$.

Let $M := \max\left(\hat{N}\left(\frac{s}{2}\right), N(1) + k\right)$. For $n \leqq -M$ we define a sequence (Q_n) of points $Q_n \in H_n\left(\frac{s}{2}\right)$ by

$$Q_{-M} := E_{-M}^+ \in H_{-M}\left(\frac{s}{2}\right)$$

$$Q_{n-1} := \hat{F}_n(Q_n)$$

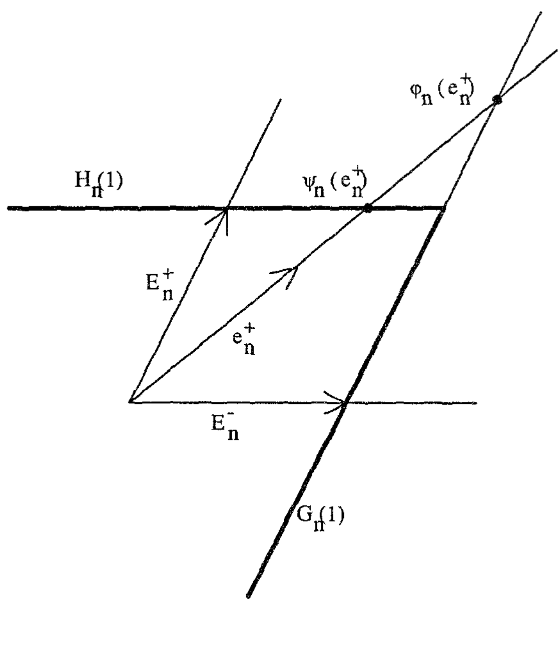

Figure 1.6

Lemma 1.5 implies $Q_n \in H_n\left(\frac{s}{2}\right)$ for all $n < -M$. Since $\psi_{-M}(e^{+}_{-M}) \in H_{-M}(1)$ we have $|\psi_{-M}(e^{+}_{-M} - Q_{-M})| \leqq 1$ and Lemma 1.5 implies $|\psi_{-M-j}(e^{+}_{-M-j}) - Q_{-M-j}| \leqq \hat{q}^j$ for all $j \geqq 0$. We choose m so large that $\hat{q}^m \leqq \frac{s}{2}$ holds. The for $n \leqq -M-m$ we have

$$|\psi_n(e_n^+) - E_n^+| \leqq |\psi_n(e_n^+) - Q_n| + |Q_n - E_n^+| \leqq s$$

or $\psi_n(e_n^+) \in H_n(s)$ which proves assertion iii)

f) Now we prove iv). E_n^+ and E_n^- being linearly independent we put $e_n^- = u_n E_n^+ + v_n E_n^-$. From (ii), (iii) follows that there is an N_0 such that

$$\varphi_n(e_n^-) = P_n(1) = \frac{1}{v_n} e_n^- = \frac{u_n}{v_n} E_n^+ + E_n^- \in G_n(1)$$

holds for all n with $|n| \geqq N_0$. On one hand i) implies

$$v_{n+1} P_{n+1}(1) = e_{n+1}^- = \frac{1}{t_n^-} Df_n(x_n)\, e_n^- = \frac{v_n}{t_n^-} Df_n(x_n)\, P_n(1)$$

and on the other we have from $P_{n+1}(1) = \tilde{F}_n(P_n(1))$ and section a)

$$\begin{aligned} P_{n+1}(1) &= \varphi_{n+1}(Df_n(x_n)\, P_n(1)) \\ &= \frac{1}{\Lambda_n^- + b_2^{(n)} + \mu a_2^{(n)}} Df_n(x_n)\, P_n(1) \quad \text{where} \quad \mu = \frac{u_n}{v_n} \in [-1, 1]. \end{aligned}$$

Comparing these two last formulas one gets

$$t_n^- = (\Lambda_n^- + b_2^{(n)} + \mu a_2^{(n)}) \frac{v_n}{v_{n+1}}$$

and according to section a)

$$\frac{2 + \Lambda^-}{3} \leqq |\Lambda_n^- + b_2^{(n)} + \mu a_2^{(n)}| \leqq |\Lambda^-| + \frac{2}{3}|\Lambda^- - 1| \leqq c_0$$

holds for $|n| \geqq N_0$ and some constant c_0 (Note that the considerations of section a) apply not only for $n < -N_0$ but for $n > N_0$ as well). Now we choose $\delta \in (0, 1)$ such that

$$\theta := \frac{3}{2 + \Lambda^-} \cdot \frac{1 + \delta}{1 - \delta} < 1\,.$$

Since $\lim_{n \to \pm\infty} v_n = 1$ holds there is an $N > N_0$ such that $|v_n - 1| \leqq \delta$ holds for $|n| \geqq N$. This implies the estimate to be proven

$$1 < \frac{1}{\theta} = \frac{2 + \Lambda^-}{3} \cdot \frac{1 - \delta}{1 + \delta} \leqq |t_n^-| \leqq c_0 \frac{1 + \delta}{1 - \delta} =: \tau\,.$$

Analogously one shows that

$$\frac{1}{\tau} \leqq 2|t_n^+| \leqq \theta < 1$$

for $|n| > N$, N sufficiently large.

This completes the proof of Proposition 1.1. □

Remark. In view of the considerations of infinitely many homoclinic orbits a slightly different formulation of the results in f) is needed. The estimates

$$\tau \geqq |t_n^-| \geqq \frac{1}{\theta} > 1$$

$$\frac{1}{\tau} \leqq |t_n^+| \leqq \theta < 1$$

hold if $|x_n - z_n|$, $|e_n^- - E_n^-|$ and $|e_n^+ - E_n^+|$ are sufficiently small.

Next we modify the length of finitely many vectors $e_n^\pm$ such that assertion iv) of Proposition 1.1 holds for *all* $n \in \mathbf{Z}$ (with modified θ and τ). We show how to do it for e_n^+. Let $Q(n) := \prod_{i=-n}^{n} |t_i^+|$ and N be the number of assertion iv). Then there is an $M \geqq N$ such that $Q(M) = \prod_{i=-M}^{M} |t_i^+| \leqq Q(N) \cdot \theta^{2(M-N)} < 1$. Choose θ^+ such that $(\theta^+)^{2M+1} = Q(M)$. We define the numbers d_n^+ by

$$d_n^+ := 1 \qquad \text{for } n \leqq -M \text{ and for } n > M$$

$$d_{n+1}^+ := \frac{t_n^+ d_n^+}{\theta^+} \qquad \text{for } -M \leqq n < M$$

and the vectors h_n^+ by

$$h_n^+ := d_n^+ e_n^+ \qquad \text{for } n \in \mathbf{Z}$$

i) of Proposition 1.1 implies

$$h_{n+1}^+ = \frac{1}{\lambda_n^+} Df_n(x_n) h_n^+ \quad \text{where} \quad \lambda_n^+ := t_n^+ \frac{d_n^+}{d_{n+1}^+}$$

and it easy to verify that

$$\lambda_n^+ = t_n^+ \quad \text{for } |n| > M$$

$$\lambda_n^+ = \theta^+ \quad \text{for } |n| \leqq M$$

holds. The scaling can be done analogously for the vectors e_n^-. Summarizing the above statements we get the following lemma:

Lemma 1.6. There are $\hat{\theta}, \hat{\tau} > 0$ such that

$$\frac{1}{\hat{\tau}} \leqq |\lambda_n^+| \leqq \hat{\theta} < 1$$

$$\frac{1}{\hat{\theta}} \leqq |\lambda_n^-| \leqq \hat{\tau}$$

holds for all $n \in \boldsymbol{Z}$.

