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Particle transport in a host medium with an external source: exact solutions for a nonlinear homogeneous Boltzmann equation

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1. Introduction

The BKW mode, found by Bobylev [1] and Krook and Wu [2], is an exact similarity solution of the nonlinear spatially homogeneous (3-dimensional) Boltzmann equation (BE). It describes the relaxation to equilibrium of a one-component gas of Maxwell molecules, undergoing elastic scattering collisions, such that both particle and energy density are conserved. The BKW solution turns out to be relevant also for other classes of Boltzmann models with spatial dimensionality d as free parameter [3, 4].

As soon as Bobylev [1] discovered this particular homogeneous distribution function, he applied to it the Nikolskii transform [5, 6] and generated in this way an exact solution for the spatially inhomogeneous (d = 3-dimensional) problem. By introducing confining external forces new classes of homogeneous and inhomogeneous similarity solutions of ($d > 1$)-dimensional Boltzmann equations were discovered [7, 8].

In the absence of external forces (and referring to the standard case $d = 3$) other classes of homogeneous similarity solutions, differing from the BKW solution, may arise in problems of rarefied gas dynamics, where at least the condition of constant number density is relaxed [6]. This may happen when an external source supplies (or subtracts) molecules at an appropriate rate. Then the Boltzmann equation must be modified by adding a source term and, as is well known, one can construct (in the case of Maxwell molecules) classes of similarity solutions which are not of BKW type [6, 9].

A different physical situation, in which the particle density is also not conserved, arises when removal events take place in the gas [10, 11]: it may occur that the collision between two molecules does not lead to elastic scattering but rather to a removal event (leading to the disappearance of incident particles). The corresponding nonlinear spatially homogeneous Boltzmann equation for Maxwell molecules with removal allows a generalized BKW solution which has been determined recently [12] by means of a Bobylev ansatz.

Following Boffi, Spiga, Nonnenmacher [13–15] one may go a step further and consider a spatially homogeneous system of test particles (t. p., mass m) embedded in an unbounded host medium of field particles (f. p., mass $\hat{m}$) with fixed total density $\hat{n}$. The corresponding Boltzmann equation for the t. p. (incorporating elastic scattering, removal events, interactions with the background host medium and the presence of an external source [13, 16, 17]) reads as [14]:

$$\frac{\partial f(v,t)}{\partial t} + (\hat{n}\hat{C} + n(t)C)f(v,t) = \frac{\hat{n}\hat{C}_s}{4\pi v^2}\int_{R_3} \hat{\Pi}_s^0(v' \to v) f(v',t)\,d\boldsymbol{v}'$$

$$+ \frac{C_s}{4\pi v^2}\int_{R_3}\int_{R_3} \Pi_s^0(v', w' \to v) f(v',t) f(w',t)\,d\boldsymbol{v}'\,d\boldsymbol{w}' + Q(v,t), \tag{1}$$

where $n(t) = \int f(v,t)\,d\boldsymbol{v}$ is the test particle density, and $C = C_r + C_s$ with positive constants C_r, C_s characterizes the isotropic removal and scattering collision frequencies for t. p.–t. p. interactions. For t. p.–f. p. interactions $\hat{C} = \hat{C}_s + \hat{C}_r$ is the analogous expression. The scattering probability functions $\hat{\Pi}_s^0$, Π_s^0 are defined in Ref. 15. This Boltzmann equation, to be studied in the present paper for Maxwell molecules, can serve as a model for such complex situations of particle transport where the test particle density is not conserved in consequence of removal collisions [18] (t. p.–t. p. as well as t. p.–f. p.) and through the influence of an external source (supplying or subtracting t. p.).

In order to solve the Boltzmann equation we introduce the moments of the distribution function $f(v,t)$ as well as the moments of the source $Q(v,t)$ $(k = 0, 1, \ldots)$:

$$\begin{pmatrix} M_k(t) \\ Q_k(t) \end{pmatrix} := \left(\frac{4}{v_0^2}\right)^k \frac{k!}{(2k+1)!}\int_{R_3} d\boldsymbol{v}\, v^{2k} \begin{pmatrix} f(v,t) \\ Q(v,t) \end{pmatrix}, \quad v_0^2 \equiv \frac{2k_B T_0}{m}. \tag{2}$$

In the limiting case where the mass ratio $m/\hat{m}$ tends to zero [19] the Boltzmann equation leads to the following system of moment equations

$$\frac{dM_k}{dt} + \hat{n}\hat{C}_r M_k(t) + C M_0(t) M_k(t) - \frac{C_s}{1+k}\sum_{j=0}^{k} M_{k-j}(t) M_j(t) = Q(t). \tag{3}$$