Now we introduce local coordinates. To each point $\xi_n = (n, x_n)$ we associate the matrix T_n with the column vectors h_n^+ and h_n^-. For the points $\xi = (n, x)$ and $f(\xi) = (n+1, f_n(x))$ we introduce local coordinates (n, u_n) and $(n+1, u_{n+1})$ by

$$x = x_n + T_n u_n, \quad f_n(x) = x_{n+1} + T_{n+1} u_{n+1}.$$

By Assumption A2) we can write for f_n

$$f_n(x) = f_n(x_n) + Df_n(x_n) T_n u_n + O(|u_n|^2)$$

leading to

$$u_{n+1} = T_{n+1}^{-1} Df_n(x_n) T_n u_n + O(|u_n|^2).$$

By i) of Proposition 1.1 and Lemma 1.6 we conclude that

$$T_{n+1}^{-1} Df_n(x_n) T_n = \begin{pmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{pmatrix}$$

is a diagonal matrix with $|\lambda_n^+| \leqq \hat{\theta} < 1 < \frac{1}{\hat{\theta}} \leqq |\lambda_n^-|$. This shows that if the vectors e_n^+ and e_n^- are linearly independent the homoclinic orbit $X = \{(n, x_n)\}$ carries a hyperbolic structure. Our considerations suggest the following definitions:

Definition 1.7. *A homoclinic point $\xi_n = (n, x_n)$ is called transversal if e_n^+ and e_n^- are linearly independent. A homoclinic orbit $X = \{\xi_n \mid \xi_n = (n, x_n),\ n \in \boldsymbol{Z}\}$ is called transversal if its elements are transversal homoclinic points.*

Definition 1.8. *A set $\Lambda \subset \boldsymbol{Z} \times \boldsymbol{R}^2$ is called hyperbolic if there are constants $\theta \in (0, 1)$ and $\tau > 1$ and if for every point $\xi = (n, x) \in \Lambda$ there is a regular matrix $T(\xi) = T(n, x)$ such that*

H1) *$\Lambda = f(\Lambda)$ is invariant.*
H2) *The sections $\Lambda_n := \{(k, x) \mid (k, x) \in \Lambda,\ k = n\}$ are compact for all $n \in \boldsymbol{Z}$.*

H3) *The matrices* $A(\xi) := T(\eta)^{-1} Df_n(x) T(\xi)$ *with* $\eta = f(\xi)$ *are diagonal with diagonal elements* $\lambda^+(\xi)$, $\lambda^-(\xi)$ *where*

$$\frac{1}{\tau} \leqq |\lambda^+(\xi)| \leqq \theta$$

$$\frac{1}{\theta} \leqq |\lambda^-(\xi)| \leqq \tau$$

holds.

H4) $T(\xi)$ *and* $T(\xi)^{-1}$ *are bounded by* τ.

H5) *The map* $\xi \in \Lambda \mapsto T(\xi)^{-1} \in GL(2, 2)$ *is uniformly continuous, i.e. there is a function* $\delta(\varepsilon)$ *with* $\delta(\varepsilon) \to 0$ *for* $\varepsilon \to 0$ *such that* $|\xi - \eta| < \varepsilon$ *implies* $|T(\xi)^{-1} - T(\eta)^{-1}| < \delta(\varepsilon)$.

It is easy to check that the hyperbolic orbit $Z = \{(\xi_n)\}$ and the union $Z \cup X = \{(\zeta_n)\} \cup \{(\xi_n)\}$ of the hyperbolic orbit Z with the transversal homoclinic orbit X are hyperbolic sets. As the following paragraphs will show the concept of hyperbolic sets becomes interesting if the considered set contains infinitely many transversal homoclinic orbits.

We want to find other equivalent formulations for the transversality condition of Definition 1.7 which will be easier to verify. To this end we introduce the space $\boldsymbol{X}$ of bounded sequences $x = (x_n)$ where $n \in \boldsymbol{Z}$. Endowing $\boldsymbol{X}$ the norm

$$\|x\| = \sup_{n \in \boldsymbol{Z}} |x_n|$$

where $|x_n|$ is just any norm in $\boldsymbol{R}^2$ (e.g. the Euclidean norm) the space $\boldsymbol{X}$ becomes a Banach space. Let $F: \boldsymbol{X} \to \boldsymbol{X}$ be the map defined by

$$(F(x))_n := x_{n+1} - f_n(x_n).$$

It is obvious that the set $X = \{(n, x_n) \mid n \in \boldsymbol{Z}\}$ is a bounded orbit iff $x = (x_n) \in \boldsymbol{X}$ and $F(x) = 0$ holds.

Proposition 1.9. If $X = \{(n, x_n)\}$ is a homoclinic orbit to the hyperbolic orbit $Z = \{(n, z_n)\}$ then the following statements are equivalent.

i) X is a transversal homoclinic orbit.

ii) The difference equation $u_{n+1} = Df_n(x_n) u_n$ has no non-trivial bounded solution.

iii) The operator $L: \boldsymbol{X} \to \boldsymbol{X}$ defined by $(Lu)_n := u_{n+1} - Df_n(x_n) u_n$ is invertible.

Proof. i) $\Rightarrow$ ii). As we have shown so far one can diagonalize the matrices $Df_n(x_n)$ by the transformations $u_n = T_n v_n$. The difference equation in assertion ii) becomes $v_{n+1} = A_n v_n$ where $A_n = \operatorname{diag}(\lambda_n^+, \lambda_n^-)$ with $|\lambda_n^+| \leqq \theta < 1 < \frac{1}{\theta} \leqq |\lambda_n^-|$. With the Ansatz $v_n = \begin{pmatrix} r_n \\ s_n \end{pmatrix}$ one gets $|s_n| \geqq \frac{|s_0|}{\theta^n}$ for $n > 0$ and $|r_n| \geqq \theta^n |r_0|$ for

$n < 0$. Thus if the sequence (v_n) is bounded then $r_0 = s_0 = 0$ holds and therefore $v_n = 0$ for all $n \in \boldsymbol{Z}$.

ii) $\Rightarrow$ i). We prove the contraposition. Assume that X is not transversal. Then $e_0^+ = \pm e_0^-$. Putting $u_0 := e_0^+$ we get by Proposition 1.1 that

$$u_n = \left(\prod_{i=0}^{n-1} t_i^+\right) e_n^+ \qquad \text{for } n > 0$$

$$u_n = \pm \left(\prod_{i=n}^{-1} (t_i^-)^{-1}\right) e_n^- \quad \text{for } n < 0$$

is a solution of the difference equation ii). To see that (u_n) is a bounded solution we notice that there is an N such that

$$|t_n^+| \leqq \theta < 1 \quad \text{for} \quad n > N$$

$$|t_n^-| \geqq \frac{1}{\theta} > 1 \quad \text{for} \quad n < -N$$

holds. If e_0^+ and e_0^- were linearly independent the above estimates would follow from Proposition 1.1 part iv). The weaker statement needed above holds even if e_0^+ and e_0^- are linearly dependent, in view of Proposition 1.1 ii) and since $|x_n - z_n| \to 0$ for $|n| \to \infty$.

ii) $\Rightarrow$ iii). By ii) the equation $Lu = 0$ implies $u = 0$. Since L is linear this implies that L is injective. Again by the transformation $u_n = T_n v_n$ we get $v_{n+1} = A_n v_n$ where $A_n = \operatorname{diag}(\lambda_n^+, \lambda_n^-)$. L is invertible iff the transformed operator L' is invertible. L' is given by

$$(L' v)_n = v_{n+1} - A_n v_n = w_n = \begin{pmatrix} w_n^+ \\ w_n^- \end{pmatrix}.$$

For any $w \in \boldsymbol{X}$ one simply computes the solution $v = (v_n) = \left(\begin{pmatrix} r_n \\ s_n \end{pmatrix}\right)$ of $L' v = w$ to be

$$r_n = w_{n-1}^+ + \lambda_{n-1}^+ w_{n-2}^+ + \lambda_{n-1}^+ \lambda_{n-2}^+ w_{n-3}^+ + \cdots$$

$$s_n = -\frac{1}{\lambda_n^-} w_n^- - \frac{1}{\lambda_n^- \lambda_{n+1}^-} w_{n+1}^- - \frac{1}{\lambda_n^- \lambda_{n+1}^- \lambda_{n+2}^-} w_{n+2}^- - \cdots.$$

Thus L' is surjective and

$$\|L'^{-1} w\| \leqq \frac{1}{1-\theta} |w|$$

holds implying that L^{-1} is bounded.

iii) $\Rightarrow$ ii) is trivial. □

Now we consider the case where there are infinitely many transversal homoclinic orbits to the hyperbolic orbit.

Theorem 1.10. *Let Assumptions* A1), A2) *and* A3) *hold. If for every $j \in Z$ there is a number m_j and a homoclinic orbit*

$$X^{(j)} = \{\xi_n^{(j)} \mid \xi_n^{(j)} = (n, x_n^{(j)}), n \in \boldsymbol{Z}\} \text{ to } Z = \{\zeta_n \mid \zeta_n = (n, z_n), n \in \boldsymbol{Z}\}$$

such that

a) *For every $\sigma > 0$ there is a $\delta > 0$ such that $x_n^{(i)} \notin U_\sigma(z^n)$ implies $|x_n^{(i)} - x_n^{(j)}| \geqq \delta$ for every $j \in \boldsymbol{Z} - \{i\}$. $U_\sigma(z_n)$ denotes the σ-neighborhood of z_n.*

b) *For all $\sigma > 0$ there exists a number $N(\sigma)$ such that $|x_n^{(j)} - z_n| < \sigma$ holds for all n with $|n - m_j| > N(\sigma)$. (Note that $N(\sigma)$ is independent of j).*

c) *The linear operators $L_j\colon \boldsymbol{X} \to \boldsymbol{X}$ defined by*

$$(L_j u)_n = u_{n+1} - Df_n(x_n^{(j)})\, u_n$$

are invertible and there exists a constant c_0 with $\| L_j^{-1} \| \leqq c_0$.

Then the set $\Lambda := Z \cup \bigcup_{j \in \boldsymbol{Z}} X^{(j)}$ is hyperbolic.

Remarks. 1) Condition a) means that the only limit point of $\{x_n^{(j)} \mid j \in \boldsymbol{Z}\}$ is z_n.

2) Condition b) means that the orbits $X^{(j)}$ stay σ-close to the hyperbolic orbit Z with $2N(\sigma) + 1$ exceptional points.

3) Condition c) implies by means of Proposition 1.9 that the homoclinic orbits $X^{(j)}$ are transversal.

4) In the case where f is 1-periodic in time, i.e. all f_n are identical, the theorem follows immediately from Proposition 1.1, Proposition 1.9 and Lemma 1.6.

Proof: According to Proposition 1.1 for every orbit $X^{(j)} = \{(n, x_n^{(j)})\}$ there are unit vectors $e_n^{(j)+}$ and $e_n^{(j)-}$ and numbers $t_n^{(j)+}$ and $t_n^{(j)-}$ with the properties i)–iv) of Proposition 1.1. Now we want to rescale these vectors in a uniform manner such that Lemma 1.6 holds for all orbits $X^{(j)}$. As in the proof of Proposition 1.1, Sect. a), we consider the maps

$$\tilde{F}_n(x, \cdot) := \varphi_{n+1} \circ Df_n(x)\colon G_n(1) \to G_{n+1}$$

and analogously the maps

$$\bar{F}_n(x, \cdot) := \psi_{n+1} \circ Df_n(x)\colon H_n(1) \to H_{n+1}.$$

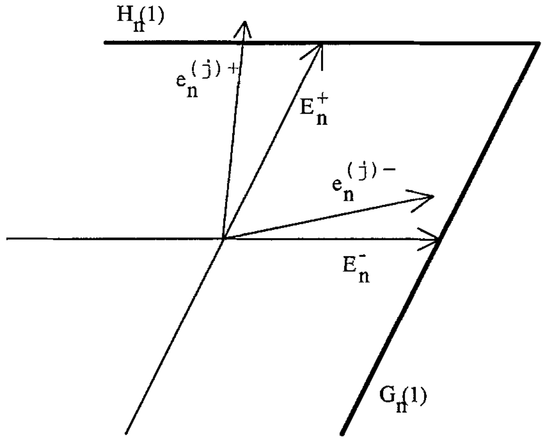

Figure 1.7

By the estimates of Sect. a) in the proof of Proposition 1.1 we know that if x is sufficiently close to z_n then the map $\tilde{F}_n(x, \cdot)$ is defined on the segment $G_n(1)$ and is contracting with contraction factor $\tilde{\theta} < 1$ and $\tilde{\theta}$ is independent of n. Analogously the map $\tilde{F}_n(x, \cdot)$ is defined on the segment $H_n(1)$ and is expanding with expansion factor $\frac{1}{\tilde{\theta}} > 1$. By Proposition 1.1 we have

$$t_n^{(j)\pm} e_{n+1}^{(j)\pm} = Df_n(x_n^{(j)}) e_n^{(j)\pm}.$$

By the remark following the proof of Proposition 1.1 the following holds: If $|x_n^{(j)} - z_n|$, $|e_n^{(j)+} - E_n^+|$ and $|e_n^{(j)-} - E_n^-|$ are sufficiently small then the estimates

$$\frac{1}{\bar{\tau}} \leqq |t_n^{(j)+}| \leqq \bar{\theta} < 1, \quad \bar{\tau} \geqq |t_n^{(j)-}| \geqq \frac{1}{\bar{\theta}} > 1$$

hold. By assumption b) there is an N_1 such that for all orbits $X^{(j)} = \{(n, x_n^{(j)})\}$ the point $x_n^{(j)}$ is a close to z_n as we wish, provided $|n - m_j| > N_1$ holds. Now we prove the following lemma:

Lemma 1.11. For every $\delta > 0$ there is an N_2 such that for every $j \in \mathbf{Z}$ and for every n with $|n - m_j| > N_2$ the estimates

$$|e_n^{(j)+} - E_n^+| < \delta$$

$$|e_n^{(j)-} - E_n^-| < \delta$$

hold.

Proof. For $w = (w_k)$ with

$$w_k := \begin{cases} e_n^{(j)+} \pm e_n^{(j)-} & \text{for } k = n-1 \\ 0 & \text{for } k \neq n-1 \end{cases}$$

the equation $L_j u = w$ has the solution $u = (u_k)$ with

$$\begin{aligned}
&\dots\\
u_{n-3} &= \pm \frac{-1}{t_{n-1}^{(j)-}\, t_{n-2}^{(j)-}\, t_{n-3}^{(j)-}}\, e_{n-3}^{(j)-}\\
u_{n-2} &= \pm \frac{-1}{t_{n-1}^{(j)-}\, t_{n-2}^{(j)-}}\, e_{n-2}^{(j)-}\\
u_{n-1} &= \pm \frac{-1}{t_{n-1}^{(j)-}}\, e_{n-1}^{(j)-}\\
u_n &= e_n^{(j)+}\\
u_{n+1} &= t_n^{(j)+}\, e_{n+1}^{(j)+}\\
u_{n+2} &= t_n^{(j)+}\, t_{n+1}^{(j)+}\, e_{n+2}^{(j)+}\\
&\dots .
\end{aligned}$$

Assumption c) yields

$$|e_n^{(j)+} \pm e_n^{(j)-}| = |w| \geqq \frac{1}{\|L_j^{-1}\|}\,|u| \geqq \frac{1}{c_0}\,|u_n| = \frac{1}{c_0}.$$

This inequality shows that $e_n^{(j)+}$ and $\pm e_n^{(j)-}$ have a minimal distance independent of n and j. Constructing $e_n^{(j)-}$ we have seen that $\varphi_n(e_n^{(j)-}) \in G_n(1)$ holds for all $n \leqq m_j - N_1$. Analogously $\psi_n(e_n^{(j)+}) \in H_n(1)$ holds for all $n \geqq m_j + N_1$. We show that there is an $N' \geqq N_1$ such that for all $j \in \boldsymbol{Z}$ and all n with $|n - m_j| \geqq N'$

$$\varphi_n(e_n^{(j)-}) \in G_n(1)$$
$$\psi_n(e_n^{(j)+}) \in H_n(1).$$

Assume that $\varphi_n(e_n^{(j)-}) \notin G_n(1)$ for some j and $n = m_j + N_1$. $\psi_n(e_n^{(j)+})$ and $\psi_n(e_n^{(j)-})$ have a minimal distance independent of n and j as follows from the previous estimate on $|e_n^{(j)+} \pm e_n^{(j)-}|$. $\bar{F}_n(x_n^{(j)}, \cdot)$ is expanding with expansion factor $\frac{1}{\tilde{\theta}}$ for $n \geqq m_j + N_1$. Thus there is an $N' > N_1$ independent of j such that for some $k \in [m_j + N_1, m_j + N']$ we have $\psi_k(e_k^{(j)-}) \notin H_k(1)$. By definition of $H_k(1)$ and $G_k(1)$ we therefore conclude $\varphi_k(e_k^{(j)-}) \in G_k(1)$ cf. Fig. 1.7. Since $G_n(1)$ *is* invariant under $\tilde{F}_n(x_n^{(j)}, \cdot)$ we have $\varphi_n(e_n^{(j)-}) \in G_n(1)$ for all $n \geqq k$ and in particular $n \geqq m_j + N'$. The claim for $e_n^{(j)+}$ is proven analogously.