Introducing two generating functions

$$G(\omega,t) = \sum_{k=0}^{\infty} \omega^k M_k(t), \quad S(\omega,t) = \sum_{k=0}^{\infty} \omega^k Q_k(t), \tag{4}$$

this infinite set of nonlinear ordinary differential equations can be reduced to a single partial differential equation (PDE). In order to obtain the PDE in a convenient form we choose $M_0(0) = n(0) := q_0$ as a reference density for the test particles, and we measure time in terms of $t^* = q_0 C_s t$. Dropping the asterisk again and defining

$$x = (1-\omega)/\omega, \quad u = \omega G/q_0, \quad \sigma = \frac{-\omega^2}{q_0^2 C_s}\frac{\partial}{\partial\omega}(\omega S(\omega,t)), \tag{5}$$

the partial differential equation can be written

$$H(u_{xt}, u_x, u, x, t) = u_{xt} + B(t)u_x + u^2 - \sigma(x, t) = 0 \tag{6}$$

with $\sigma(x, t)$ being the characteristic function of the source and

$$B(t) = \hat{\gamma}_r + (1 + \delta_r) M_0(t)/q_0, \quad \hat{\gamma}_r = \hat{n}\hat{C}_r/q_0 C_s, \quad \delta_r = C_r/C_s. \tag{7}$$

Equation (6) possesses particular solutions, called similarity solutions [19], for which H is invariant unter a Lie group of transformations containing 4 free parameters. In varying these parameters (Lie group constants) one obtains various subclasses of similarity solutions of Eq. (6) with different types of similarity variable. For each subclass there remains the twofold task of reconstructing (i) the source term in the BE, and (ii) the corresponding type of distribution function for the test particles.

At present this program has been carried out completely [20] only for the special class of similarity solutions where two of the group constants vanish and where time t represents the similarity variable. However, in the most general case the similarity variable is $\log(\gamma x + \delta)/\gamma - \gamma F(t)$ [21], and we intend to discuss here similarity solutions of this type referring to the two-component case. Under the assumption that neither of the Lie group constants vanishes we show that the Lie group constants can be fixed such that the source term in the BE vanishes. In this case we derive systematically the generalized BKW solutions found by Spiga [12] and Nonnenmacher [11]. With aid of an appropriate limiting process (where all removal parameters tend to zero) we can discover also the original BKW mode with similarity variable $\log(x) + ct$. Other allowed choices of the Lie group constants, belonging to non-zero source terms, are discussed, and the class of known exact homogeneous distributions is enlarged in this way. Special attention is focused, thereby, to the role played by the characteristic constants and the Lie group constants with respect to the positivity of homogeneous t.p. distribution functions.

2. Results of the similarity analysis

An important result of previous investigations [9, 19–22] is that similarity solutions exist only if the characteristic function of the source, $\sigma = \sigma(x, t)$, satisfies a compatible condition – a first-order partial differential (Eq. (20a) in Ref. 9) involving the generators $\xi(x)$ and $\tau(t)$. In the case where these generators do not vanish identically an admissible class of characteristic functions is given by [19]

$$\sigma = \frac{\varphi_1(\alpha, \gamma)\,\varphi_2(\alpha, \gamma)}{(\xi(x)\,\tau(t))^2}, \tag{8}$$

$$\varphi_1(\alpha, \gamma) = -\tfrac{3}{25}(\gamma - \tfrac{3}{2}\alpha)(\gamma - \tfrac{2}{3}\alpha), \quad \varphi_2(\alpha, \gamma) = -\tfrac{3}{25}(\gamma + 6\alpha)(\gamma + \tfrac{1}{6}\alpha), \tag{9}$$

and the corresponding class of similarity solutions is

$$u = \varphi(\zeta)/(\xi(x)\,\tau(t)), \tag{10}$$

where ζ represents the similarity variable

$$\xi = \log \xi(x)/\gamma - \gamma F(t), \quad F(t) = \int_0^t dt'/\tau(t'). \tag{11}$$

$\varphi(\zeta)$ is the similarity function

$$\varphi(\zeta) = \varphi_1(\alpha, \gamma) + 6\gamma^2 W^2(\zeta), \tag{12}$$

$$W(\zeta) = \frac{\frac{1}{5}\left(1 + \frac{\alpha}{\gamma}\right)}{1 - g_1 g_4 \frac{1}{5}\left(1 + \frac{\alpha}{\gamma}\right) \exp\left\{-\frac{1}{5}\left(1 + \frac{\alpha}{\gamma}\right)\zeta\right\}} \tag{13}$$

which has to satisfy the side-condition

$$\lim_{\zeta \to \infty} \varphi(\zeta) = \beta\gamma. \tag{14}$$

The generators $\xi(x)$, $\tau(t)$ read as

$$\xi(x) = \gamma x + \delta, \tag{15}$$

$$\tau(t) = \beta + \lambda t, \quad \hat{\gamma}_r = 0, \tag{16}$$

$$\tau(t) = (\beta + \lambda/\hat{\gamma}_r) \exp(\hat{\gamma}_r t) - \lambda/\hat{\gamma}_r, \quad \hat{\gamma}_r \neq 0 \tag{17}$$

with $\lambda = (1 + \delta_r)\beta - \alpha$. The parameters $\alpha, \beta, \gamma, \delta$ (Lie group constants) and g_1, g_4 (integration constants) in these equations are arbitrary constants introduced by the similarity analysis. With respect to the side condition (14) we remark that the Eqs. (12–13) contain similarity solutions of the PDE (6) for which $\varphi(\infty) \neq \beta\gamma$. These solutions lead, however, to a contradiction in the moment equation of order $k = 0$, Eq. (3), and must be discarded in the context of our problem. Using the above expressions for $u(x, t)$, $\sigma(x, t)$ we adress to the problem of calculating explicitly (a.) the infinite sets of moments $\{M_n(t)\}$, $\{Q_n(t)\}$ as well as (b.) the final form of the distribution function $f(v, t)$ and the corresponding source term $Q(v, t)$.

The restrictions, imposed by Eq. (14), can be worked out explicity by analyzing the asymptotic behavior of $W(\zeta)$: According to the sign of the argument in the exponential function the limit $\zeta \to +\infty$ yields two different asymptotic values for the similarity function $\varphi(\zeta)$, namely $\varphi_1(\alpha, \gamma)$ if $\alpha/\gamma < -1$ and $-\varphi_2(\alpha, \gamma)$ if $\alpha/\gamma \geqq -1$. Inserting these asymptotic values into Eq. (14) one obtains:

$$\beta = \frac{1}{\gamma}\varphi_1(\alpha, \gamma) \qquad \text{for } \alpha/\gamma < -1 \tag{18}$$

$$\beta = -\frac{1}{\gamma}\varphi_2(\alpha, \gamma) \quad \text{for } \alpha/\gamma \geqq -1. \tag{19}$$

The side condition (14) evidently gives rise to two different subclasses of similarity solutions: those for which $\alpha/\gamma < -1$ (class C_1) and $\alpha/\gamma \geqq -1$ (class C_2), respectively. Within each subclass the group constant β can be expressed in terms of the group constants α and γ alone; this implies that the number of independent group constants is reduced by one, and we may choose $\alpha, \gamma, \delta, g_1, g_4$ as a convenient set of independent parameters within each subclass C_1, C_2. A further reduction of the set of free parameters arises from the following observation: All the relations (8–19) are homogeneous with respect to the group constants. This implies that we may set $\gamma = 1$ without loss of generality.

3. Reconstruction of the source term $Q(v, t)$ and preliminary remarks on the t.p. distribution $f(v, t)$

When $\gamma = 1$ the characteristic function of the source, Eq. (8), yields for any choice of α, β, δ a generating function of the type

$$S(\omega, t; \alpha, \beta, \delta) = \sum_{k=0}^{\infty} [-q_0^2 C_s \varphi_1(\alpha, 1) \varphi_2(\alpha, 1)(1-\delta)^k/\tau^2(t; \alpha, \beta)] \omega^k \tag{20}$$

which contains all information about the moments Q_k – according to the basic definition $S = \Sigma Q_k \omega^k$.

By the standard Fourier-Cosine-transform method [2] the generating function (20) leads to the source distribution

$$Q(v, t; \alpha, \beta, \delta) = -q_0^2 C_s \varphi_1(\alpha, 1) \varphi_2(\alpha, 1) m(v; 1-\delta)/\tau^2(t; \alpha, \beta) \tag{21}$$

where the velocity-dependent term $m(v; 1-\delta)$ represents a generalized Maxwellian

$$m(v; y) = \frac{1}{(\pi v_0^2 y)^{3/2}} e^{-v^2/v_0^2 y} \tag{22}$$

with argument $y = 1 - \delta$. Hence, the type of source admitted in the Boltzmann equation is completely specified by fixing α, β, δ. From Eqs. (22) and (2) one can see that the temperature of the source is $T^* = T_0(1-\delta)$. Since T^* has to be non-negative the group constant δ is restricted to $\delta \leqq 1$. As we shall see in the following investigation of the distribution function $f(v, t)$, also the remaining group parameters α, β cannot be chosen arbitrarily. In particular, they can take discrete values only.

All information about the distribution function f is contained in Eq. (10) when inserting for φ, ξ, τ the expressions given in Eqs. (12–13), (15–17) and eliminating the similarity variable ζ via Eq. (11). Replacing $x \to (1-\omega)/\omega$ everywhere we obtain the generating function $G = q_0 u/\omega$ in the appropriate variables $\omega, t, \ldots$ etc. Since the basic definition (4) requires for G a power series in ω it is important to analyze the behavior of $1/\xi$ and φ under this substitution: (i) $1/\xi(x;\delta)$ leads simply to ω for $\delta = 1$ and, otherwise, to a power series in ω; (ii) the similarity function φ yields, however, a more extensive class of series involving non-integer powers of ω since

$$W = \frac{A}{1 + Z\left(\dfrac{\omega}{1-(1-\delta)\omega}\right)^{A}}, \quad Z = +\,\hat{g}_1 \hat{g}_4 A\, e^{AF(t;\alpha,\beta)}(1-\delta)^{A} \tag{23}$$

where $\hat{g}_1 = g_1(1-\delta)^{-A}$, $\hat{g}_4 = -g_4$, $A(\alpha) = (1+\alpha)/5$. In the limit $\omega \to 0$ we have $W \doteq A(1 - Z\omega^{A}\ldots)$ for $A \geqq 0$ and $W \doteq (A/Z)\omega^{|A|}\ldots$ in the case $A \leqq 0$.