Given $\delta > 0$ we show that there is an $N'' > N'$ such that $|\varphi_n(e_n^{(j)-}) - E_n^-| < \delta$ holds for all $n \geqq m_j + N''$.

There is an $N\left(\frac{\delta}{2}\right)$ such that $\tilde{F}_n(x_n^{(j)}, \cdot)$ maps $G_n\left(\frac{\delta}{2}\right)$ into $G_n\left(\frac{\delta}{2}\right)$ for all j and all $n \geqq m_j + N\left(\frac{\delta}{2}\right)$. This follows from assumption b). Put $M := \max\left(N\left(\frac{\delta}{2}\right), N'\right)$.

For every j we define a sequence of points $Q_n^{(j)}$, $n \geqq m_j + M$ by

$$Q_{m_j+M}^{(j)} := E_{m_j+M}^{-} \in G_{m_j+M}\left(\frac{\delta}{2}\right)$$

$$Q_{n+1}^{(j)} := \tilde{F}_n(x_n^{(j)}, Q_n^{(j)}) \quad \text{for } n \geqq m_j + M.$$

It follows $Q_n^{(j)} \in G_n\left(\frac{\delta}{2}\right)$ for all $n \geqq m_j + M$. For $n = m_j + M$ we have $|\varphi_n(e_n^{(j)-}) - E_n^-| < 1$. Since $\tilde{F}_n(x_n^{(j)}, \cdot)$ are contractions with contraction factor $\tilde{\theta}$ there is an N'' independent of n and j such that $|\varphi_n(e_n^{(j)-}) - Q_n^{(j)}| \leqq \frac{\delta}{2}$ for $n \geqq m_j + N''$ and therefore

$$|\varphi_n(e_n^{(j)-}) - E_n^-| \leqq |\varphi_n(e_n^{(j)-}) - Q_n^{(j)}| + |Q_n^{(j)} - E_n^-| \leqq \delta$$

Similarly one shows that $|\varphi_n(e_n^{(j)-}) - E_n^-| < \delta$ holds for all $n \leqq m_j - N''$. Now the lemma follows immediatly from Fig. 1.5. □

Thus by the remark after the proof of Proposition 1.1 there is an N_2 such that the estimates

$$\frac{1}{\bar{\tau}} \leqq |t_n^{(j)+}| \leqq \bar{\theta} < 1 \quad \text{and} \quad \bar{\tau} \geqq |t_n^{(j)-}| \geqq \frac{1}{\bar{\theta}} > 1 \tag{4}$$

hold for $|n - m_j| > N_2$. Now we build the products

$$Q_j^{\pm}(N) := \prod_{i=-N}^{N} |t_{m_j+i}^{(j)\pm}|.$$

Since the derivatives of f and f^{-1} are bounded and regular there are bounds S and T independent of j such that the estimates

$$\frac{1}{T} < Q_j^{+}(N_2) < S \quad \text{and} \quad T > Q_j^{-}(N_2) > \frac{1}{S}$$

hold. According to Eq. (4) there is a number $M \geqq N_2$ independent of j such that

$$\frac{1}{T'} < Q_j^{+}(M) < \frac{1}{2} \quad \text{and} \quad T' > Q_j^{-}(M) > 2$$

holds for some T'.

Now we proceed as in the derivation of Lemma 1.6. We choose $\bar{\theta}^{(j)\pm}$ and $\bar{\tau}^{(j)\pm}$ such that $(\bar{\theta}^{(j)\pm})^{2M+1} = Q_j^{\pm}(M)$ and put

$$d_n^{(j)\pm} := 1 \qquad \text{for } n \leqq m_j - M \text{ and for } n > m_j + M$$

$$d_{n+1}^{(j)\pm} := \frac{t_n^{(j)\pm} d_n^{(j)\pm}}{\bar{\theta}^{(j)\pm}} \quad \text{for } m_j - M \leqq n < m_j + M$$

and

$$h_n^{(j)\pm} := d_n^{(j)\pm} e_n^{(j)\pm} \quad \text{for } n \in \mathbf{Z}.$$

For the vectors $h_n^{(j)\pm}$ the relations

$$\lambda_n^{(j)\pm} h_{n+1}^{(j)\pm} = Df_n(x_n^{(j)}) h_n^{(j)\pm} \tag{5}$$

hold with $\lambda_n^{(j)\pm} = t_n^{(j)\pm} \frac{d_n^{(j)\pm}}{d_{n+1}^{(j)\pm}}$. The estimates

$$\begin{aligned} \frac{1}{\tau} &\leqq |\lambda_n^{(j)+}| < \theta \\ \tau &\geqq |\lambda_n^{(j)-}| > \frac{1}{\theta} \end{aligned} \tag{6}$$

with $\theta = \max\{\tilde{\theta}, \bar{\theta}\}$ and $\tau = \max\left\{\tilde{\tau}, \frac{1}{\bar{\theta}}\right\}$ are easy to verify. We now are at the point where we can show that Λ is a hyperbolic set. We define

$$T(n, x_n^{(j)}) := (h_n^{(j)+}, h_n^{(j)-})$$

to be the matrix consisting of the column vectors $h_n^{(j)+}$ and $h_n^{(j)-}$.

H1) Λ is invariant since it is the union of invariant sets.
H2) The sections $\Lambda_n = \{z_n\} \cup \{x_n^{(j)} \mid j \in \mathbf{Z}\}$ are compact since by assumption a) $x_n^{(j)} \to z_n$ holds for $j \to \pm\infty$.
H3) According to Eq. (5) and Eq. (6)

$$A(n, x_n^{(j)}) = T(n+1, y)^{-1} Df_n(x_n^{(j)}) T(n+1, x_n^{(j)}) = \begin{pmatrix} \lambda_n^{(j)+} & 0 \\ 0 & \lambda_n^{(j)-} \end{pmatrix}$$

holds with the estimates

$$\frac{1}{\tau} < |\lambda_n^{(j)+}| < \theta$$

$$\tau > |\lambda_n^{(j)-}| > \frac{1}{\theta}.$$

H4) Since $h_n^{(j)+}$ and $h_n^{(j)-}$ are bounded and bounded away from zero and since the angle between them is bounded from below the matrices $T(n, x)$ and $T(n, x)^{-1}$ are bounded.

H5) The angle between $h_n^{(j)+}$ and $h_n^{(j)-}$ being bounded from below it suffices to show that the vectors $h_n^{(j)\pm}$ are uniformly continuous. Since by Assumption A2) a neighbourhood of z_n (the width of which is independent of n) is hyperbolic there is a $\delta > 0$ such that $|x_{m_j}^{(j)} - z_{m_j}| > \delta$ holds for all $j \in \mathbf{Z}$. And therefore there is a δ' such that $|x_n^{(j)} - z_n| > \delta'$ holds for all n with $|n - m_j| \leqq M$.