It is obvious that the expressions for W, φ, u as well as for G reduce to power series in ω only if $|A(\alpha)| \in N_0$. This condition is met when α takes the following discrete values:

$$\alpha = -1 - 5(n+1), \quad \text{class } C_1; \quad \alpha = 5n - 1, \quad \text{class } C_2 \tag{24}$$

with $n = 0, 1, 2, \ldots$ Then φ, u, G reduce to rational functions of the variable ω, and the decomposition of G into partial fractions provides an important step for the final representation of the generating function as a power series in ω. In this paper we do not discuss systematically the representation of G in terms of partial fractions and the subsequent power-series expansion in ω for arbitrary n. We concentrate rather, on three illustrate examples: $n = 0$ in class C_1 and $n = 0, 1$ in class C_2.

4. Exact distribution functions

In the classes C_1 and C_2, α can take only the discrete values given in the last section. From Eqs. (18) and (19), respectively, one can infer that β is also restricted to discrete values. This discretization must be observed in order to remain within the class of generating functions, Eqs. (4), allowed for the modified Krook and Wu procedure (and to be consistent with the basic set of moment Eqs. (3)). The remaining parameters δ, $\hat{g}_1$ and $\hat{g}_4$, however, are not restricted to discrete values. We investigate now the peculiar role which these parameters and the removal constants $\hat{\gamma}_r$, δ_r play with respect to the positivity of the distribution function $f(v,t)$. For this purpose we analyze the three examples given in Sect. 3. In the discussion of these examples the lowest order moment of the distribution function, the particle density M_0, plays an exceptional role. Since all information about M_0 can be obtained from the similarity analysis alone (i.e. without calculating $f(v,t)$ explicitly) we give here some remarks on the time evolution of the density [19] and its dependence on the group constants:

$$M_0(t;\alpha,\beta) = \beta q_0/\tau(t;\alpha,\beta). \tag{25}$$

This relation shows that the time evolution of the particle density is intimately connected with the time dependence of the generator τ as given by Eqs. (16) and (17), in the cases $\hat{\gamma}_r = 0$ and $\hat{\gamma}_r \neq 0$, respectively. For physical reasons the particle must be non-negative for all $t \geqq 0$. Using Eqs. (18) and (19) one can easily show that this condition is met for any $\hat{\gamma}_r \geqq 0$ if

$$\delta_r \geqq -\varphi_2(\alpha, 1)/\varphi_1(\alpha, 1) \text{ in } C_1, \quad \delta_r \geqq -\varphi_1(\alpha, 1)/\varphi_2(\alpha, 1) \text{ in } C_2, \tag{26}$$

with φ_1, φ_2 given in Eq. (9).

4.1 The case $n = 0$ in class C_2

The group constants are now $(\alpha, \beta, \gamma, \delta) = (-1, -1/2, 1, \delta)$, and the similarity function φ, Eq. (12), becomes independent of x and $t\,(\varphi = -1/2)$. With the zeroth order moment $M_0(t) = M_0(t; -1, -1/2)$, Eq. (25), the moments and the source moments can be written as

$$M_k(t) = M_0(t)(1-\delta)^k, \quad Q_k(t) = C_s M_k^2(t)(1-\delta)^{-k}. \tag{27}$$

The scaled distribution functions (evaluated by the Fourier transform method)

$$f(v, t)/M_0 = Q(v, t)/Q_0 = m(v; 1-\delta) \tag{28}$$

are of Maxwellian type.

From Eq. (28) it is obvious that the time dependence in the distributions $f(v, t)$ and $Q(v, t)$ enters only via $M_0(t)$ and $Q_0(t)$ as given by Eqs. (27) for $k = 0$.

4.2 The case $n = 0$ in class C_1

The relevant set of group constants reads as $(\alpha, \beta, \gamma, \delta) = (-6, -6, 1, \delta)$, and this is the only choice for which $\{Q_k(t) \equiv 0\}$ and hence $Q(v, t) \equiv 0$. In the case δ_r, $\hat{\gamma}_r \neq 0$ the complete sequence of moments can be written as

$$M_k(t) = \frac{-6 q_0}{\tau(t; -6, -6)} \left\{1 - k \frac{g(t)}{1 + g(t)}\right\} (1 + g(t))^k (1-\delta)^k, \; k = 0, 1, \tag{29a}$$

where

$$g(t) = \frac{1}{\hat{g}_1 \hat{g}_4} \left\{\frac{-6 \exp(\hat{\gamma}_r t)}{\tau(t; -6, -6)}\right\}^{1/6\delta_r} \tag{29b}$$

and $\tau(t; -6, -6)$ is given by Eq. (17). The formulae for the case $\hat{\gamma}_r = 0$ and/or $\delta_r = 0$ can be obtained from these expressions by carrying out (appropriately) the limits $\hat{\gamma}_r \to 0$ and/or $\delta_r \to 0$.

Introducing the particle density via Eq. (25) we obtain from Eqs. (29), using the Fourier transform method, the distribution function in a form which is very convenient for discussing the problem of positivity:

$$f(v,t) = M_0(t; -6, -6)\left\{(2b(t)^2 - b(t))\frac{v^2}{v_0^2(1-\delta)} + \frac{5}{2} - 3b(t)\right\} m(v; (1-\delta)/b(t)) \tag{30}$$

with

$$b(t) = 1/2(1 + g(t)).$$

According to inequality (26), the first factor in Eq. (30), the test-particle density, is non-negative for $0 \leqq t \leqq \infty$ and any choice of the removal parameters δ_r, $\hat{\gamma}_r \geqq 0$. At a given time t, the distribution function $f(v, t)$ is a non-negative function of v^2 if the following conditions are met: (i) $\delta \leqq 1$ (the physically relevant range for the group constant δ) and (ii): $1/2 \leqq b(t) \leqq 5/6$. Condition (ii) imposes a restriction on the parameters $\hat{g}_1, \hat{g}_4$, namely: $\hat{g}_1 \hat{g}_4 \leqq -5/2$. Hence, when α, β, γ are fixed as above, the natural restrictions upon the remaining continuous parameters are $\delta \leqq 1$ and $\hat{g}_1 \hat{g}_4 \leqq -5/2$. Under these conditions M_0 and f are non-negative functions in the time interval $0 \leqq t \leqq \infty$, and – in contrast to the other cases considered in this paper (Sects. 4.1, 4.3) – the removal parameters may take any values $\hat{\gamma}_r, \delta_r \geqq 0$. The solution, obtained in Ref. 11 with a quite different method, follows from Eq. (30) by setting $\delta = 0$ and $\hat{g}_1 \hat{g}_4 = -\exp(\tau_0/6)$. It is clear that the equations of this section imply various limiting cases, where the removal parameters tend to zero. In this sense the distribution function, Eq. (30), implies for $\hat{\gamma}_r \to 0$ the result, obtained previously by Spiga [12]. This limit corresponds to the physical situation, where the test particles are not influenced by the particles of the host medium [14]. Therefore, $\hat{\gamma}_r \to 0$ is formally equivalent to the transition of our two-component system to a one-component gas of test particles: Setting $\delta = 0$, we rediscover the generalized BKW solution for a single gas with removal between the test-particles ($\delta_r \neq 0$) which has been found by Spiga, using a Bobylev ansatz [12]. In order to obtain identical results the integration constant β_0 in Ref. 12 has to be correlated to $\hat{g}_1, \hat{g}_4$ as follows: $\hat{g}_1 \hat{g}_4 = -(1 + \beta_0)/\beta_0$.

Furthermore, if both $\hat{\gamma}_r$ and δ_r tend to zero, we obtain the particle conservation law $M_0 = \text{const} = q_0 > 0$ and the famous BKW mode [2] for the relaxation of a classical gas of Maxwell molecules towards an equilibrium state. Since the BKW mode corresponds, however, to the choice $\hat{g}_1 \hat{g}_4 = -1$, the positivity of $f(v, t)$ in the range $0 \leqq t \leqq \infty$ is violated. As is well known [2, 6] this problem can be overcome by considering the BKW-mode in a shifted time interval $t_0 \leqq t \leqq \infty$. The appropriate initial time t_0 is given by $t_0 = 6\log(-5/2\,\hat{g}_1 \hat{g}_4)$. In Fig. 1 the time evolution of the t.p. distribution $v^2 f(v, t) \equiv h(v, t)$ is illustrated for $t \geqq 0$, $\hat{g}_1 \hat{g}_4 = -5/2$, $q_0 = 1$, $\delta = 0$ and two different choices of the removal parameters $\hat{\gamma}_r = \delta_r = 0$ (Fig. 1 a) and $\hat{\gamma}_r = \delta_r = 0.001$ (Fig. 1 b). When the removal parameters vanish both particle and energy density are conserved, and we have the relaxation of the distribution function towards a Maxwellian (limiting