This means that in the δ'-neighbourhood of z_n we have $h_n^{(j)\pm} = e_n^{(j)\pm}$. By Assumption a) the points outside these neighbourhoods have a minimal distance independent of j and n. Thus it suffices to show that the vectors $e_n^{(j)\pm}$ are uniformly continuous, i.e. that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x_n^{(j)} - x_n^{(i)}| < \delta$ implies $|e_n^{(j)\pm} - e_n^{(i)\pm}| < \varepsilon$. By Assumption a) it suffices to show that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|x_n^{(j)} - z_n| < \delta$ implies $|e_n^{(j)\pm} - E_n^{\pm}| < \varepsilon$. One can conclude that if $|x_n^{(j)} - z_n|$ is small then $|n - m_j|$ is large. Thus the last statement follows from the uniform contraction property of $\tilde{F}_n(x, \cdot): G_n(1) \to G_{n+1}(1)$ and the uniform expansion property of $\bar{F}_n(x, \cdot): H_n(1) \to H_{n+1}(1)$. □

2. Pseudo orbits and the shadowing lemma

In this paragraph we prove that the Shadowing Lemma holds for maps admitting a time dependent hyperbolic set. The proof follows essentially the ideas of Kirchgraber [2] where the Shadowing Lemma is proved for autonomous maps. Now we have to work in the $t - x$ space instead of the x-space only. All considerations can be carried over from one case to the other. For completeness we give the full proof of the Shadowing Lemma for the new situation.

We consider a map $f: \boldsymbol{Z} \times \boldsymbol{R}^2 \to \boldsymbol{Z} \times \boldsymbol{R}^2$ with $f(n, z) = (n + 1, f_n(z))$ admitting a hyperbolic set Λ (see Definition 1.8).

Definition 2.1. *A set $P = \{(n, p_n) \mid n \in \boldsymbol{Z}, p_n \in \boldsymbol{R}^2\}$ is called orbit if $p_{n+1} = f_n(p_n)$ holds for all $n \in \boldsymbol{Z}$.*

A set $Q = \{(n, q_n) \mid n \in \boldsymbol{Z}, q_n \in \boldsymbol{R}^2\} \subset \Lambda$ is called pseudo orbit.

A set $Q = \{(n, q_n) \mid n \in \boldsymbol{Z}, q_n \in \boldsymbol{R}^2\} \subset \Lambda$ is called ε-pseudo orbit if Q is a pseudo orbit and if additionally $|q_{n+1} - f_n(q_n)| \leqq \varepsilon$ holds for all $n \in \boldsymbol{Z}$.

A set $P = \{(n, p_n) \mid n \in \boldsymbol{Z}, p_n \in \boldsymbol{R}^2\}$ is called ϱ-shadowing orbit of the pseudo orbit Q if P is an orbit and if in addition $|p_n - q_n| < \varrho$ holds for all $n \in \boldsymbol{Z}$.

We give the set Ψ of all pseudo orbits a topology. To every pseudo orbit $Q = \{(n, q_n)\}$ we associate the unique sequence $(q_n) \in \prod_{n \in \boldsymbol{Z}} \Lambda_n$. For the set $\Omega := \prod_{n \in \boldsymbol{Z}} \Lambda_n$ we choose the product topology. The map which takes the sequence (q_n) to the pseudo orbit $Q = \{(n, q_n)\}$ induces our desired topology in Ψ. The sets

$$B_L := \{Q' = \{(n, q_n')\} \mid |q_n' - q_n| < \frac{1}{L} \text{ for all } n \text{ with } |n| < L\}$$

form a base of neighbourhoods of $Q = \{(n, q_n)\}$.

Proposition 2.2. i) The set Ψ of the pseudo orbits is compact.

ii) The set Ψ_ε of the ε-pseudo orbits is compact.

Proof. i) As product of compact sets the set Ω *is* compact. Hence the set Ψ is compact as well.

ii) The subset Ψ_ε of the compact space Ψ is compact iff it is closed. We show that the complement $C\,\Psi_\varepsilon$ of Ψ_ε is open. We have $q \in C\,\Psi_\varepsilon$ iff there is an $n \in \mathbf{Z}$ such that $|q_{n+1} - f_n(q_n)| = d > \varepsilon$ holds. Since f_n is continuous there is a $\delta > 0$ such that

$$|q'_{n+1} - q_{n+1}| < \delta \quad \text{and} \quad |q'_n - q_n| < \delta \tag{1}$$

implies $|q'_{n+1} - f_n(q'_n)| > \varepsilon$. The set of all pseudo orbits satisfying (1) form an open neighbourhood of q which is contained in $C\,\Psi_\varepsilon$. Thus $C\,\Psi_\varepsilon$ is open and therefore Ψ_ε is closed implying that Ψ_ε is compact. □

Theorem 2.3. (*Shadowing Lemma*). *Let assumptions* A1) *and* A2) *of Paragraph* 1 *be satisfied. If f admits a hyperbolic set Λ then there is a ϱ_0 such that for each ϱ with $0 < \varrho < \varrho_0$ there is an $\varepsilon > 0$ such that the following statement holds: For every ε-pseudo orbit $Q = \{(n, q_n)\}$ there is exactly one ϱ-shadowing orbit $P = \{(n, p_n)\}$ of Q.*

Proof. Let $Q = \{(n, q_n)\}$ be an ε-pseudo orbit. Since $(n, q_n) \in \Lambda$ holds there are matrices $T(n, q_n)$ such that statements H1)–H5) of Definition 1.8 hold. To simplify the notation we introduce the following abreviations:

$$\bar{q}_{n+1} := f_n(q_n) \quad T_n := T(n, q_n) \quad \bar{T}_{n+1} := T(n+1, \bar{q}_{n+1})$$
$$A_n := A(n, q_n) = \bar{T}_{n+1}^{-1} Df_n(q_n)\, T_n = \operatorname{diag}(\lambda_n^+, \lambda_n^-).$$

We introduce local coordinates u_n by $p_n = q_n + T_n u_n$. The set $P = \{(n, p_n)\}$ is a ϱ-shadowing orbit of $Q = \{(n, q_n)\}$ iff for all $n \in \mathbf{Z}$

$$|T_n u_n| = |p_n - q_n| \leqq \varrho \quad \text{and} \quad u_{n+1} = A_n u_n + g_n(T_n u_n)$$

holds where

$$g_n(x) = T_{n+1}^{-1}(\bar{q}_{n+1} - q_{n+1}) + (T_{n+1}^{-1} - \bar{T}_{n+1}^{-1}) Df_n(q_n)\, x + T_{n+1}^{-1} \hat{f}_n(q_n, x)$$

and $\hat{f}_n$ has the same meaning as in Paragraph 1, i.e.

$$f_n(x+y) = f_n(x) + Df_n(x)\, y + \hat{f}_n(x, y).$$

The functions $g_n(x)$ satisfy the following estimates

$$\begin{aligned} |g_n(x)| &\leqq \tau \cdot \varepsilon + \delta(\varepsilon) \cdot c \cdot \varrho + \tau \cdot c \cdot \varrho^2 \\ |D\, g_n(x)| &\leqq \delta(\varepsilon) \cdot c \cdot \tau + \tau^2 \cdot c \cdot \varrho \end{aligned} \tag{2}$$

for all x with $|x| \leqq \varrho$. The constants τ, $\delta(\varepsilon)$ and c have the following meaning: τ is an upper bound of $T(n, q_n)$ and $T(n, q_n)^{-1}$ and $\delta(\varepsilon)$ is the modulus of continuity of $T(n, q_n)^{-1}$. c is the constant of Assumption A2).

Let X be the space of bounded sequences $x = (x_n)$, $n \in \mathbf{Z}$. X is given the sup-norm and this way it becomes a Banach space. Let us define the following operators in X:

$$T: X \to X, \qquad (Tu)_n := T_n u_n.$$

$$G: B_\varrho := \{u \in X \mid |T_n u_n| < \varrho\} \to X, \quad (G(v))_n := g_n(v_n).$$

$$L: X \to X, \qquad (Lu)_n := u_{n+1} - A_n u_n.$$

There exists a unique ϱ-shadowing orbit iff there is a unique $u \in X$ with $\|Tu\| \leqq \varrho$ and $Lu = G(Tu)$ or iff there is a unique $v \in X$ with $\|v\| \leqq \varrho$ and $LT^{-1}v = G(v)$.