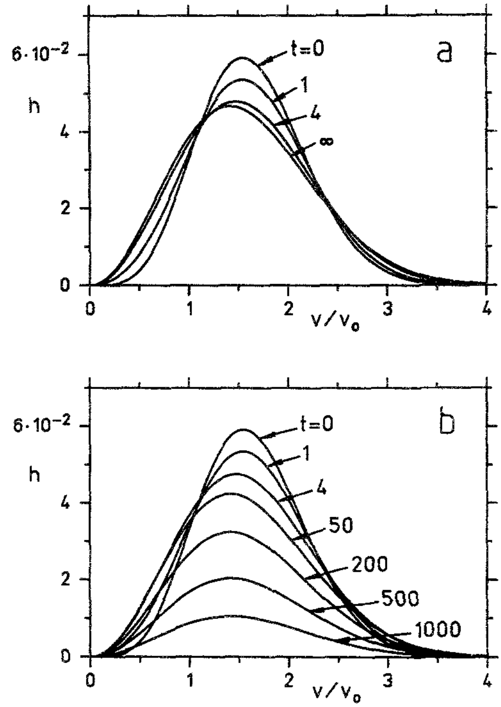

Figure 1
The distribution function $h = v^2 f$ (in units of v_0^3) versus v/v_0, Eq. (30), for various times t and the following values of parameters: $\hat{g}_1 \hat{g}_4 = -5/2$, $\delta = 0$, $q_0 = 1$. The removal constants are chosen as a) $\delta_r = \hat{\gamma}_r = 0$ and b) $\delta_r = \hat{\gamma}_r = 0.001$.

curve $t = \infty$ in Fig. 1a). Due to our assumption $\hat{g}_1 \hat{g}_4 = -5/2$ the distribution function does not violate positivity for $0 < t < \infty$ and $|v| < \infty$ – in contrast to the original BKW-mode where $\hat{g}_1 \hat{g}_4 = -1$. A slight change in the values of the removal parameters (here by an amount of 1/1000) leads to a completely different situation where particle and energy density are no longer conserved. Since the t.p. density $M_0(t; -6, -6)$ vanishes for $\hat{\gamma}_r, \delta_r \neq 0$, according to Eqs. (24) and (17), the distribution function does not reach a Maxwellian (as in Fig. 1a) but tends, rather, towards zero in the limit $t \to \infty$. Due to the fact $h(v, \infty) = v^2 f(v, \infty) \equiv 0$ it is clear that the normalization $f(v, t)/f(v, \infty)$, known from the Krook and Wu study, does not make sense in the presence of removal effects.

4.3 Distribution function for $n = 1$ in class C_2

The set of the group constants is now $(\alpha, \beta, \gamma, \delta) = (4, 5, 1, \delta)$. For $\hat{\gamma}_r, \delta_r \neq 0$ the following set of moments is obtained:

$$M_k(t) = \frac{M_0(t; 4, 5)}{5} \{6\, q(t)^k (k + 1 - k\, q(t)^{-1}) - 1\} (1 - \delta)^k \tag{31a}$$

where

$$q(t) = 1 - \hat{g}_1 \hat{g}_4 \left(\left\{ 1 + (1 - e^{-\hat{\gamma}_r t}) \frac{1 + 5\delta_r}{5\hat{\gamma}_r} \right\}^{1/(1+5\delta_r)} , \right. \tag{31b}$$

and the corresponding source moments (written in terms of M_0) are

$$Q_k(t) = -\frac{C_s}{5} M_0^2(t; 4, 5)(1-\delta)^k. \tag{32}$$

Then the function

$$f(v, t) = \frac{3 M_0(t; 4, 5)}{5} \left[\left\{ \left(\frac{3}{q(t)} - 1 \right) + \left(\frac{2}{q(t)} - \frac{2}{q^2(t)} \right) \frac{v^2}{v_0^2(1-\delta)} \right\} \right.$$

$$\left. \cdot\, m(v; q(t)(1-\delta)) - \frac{1}{3} m(v; 1-\delta) \right] \tag{33a}$$

is the exact solution of the Boltzmann Eq. (1) for

$$Q(v, t) = -\frac{C_s}{5} M_0^2(t; 4, 5)\, m(v; 1-\delta) \tag{33b}$$

which is, in fact, a sink term, because $C_s > 0$.

A first insight to the physical content of these equations can be obtained as follows: We think of a concrete two-component system with all characteristic constants being fixed such that the removal parameters $\hat{\gamma}_r$, δ_r are strictly positive. Furthermore we assume the initial t.p. density as $M_0(t=0) = q_0 = 1$, and (for sake of simplicity) we may suppose that the source subtracts t.p. at constant temperature T_0 (corresponding to $\delta = 0$). Under these conditions (i) the time evolution of the t.p. density, and (ii) the behavior of the source term (33b), separable in v and t, are completely determined (in the sense that these two functions contain no arbitrary constants). The initial distribution of t.p. $f(v, 0)$, however, depends on the remaining free constants $\hat{g}_1, \hat{g}_4$ since $q(t=0) = 1 - \hat{g}_1 \hat{g}_4$. Therefore, the only way of changing the initial velocity distribution of t.p. in the present case consists of varying $\hat{g}_1 \hat{g}_4$. For $t \neq 0$ the auxiliary function $q(t)$ as well as the distribution function depend on the product $\hat{g}_1 \hat{g}_4$. Hence, there arises the question whether $\hat{g}_1 \hat{g}_4$ can be chosen such that the similarity solution (33a) of the BE is physically acceptable, i.e. a non-negative function of the variables v and t for any choice $|v| \leqq \infty$ and $t \geqq 0$. With respect to the discussion of this problem we proceed as follows: Leaving aside the trivial case $\hat{g}_1 \hat{g}_4 = 0$, we show in a first step that the inequality $f \geqq 0$ is violated in the limit $|v| \to \infty$ for any choice $\hat{g}_1 \hat{g}_4 > 0$. Hence, the choice $\hat{g}_1 \hat{g}_4 > 0$ is physically irrelevant. In the opposite case $\hat{g}_1 \hat{g}_4 < 0$ the inequality $f \geqq 0$ may be violated. Therefore, we aim at finding a sufficient condition – an inequality between the quantities $\hat{g}_1 \hat{g}_4$, $\hat{\gamma}_r$, δ_r alone – such that $f \geqq 0$ for any $|v| \leqq \infty$ and $t \geqq 0$.

As to the first step we note that the second argument in the generalized Maxwellians, occuring in Eq. (33a) and Eq. (33b), must be positive for physical reasons. Hence, we have to choose the group constant δ in general such that $\delta > 1$, and for the auxiliary function, defined in Eq. (31b), we have to require

$q(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) > 0$. For any choice $t, \hat{\gamma}_r, \delta_r \geqq 0$ one can see that there exist upper and lower bounds for q, according to the sign of $\hat{g}_1 \hat{g}_4$, namely (i) $q(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) < 1$ for $\hat{g}_1 \hat{g}_4 > 0$ and (ii) $q(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) > 1$ for $\hat{g}_1 \hat{g}_4 < 0$. Analyzing the high speed limit of the distribution function (33 a) and observing the condition $\lim_{v \to \infty} f(v, t) \geqq 0$, one can rule out case (i).

Even for $\hat{g}_1 \hat{g}_4 < 0$, which will be assumed now, the condition $f \geqq 0$ is not met for all choices of $\hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r$ and v, t. In order to obtain a sufficient condition for $f \geqq 0$ in the whole range $|v| \leqq \infty, t \geqq 0$ we start with the following observation: The auxiliary function q, Eq. (31 b), has a positive lower bound, i.e. is strictly positive for the allowed removal parameters $\hat{\gamma}_r, \delta_r \geqq 0$ and for $t \geqq 0$. Therefore, we may rewrite Eq. (33 a) in the following form, which is convenient for discussing positivity:

$$f(v, t) = \tfrac{1}{5} M_0(t; 4, 5) q(t)^{3/2} m(v; q(t)(1-\delta)) \chi(v^2, t), \tag{34}$$

with

$$\chi(v^2, t) = a(t) + b(t) X(v^2, t) - \exp\{-X(v^2, t)\},$$

and the auxiliary functions

$$X(v^2, t) = \frac{v^2}{v_0^2(1-\delta)}\left(1 - \frac{1}{q(t)}\right), \tag{35a}$$

$$a(t) = \frac{3}{q(t)^{3/2}}\left(\frac{3}{q(t)} - 1\right), \quad b(t) = \frac{6}{q(t)^{5/2}}. \tag{35b}$$

Each of the first three functions on the *rhs* of Eq. (34) is non-negative for any choice $t, \hat{\gamma}_r, \delta_r \geqq 0$. Hence, $\chi(v^2, t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq 0$ is necessary and sufficient for f to be non-negative. Since Eq. (35 a) allows to express v^2 in terms of X we see that $\chi(X, t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq 0$ is also necessary and sufficient for f to be non-negative. The implications of this inequality can be worked out easily in the limiting cases, where $X \to 0$ and $X \to \infty$, respectively: one obtains two inequalities $a(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq 1$ and $q(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq 1$, respectively. The last inequality is met whenever $\hat{g}_1 \hat{g}_4 < 0$. In order to find conditions, where the first inequality is also met, we note the following property $a(0; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq a(t; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq a(\infty; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r)$, holding for $\hat{g}_1 \hat{g}_4 < 0$. Hence, by assuming $a(\infty, \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) \geqq 1$ we ensure that both inequalities are met. Observing the relation between a and q, Eq. (35 b), the last inequality is equivalent to

$$q(\infty; \hat{g}_1 \hat{g}_4, \hat{\gamma}_r, \delta_r) = 1 - \hat{g}_1 \hat{g}_4 \left\{1 + \frac{1 + 5\delta_r}{5\hat{\gamma}_r}\right\}^{1/(1+5\delta_r)} \leqq \bar{q}, \tag{36}$$

where $\bar{q}$ is given approximately by 1.715374931.