Proposition 2.4. Let (A_n) be a sequence of 2×2-matrices with

$$A_n = \operatorname{diag}(\lambda_n^+, \lambda_n^-), \quad \frac{1}{\tau} \leqq |\lambda_n^+| \leqq \theta \quad \text{and} \quad \frac{1}{\theta} \leqq |\lambda_n^-| \leqq \tau$$

for some constants $\theta \in (0, 1)$ and $\tau > 1$ and let $g_n(x)$, $n \in \mathbf{Z}$ be continuously differentiable functions which satisfy the following conditions:

$$|g_n(0)| \leqq \frac{\alpha \varrho}{\tau}$$

$$|D g_n(x)| \leqq \alpha \quad \text{for all } x \text{ with } |x| \leqq \varrho$$

where $\alpha = \frac{1}{2\tau}(1 - \theta)$. Then there exists exactly one sequence $v = (v_n) \in X$ such that

$$\|v\| \leqq \varrho \quad \text{and} \quad LT^{-1} v = G(v)$$

holds.

Equation (2) implies that this proposition applies. One first fixes ϱ suitably and afterwards chooses ε sufficiently small. Thus Theorem 2.3. is proven up to Proposition 2.4. □

Proof of Proposition 2.4. We show that L is invertible. From the structure of the matrices A_n it follows that the equation $Lu = 0$ only admits the trivial solution. We show that $Lu = w$ has a solution for all $w \in X$. Putting

$$u_n = \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}, \quad w_n = \begin{pmatrix} w_n^+ \\ w_n^- \end{pmatrix}$$

we get the solution

$$u_n^+ = w_n^+ + \lambda_{n-1}^+ w_{n-1}^+ + \lambda_{n-1}^+ \lambda_{n-2}^+ w_{n-2}^+ + \cdots$$

$$u_n^- = -\frac{1}{\lambda_n^-} w_{n+1}^- - \frac{1}{\lambda_n^- \lambda_{n+1}^-} w_{n+2}^- - \frac{1}{\lambda^- \lambda_{n+1}^- \lambda_{n+2}^-} w_{n+3}^- - \cdots .$$

One verifies easily that the series converge (this follows from

$$|\lambda_n^+| \leqq \theta < 1 < \frac{1}{\theta} \leqq |\lambda_n^-|),$$

that $u = (u_i)$ solves the equation $Lu = w$ and that $|u| \leqq \frac{1}{(1-\theta)} |w|$ holds. Thus L is invertible and L^{-1} satisfies the estimate $|L^{-1}| \leqq \frac{1}{(1-\theta)}$. By the assumptions of the proposition the map G has Lipschitz constant α and $|G(0)| \leqq \frac{\alpha \varrho}{\tau}$ holds. It remains to show that the equation $LT^{-1} v = G(v)$ has exactly one solution in B_ϱ. An equivalent equation is $H(v) := TL^{-1} G(v) = v$. The function H is Lipschitz with Lipschitz constant $\frac{1}{2}$ and for $v \in B_\varrho$

$$|H(v)| \leqq |H(0)| + |H(v) - H(0)| \leqq \frac{\varrho}{2} + \frac{\varrho}{2} \leqq \varrho$$

holds. Consequently $H(v) \in B_\varrho$ holds. Now the proposition follows from the Banach Point Theorem. □

We derive a first application of the Shadowing Lemma. Let ϱ and ε be chosen according to the Shadowing Lemma. For every $j \in \boldsymbol{Z}$ the orbit $P = \{(n, p_n)\}$ is uniquely determined by the point (j, p_j). The Shadowing Lemma says that for every ε-pseudo orbit $Q = \{(n, q_n)\}$ there is exactly one p_j with $|p_j - q_j| < \varrho$ such that the orbit $P = \{(n, p_n)\}$ generated by (j, p_j) is a ϱ-shadowing orbit of Q. We denote the map from the ε-pseudo orbits to (j, p_j) by π_j.

$$\pi_j \colon \Psi_\varepsilon \to \boldsymbol{Z} \times \boldsymbol{R}^2, \quad Q \to \pi_j(Q) = (j, p_j).$$

In other words: π_j takes the ε-pseudo orbit Q to the point (j, p_j) which generates the uniquely determined ϱ-shadowing orbit of Q.

Proposition 2.5. For every $j \in \boldsymbol{Z}$ the map π_j is continuous.

Proof. Since in Ψ every point has a countable base of neighbourhoods it suffices to show that for every sequence $(Q^{(k)})$ of ε-pseudo orbits which converges to Q the sequence $(\pi_j(Q^{(k)})) = ((j, p_j^{(k)}))$ converges to $\pi_j(Q) = (j, p_j)$. For every

$N \in \boldsymbol{N}$ there is a k_0 such that $k > k_0$ implies $|q_j^{(k)} - q_j| \leqq \frac{1}{N}$ for all j with $|j| < N$. For the ϱ-shadowing orbit $P^{(k)}$ of $Q^{(k)}$ every $p_j^{(k)}$ lies in the ϱ-neighbourhood of $q_j^{(k)}$ and therefore in the $\varrho + \frac{1}{N}$-neighbourhood of q_j if $k > k_0$ and $|j| < N$ holds. Thus for $j = 0$ the sequence $(p_0^{(k)})$, $k \in \boldsymbol{N}$, is bounded and therefore it suffices to show that p_0 is the only limit point of that sequence. Let $\hat{p}_0$ be any limit point of $(p_0^{(k)})$, $k \in \boldsymbol{N}$. Then there is a subsequence $(p_0^{(k_i)})$ which converges to $\hat{p}_0$. We consider the ϱ-shadowing orbits $P^{(k_i)} = \{(n, p_n^{(k_i)})\}$ of the ε-pseudo orbits $Q^{(k_i)} = \{(n, q_n^{(k_i)})\}$. Since f_n and f_n^{-1} are continuous one concludes by induction with respect to n that $p_n^{(k_i)}$ converges to some $\hat{p}_n$ as $k_i \to \infty$ and that $f(\hat{p}_n) = \hat{p}_{n+1}$ holds. Thus the set $\hat{P} := \{(n, \hat{p}_n)\}$ is an orbit. Moreover

$$|\hat{p}_n - q_n| \leqq \underbrace{|\hat{p}_n - p_n^{(k_i)}|}_{\to 0} + \underbrace{|p_n^{(k_i)} - q_n^{(k_i)}|}_{\leqq \varrho} + \underbrace{|q_n^{(k_i)} - q_n|}_{\to 0}$$

holds for $i \to \infty$. Since the l.h.s. doesn't depend on k_i one concludes $|\hat{p}_n - q_n| \leqq \varrho$. That means that $\hat{P}$ is a ϱ-shadowing orbit of Q. Since by the Shadowing Lemma the ϱ-shadowing orbit is uniquely determined the orbits $\hat{P}$ and $P = \{(n, p_n)\}$ coincide and $\hat{p}_n = p_n$ holds for all $n \in \boldsymbol{Z}$. Thus p_0 is the only limit point of the sequence $(p_0^{(k)})$, $k \in \boldsymbol{Z}$. □

3. The shift map as a subsystem for non-autonomous maps

We consider the following situation. Let Assumptions A1)–A3) of Paragraph 1 be satisfied, i.e. the differentiable map $f: \boldsymbol{Z} \times \boldsymbol{R}^2 \to \boldsymbol{Z} \times \boldsymbol{R}^2$ with $f(n, x) = (n + 1, f_n(x))$ admits a hyperbolic orbit $Z = \{n, z_n)\}$. Moreover we assume that there are infinitely many homoclinic orbits $X^{(j)} = \{(n, x_n^{(j)})\}$ which together with Z form a hyperbolic set Λ. We make an additional assumption on these homoclinic orbits: There are infinitely many points $x_n^{(j)}$ on the local stable manifold M_{loc}^+ of z_n as well as on the local unstable manifold M_{loc}^-. By this statement we mean the following: in any neighbourhood of z_n there are infinitely many indices j for which

$$|x_{n+k}^{(j)} - z_{n+k}| \leqq q^k \, |x_n^{(j)} - z_n| \quad \text{for } k \in \boldsymbol{N} \text{ and some } q \in (0, 1)$$

holds,

$$|x_{n-k}^{(j)} - z_{n-k}| \leqq q^k \, |x_n^{(j)} - z_n| \quad \text{for } k \in \boldsymbol{N}, \text{ respectively.}$$