The above conditions are sufficient for χ and f to be non-negative in the limits $X \to 0$ and $X \to \infty$ (corresponding to low- and high-speed limits of χ, f).

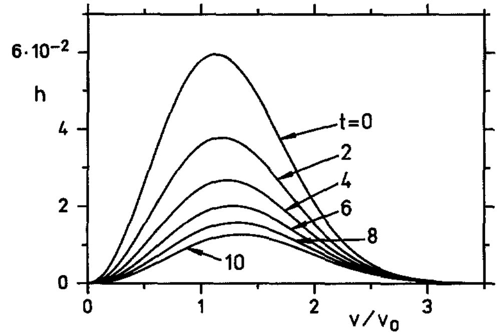

Figure 2
The distribution function $h = v^2 f$ (in units of v_0^3) versus v/v_0, Eq. (33 a), for various times t and the following values of parameters: $\delta_r = 0, \hat{\gamma}_r = 0.05, \hat{g}_1 \hat{g}_4 = -0.1, \delta = 0, q_0 = 1$.

A more detailed investigation shows that these conditions ensure also $\chi \geqq 0$ in the whole range $0 \leqq X \leqq \infty$ and for all $t \geqq 0$. Hence, in the case $\hat{g}_1 \hat{g}_4 < 0$ the inequality (36) represents a sufficient condition on χ and f to be non-negative functions of the physical variables v and t. In contrast to the previous examples positivity of $f(v, t)$ requires specific restrictions on both, removal parameters $\hat{\gamma}_r, \delta_r$ and the free constants $\hat{g}_1 \hat{g}_4$.

As an example illustrating the behavior of the one-particle distribution function, Eq. (33a), we have plotted $h(v, t) \equiv v^2 f(v, t)$ versus v/v_0 in Fig. 2 for various times and a set of parameters where the above conditions are satisfied.

5. Conclusions

In this paper we have studied the problem of constructing exact homogeneous distributions of t.p. in a host medium on the basis of a nonlinear BE. The difficulties in treating analytically the influence of an external source upon these distributions may be overcome at least in certain cases of interest where (i) the BE can be transformed into a PDE, and (ii) the source term in the BE is such that the associated PDE admits a Lie group. We have focused attention to a particular type of source term in the BE involving 4 Lie group constants. To each allowed source term, obtained by obseving certain selection rules for the group constants, there corresponds a specific kind of distribution function. These homogeneous distributions, constructed via similarity methods, may be called similarity solutions of the BE. They can be arranged in two distinct infinite sequences determined by Eq. (24). It should be noticed here that the term "similarity solution" is also used in the literature in a different context – namely for spatially inhomogeneous solutions (of other types of Boltzmann equations) which depend on the moving wave variable $\boldsymbol{r} - \boldsymbol{v}t$ or on the more general variable $\gamma(\boldsymbol{r}, t)(\boldsymbol{v} - \boldsymbol{v}_0(\boldsymbol{r}, t))$ [8]. The three examples, studied in the present work, give a first insight to possible homogeneous distributions of t.p. in the host medium. In our considerations the (generalized) BKW mode appears as a special member in one of the infinite sequences of possible distributions when the source term vanishes (class C_1, $n = 0$). For non-zero source terms we have demonstrated that the similarity solutions contained in the classes C_1 and C_2 ($n = 0, 1$) are markedly different from the well-known BKW-type. Sinces these distributions do not violate the requirement of positivity – when removal effects are present in the system – it seems reasonable to continue the analysis for higher n in order to improve our understanding of these two sequences of similarity solutions.

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Abstract

The diffusion of Maxwell molecules in an unbounded host medium is considered in the case where the particle density is not conserved in consequence of removal events and through the influence of an external source. Heretofore undiscovered classes of similarity solutions of the corresponding nonlinear isotropic and homogeneous Boltzmann equation are presented and already discovered special solutions are classified in terms of Lie group constants and removal parameters. The conditions under which these homogeneous distribution functions do not violate positivity are discussed, and the restrictions upon the parameters of the underlying collision model are determined.

Zusammenfassung

Die Diffusion von Maxwellmolekülen in einem unbegrenzten Hintergrundmedium wird untersucht für den Fall, daß die Teilchenzahldichte aufgrund von „removal"-Effekten und durch den Einfluß einer äußeren Quelle keine Erhaltungsgröße ist. Bisher unbekannte Klassen von Similarity-Lösungen der entsprechenden nichtlinearen, homogenen und isotropen Boltzmanngleichung werden hergeleitet und bereits bekannte spezielle Lösungen werden anhand von Lie-Gruppenkonstanten und removal-Parametern klassifiziert. Die Bedingungen, unter denen diese homogenen Verteilungsfunktionen positiv sind, werden diskutiert, und die Einschränkungen an die Parameter des zugrundeliegenden Stoßmodells werden bestimmt.

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