We prove a theorem of Smale for non-autonomous systems for which the map f admits a hyperbolic set. To this end we have to extend the symbolic sequences to the t-s-space. Let $A := \{0, 1, \ldots, N - 1\}$ be the set of symbols called alphabeth ($N \geqq 2$). Let $\Sigma := \{s \mid s = (\ldots, s_{-1}; s_0, s_1, \ldots), \; s_i \in A\} = A^{\boldsymbol{Z}}$ be the usual set of symbol sequences and $\sigma: \Sigma \to \Sigma$ the (Bernoulli) shift map $(\sigma(s))_n = s_{n+1}$. A is given the discrete topology and $\Sigma = A^{\mathbf{Z}}$ the product topology. We consider the extended shift space

$$\tilde{\Sigma} := \mathbf{Z} \times \Sigma$$

and the extended shift map

$$\tilde{\sigma}: \tilde{\Sigma} \to \tilde{\Sigma}$$

defined by

$$\tilde{\sigma}(n, s) := (n + 1, \sigma(s)).$$

Definition 3.1. *For an infinite set* $T = \{\ldots, t_{-1}, t_0, t_1, \ldots\} \subset \mathbf{Z}$ *of integer time values* t_i *with* $t_i < t_{i+1}$ *we define the generalized Poincaré map*

$$P_T: T \times \mathbf{R}^2 \to T \times \mathbf{R}^2$$

by

$$P_T(t_i, x) := (t_{i+1}, f_{(t_{i+1}-1)} \circ f_{(t_{i+1}-2)} \circ \cdots \circ f_{(t_i+1)} \circ f_{t_i}(x)).$$

Remark. P_T is the return map of f with respect to the set $T \times \mathbf{R}^2$. If $X = \{(n, x_n)\}$ is an orbit of f then the map P_T takes (t_i, x_{t_i}) to $(t_{i+1}, x_{t_{i+1}})$. The map P_T describes the change of the state of an orbit $X \subset \mathbf{Z} \times \mathbf{R}^2$ on its reduction to the set $T \times \mathbf{R}^2$.

Theorem 3.2. *Let Assumptions* A1)–A3) (*see Paragraph* 1) *be satisfied and let there be infinitely many transversal homoclinic points on the local stable and on the local unstable manifold. Let the union of all homoclinic orbits together with the hyperbolic orbit form a hyperbolic set. Then there is a set* T *of integer time values* t_i *such that the following holds*:

i) *The generalized Poincaré map* P_T *admits the shift map* $\tilde{\sigma}$ *as a subsystem, i.e. there is a homeomorphism* $\tau: \tilde{\Sigma} \to \varphi(\tilde{\Sigma}) \subset T \times \mathbf{R}^2$ *such that the following diagram commutes*:

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\;\tilde{\sigma}\;} & \tilde{\Sigma} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \tau(\tilde{\Sigma}) \subset T \times \mathbf{R}^2 & \xrightarrow{\;P_T\;} & \tau(\tilde{\Sigma}) \subset T \times \mathbf{R}^2 \end{array}$$

(ii) *If* $s_0 = 0$, $s_0 = 1$ *respectively, then* $|\tau(n, s) - z_{t_n}| \leqq \varrho$, $|\tau(n, s) - z_{t_n}| > 2\varrho$ *respectively.*

(iii) *If* $s_0 = 0$ *and* $s_1 = 0$ *and if* $P = \{(k, p_k)\}$ *is the orbit generated by* $\tau(n, s)$ *then*

$$|p_k - z_k| \leqq \varrho \quad \textit{for} \quad k \in [t_n, t_{n+1}].$$

Proof. We prove the theorem for the case where $A = \{0, 1\}$. (Note that for every N there is a generalized Poincaré map of the shift of two symbols which admits the shift of N symbols as a subsystem). In the first part we determine time values r_i and t_i for $i \in \boldsymbol{Z}$. In the second part we associate to every element $\tilde{s} = (n, s) \in \tilde{\Sigma}$ an ε-pseudo orbit $Q = \{(j, q_j)\}$. According to the Shadowing Lemma we apply the map π_{t_n} (see Proposition 2.5) to Q. Finally we show that the composed map is the homeomorphism we are looking for.

By Assumption A2), A3) there exists a $\varrho_0 > 0$ such that for every orbit $X^{(j)} = \{(n, x_n^{(j)})\}$ different from the hyperbolic orbit $Z = \{(n, z_n)\}$ there is a time value m such that $|x_m^{(j)} - z_m| > \varrho_0$ holds. We fix $\varrho < \frac{1}{3}\varrho_0$ and $\varepsilon \leqq \varrho$ according to the Shadowing Lemma and define the time values r_i and the set T of time values t_i as follows: We put $r_0 = 0$ and choose a transversal homoclinic point $x_0^{(j_0)}$ on the local unstable manifold M_{loc}^- with $|x_0^{(j_0)} - z_0| < \frac{\varepsilon}{2}$. We consider the images $x_1^{(j_0)} = f_0(x_0^{(j_0)})$, $x_2^{(j_0)} = f_1 \circ f_0(x_0^{(j_0)}), \ldots, x_k^{(j_0)} = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0(x_0^{(j_0)})$. At the beginning as k increases the distance $|x_k^{(j_0)} - z_k|$ gets larger and larger and there is a time value $t_0 > r_0$ such that $|x_{t_0}^{(j_0)} - z_{t_0}| > \varrho_0$ holds. But since $x_0^{(j_0)}$ is a homoclinic point there is a $r_1 > t_0$ such that $|x_{r_1}^{(j_0)} - z_{r_1}| < \frac{\varepsilon}{2}$. Again there is a transversal homoclinic point $x_{r_1}^{(j_1)}$ on the local unstable manifold M_{loc}^- with $|x_{r_1}^{(j_1)} - z_{r_1}| < \frac{\varepsilon}{2}$. The distance between $x_k^{(j_1)} = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_{r_1}(x_{r_1}^{(j_1)})$ and z_k is increasing for k slightly larger than r_1 and for some $t_1 > r_1$ the estimate $|x_{t_1}^{(j_1)} - z_{t_1}| > \varrho_0$ holds. There is a $r_2 > t_1$ such that $|x_{r_2}^{(j_0)} - z_{r_2}| < \frac{\varepsilon}{2}$ holds since $x_{r_1}^{(j_1)}$ is a homoclinic point. Going on in the same manner one gets the time values $r_0, r_1, r_2, \ldots$ and $t_0, t_1, t_2, \ldots$. The time values $r_{-1}, r_{-2}, \ldots$ and $t_{-1}, t_{-2}, \ldots$ are constructed analogously. One takes the map f^{-1} instead of the map f and the homoclinic points are to be chosen on the local stable manifold M_{loc}^+, i.e. the local unstable manifold of f^{-1}. Finally the following holds:

For all $i \in \boldsymbol{Z}$ there are time values r_i and t_i and a transversal homoclinic point $x_{r_i}^{(j_i)}$ with the following properties:

1) $\cdots < r_{-1} < t_{-1} < r_0 < t_0 < r_1 < t_1 < \cdots$

2) $|x_{r_i}^{(j_i)} - z_{r_i}| < \frac{\varepsilon}{2}$ and $|x_{r_{i+1}}^{(j_i)} - z_{r_{i+1}}| < \frac{\varepsilon}{2}$

3) $|x_{t_i}^{(j_i)} - z_{t_i}| > \varrho_0$.

Now we define the following finite sections of orbits

$$Q_0^{(n)} := \{(k, x) \mid r_n \leqq k < r_{n+1}, x = z_k\}$$

$$Q_1^{(n)} := \{(k, x) \mid r_n \leqq k < r_{n+1}, x = x_k^{(j_n)}\}.$$

We associate to every element $\tilde{s} = (n, s) \in \tilde{\Sigma}$ an ε-pseudo orbit in the following way:

$$\varphi: \tilde{\Sigma} \to \Psi_\varepsilon$$

$$\tilde{s} = (n, s) \mapsto \varphi(\tilde{s}) := \bigcup_{i \in \mathbf{Z}} Q_{s_i}^{(n+i)} = \cdots \cup Q_{s_{-1}}^{(n-1)} \cup Q_{s_0}^{(n)} \cup Q_{s_1}^{(n+1)} \cup \cdots$$

Property 2) guarantees that $\varphi(\tilde{s})$ indeed is an ε-pseudo orbit. We define $\tau: \tilde{\Sigma} \to \mathbf{Z} \times \mathbf{R}^2$ by

$$\tau(n, s) := \pi_{t_n} \circ \varphi(n, s)$$

where π_j denotes the map defined in Paragraph 2 which takes the ε-pseudo orbit Q to the point (j, x_j) generating the ϱ-shadowing orbit of Q.

We show that τ is the homeomorphism we are looking for. First we show that the diagram commutes. By definition of φ we have $\varphi(\tilde{s}) = \varphi(\tilde{\sigma}(\tilde{s}))$. Thus the shadowing orbits $\{(k, p_k)\}$ of $\varphi(\tilde{s})$ and of $\varphi(\tilde{\sigma}(\tilde{s}))$ are the same and hence

$$\tau \circ \tilde{\sigma}(n, s) = \tau(n + 1, \sigma(s)) = (t_{n+1}, p_{t_{n+1}}) = P_T(t_n, p_{t_n}) = P_T \circ \tau(n, s)$$

holds. Now we show that τ is one to one. Let $\tilde{s} = (n, s) \neq \tilde{s}' = (n', s')$ be given. If $n \neq n'$ then $\tau(\tilde{s}) = (t_n, p_{t_n}) \neq (t_{n'}, p_{t_{n'}}) = \tau(\tilde{s}')$ since $t_n \neq t_{n'}$. Therefore it suffices to show that $s \neq s'$ implies $\tau(n, s) \neq \tau(n, s')$. If $s \neq s'$ then there is an index k with $s_k \neq s'_k$ and hence $Q_{s_k}^{(n+k)} \neq Q_{s'_k}^{(n+k)}$. Thus by property 1) and 3) the ε-pseudo orbits $\varphi(n, s) = \{(j, q_j)\}$ and $\varphi(n, s') = \{(j, q'_j)\}$ are different and for the index $m := t_{n+k}$

$$|q_m - q'_m| \geqq \varrho_0 \geqq 3\varrho$$

holds. Now the estimate

$$|p_m - p'_m| \geqq |q_m - q'_m| - |p_m - q_m| - |q'_m - p'_m| \geqq 3\varrho - 2\varepsilon \geqq \varrho$$

follows implying $p_m \neq p'_m$ or $\tau(n, s) \neq \tau(n, s')$. It remains to show that τ and τ^{-1} are continuous. τ is continuous since it is the composition of two continuous maps. The continuity of τ^{-1} follows from the following Lemma the proof of which can be found in G. Preuss [9].

Lemma 3.3. Let $f: X \to Y$ be a bijective map from a compact topological space onto a Hausdorff space. Then f is a homeomorphism iff f is continuous.

This completes the proof of (i).

Let $\tau(n, s) = \pi_{t_n}(\varphi(n, s)) = (t_n, p_{t_n})$ with $\varphi(n, s) = Q$. In the time interval $[r_n, r_{n+1})$ we have $Q = Q_{s_0}^{(n)}$. Thus if $s_0 = 0$ then by the Shadowing Lemma $|p_{t_n} - z_{t_n}| \leqq \varrho$ holds and if $s_0 = 1$ then $|q_{t_n} - z_{t_n}| \geqq \varrho_0$ holds implying

$$|p_{t_n} - z_{t_n}| \geqq |q_{t_n} - z_{t_n}| - |p_{t_n} - q_{t_n}| \geqq 3\varrho - \varepsilon \geqq 2\varrho.$$

This proves (ii).

To prove (iii) we note that $s_0 = s_1 = 0$ implies $q_i = z_i$ for $i \in [r_n, r_{n+2})$ containing the interval $[t_n, t_{n+1}]$. Now the assertion follows from the Shadowing Lemma. □

In analogy to the notion of Li-Yorke chaos (see Li and Yorke [4]) one can conclude from Theorem 3.2:

Corollary 3.4. *Let the assumptions of Theorem* 3.2 *be satisfied. Then there is an uncountable set M and a constant* ϱ *with the following properties*:

(i) $f(M) = M$.

(ii) $\liminf_{n \to \pm\infty} |f^n(k, x) - f^n(k, y)| = 0$ *for all* $k \in \mathbf{Z}$, $(k, x) \in M$, $(k, y) \in M$.

(iii) $\limsup_{n \to \pm\infty} |f^n(k, x) - f^n(k, y)| \geqq \varrho > 0$ *for all* $k \in \mathbf{Z}$, $(k, x) \in M$, $(k, y) \in M$, $x \neq y$.

Sketch of the proof. In Stoffer [12] it is proven that the shift space Σ is chaotic in the sense of Li and Yorke. Thus there is a uncountable subset M_Σ *of* Σ with the above properties. Now put $\tilde{M} := \mathbf{Z} \times M_\Sigma$ and define $M_P := \tau(\tilde{M})$. From Theorem 3.2 assertion (i) it follows that M_P is invariant under the Poincaré map P_T. Now let $(t_k, x) \in M_P$ and $(t_k, y) \in M_P$ with $x \neq y$. There are sequences u, $v \in M_\Sigma$ such that $\tau(k, u) = (t_k, x)$ and $\tau(k, v) = (t_k, y)$. Since M_Σ is Li-Yorke chaotic with respect to the shift map σ there are infinitely many indices i with $u_i \neq v_i$. Property (ii) of Theorem 3.2 implies

$$|P_T(t_{k+i}, x) - P_T(t_{k+i}, y)| \geqq |q_{t_{k+i}} - z_{t_{k+i}}| - |p_{t_{k+i}} - q_{t_{k+i}}| \geqq 3\varrho - \varepsilon \geqq 2\varrho$$

and thus assertion (iii) holds for the Poincaré map P_T. In Stoffer [12] it is proven that there are intervals I of arbitrary length with $u_i = v_i = 0$. By property (iii) of Theorem 3.2 it follows that the orbits stay simultaneously ϱ-close to the Hyperbolic orbit for arbitrary long time. Thus they are arbitrary close together for a certain time value. Hence the set M_P satisfies (i), (ii) and (iii) for the Poincaré map P_T. Now extend the set $M_P \subset T \times \mathbf{R}^2$ to the space $\mathbf{Z} \times \mathbf{R}^2$, i.e. put $M := \bigcup_{i \in \mathbf{N}} f^i(M_P)$ and the corollary follows immediately. □

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Abstract

A concept of generalized hyperbolic sets for non-autonomous maps is developed. Starting from transversal homoclinic orbits such generalized hyperbolic sets are constructed. The Shadowing Lemma is proven for maps admitting a generalized hyperbolic set. Time dependent symbolic dynamics is introduced and related to non-autonomous maps.

Zusammenfassung

Das Konzept von verallgemeinerten hyperbolischen Mengen für nicht-autonome Abbildungen wird entwickelt. Ausgehend von transversalen homoklinen Bahnen werden solche verallgemeinerte hyperbolische Mengen konstruiert. Das Shadowing Lemma wird für Abbildungen bewiesen, welche eine verallgemeinerte hyperbolische Menge haben. Es wird zeitabhängige symbolische Dynamik eingeführt und der Zusammenhang mit nicht-autonomen Abbildungen dargestellt.

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