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On finite deformations of space-curved beams

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Introduction

We are concerned in what follows with the manner of derivation and with an application of large-displacement finite-strain theory of space-curved beams, as previously considered in [5].

In regard to the manner of derivation we have two objects. One of these is of an expository nature, with a clarification of the way in which our descriptions of the state of strain and of the state of stress are shown to be consistent without the necessity of a "tentative assumption of an implicit representation of force strains". The other is an approach to the problem of relations for components of moment strain in terms of components of rotational displacement, without use of Rodriguez' formula, in a way which involves a symmetric treatment of the two components of bending strain without a participation in this of the one component of twisting strain.

As an example of application of the general theory we consider the problem of helical deformations of a helical rod for the case of a simply symmetric cross section with unequal principal bending stiffnesses and with non-coincident centroid and shear center locations, in generalization of an analysis in Love's Treatise [3].

Vectorial one-dimensional equilibrium equations and virtual strain displacement relations

We have as equations of equilibrium for a cross sectional force $\boldsymbol{P}$ and a cross sectional moment $\boldsymbol{M}$ the two vectorial relations

$$\boldsymbol{P}' + \boldsymbol{p} = 0, \qquad \boldsymbol{M}' + \boldsymbol{R}' \times \boldsymbol{P} + \boldsymbol{m} = 0\,, \tag{1 a, b}$$

with primes indicating differention with respect to arc length s along the *un*deformed "center" line of the rod, and with $\boldsymbol{R} = \boldsymbol{R}(s)$ being the radius vector to points of the deformed center line.

We obtain vectorial virtual strain displacement relations for a force strain vector γ and a moment strain vector $\varkappa$ in terms of virtual translational and rotational displacement components $\delta\boldsymbol{R}$ and $\delta\boldsymbol{\Phi}$, as in [5], through use of the virtual work equation

$$\int_{s_1}^{s_2} (\boldsymbol{P}\cdot\delta\boldsymbol{\gamma}+\boldsymbol{M}\cdot\delta\boldsymbol{\varkappa})\,\mathrm{d}s=\int_{s_1}^{s_2}(\boldsymbol{p}\cdot\delta\boldsymbol{R}+\boldsymbol{m}\cdot\delta\boldsymbol{\Phi})\,\mathrm{d}s+(\boldsymbol{P}\cdot\delta\boldsymbol{R}+\boldsymbol{M}\cdot\delta\boldsymbol{\Phi})_{s_1}^{s_2}, \tag{2}$$

in conjunction with the equilibrium equations (1), in the form

$$\delta\boldsymbol{\gamma}=(\delta\boldsymbol{R})'+\boldsymbol{R}'\times\delta\boldsymbol{\Phi},\qquad \delta\boldsymbol{\varkappa}=(\delta\boldsymbol{\Phi})'. \tag{3a, b}$$

In this we have that $(\delta\boldsymbol{R})'=\delta(\boldsymbol{R}')$, but we cannot write $\delta(\boldsymbol{\Phi}')$ in place of $(\delta\boldsymbol{\Phi})'$, inasmuch as we do not have the existence of a function $\boldsymbol{\Phi}$ in association with the stipulated $\delta\boldsymbol{\Phi}$.

Derivation of scalar strain displacement relations and equilibrium equations

Given the radius vectors $\boldsymbol{r}(s)$ and $\boldsymbol{R}(s)$ to the undeformed and the deformed center lines, respectively, we introduce in association with these two radius vectors two triads of mutually perpendicular unit vectors $(\boldsymbol{t},\boldsymbol{n}_1,\boldsymbol{n}_2)$ and $(\boldsymbol{T},\boldsymbol{N}_1,\boldsymbol{N}_2)$. In this $\boldsymbol{t}$ is tangent and $\boldsymbol{n}_1,\boldsymbol{n}_2$ are perpendicular to the curve $\boldsymbol{r}(s)$ but no such stipulation is made relative to the triad $(\boldsymbol{T},\boldsymbol{N}_1,\boldsymbol{N}_2)$ and the curve $\boldsymbol{R}(s)$, with the determination of $\boldsymbol{R}$ *and* $\boldsymbol{T},\boldsymbol{N}_1,\boldsymbol{N}_2$ being part of the problem of the rod, in a manner which will become apparent.

In order to derive from the vectorial virtual strain displacement relations (3) actual scalar strain displacement relations we now take $\delta\boldsymbol{\gamma}$ and $\delta\boldsymbol{\varkappa}$ in the form

$$\delta\boldsymbol{\gamma}=\boldsymbol{T}\,\delta\gamma_t+\boldsymbol{N}_i\,\delta\gamma_i,\qquad \delta\boldsymbol{\varkappa}=\boldsymbol{T}\,\delta\varkappa_t+\boldsymbol{T}\times\boldsymbol{N}_i\,\delta\varkappa_i, \tag{4a, b}$$

and $\boldsymbol{R}'$ in the form

$$\boldsymbol{R}'=a_t\,\boldsymbol{T}+a_i\,\boldsymbol{N}_i, \tag{5}$$

with the choice of a_t, a_i left open. In addition to this we define virtual triad vectors $\delta\boldsymbol{T}$, $\delta\boldsymbol{N}_i$ in terms of the virtual rotational displacement $\delta\boldsymbol{\Phi}$ in the form

$$\delta\boldsymbol{T}=\delta\boldsymbol{\Phi}\times\boldsymbol{T},\qquad \delta\boldsymbol{N}_i=\delta\boldsymbol{\Phi}\times\boldsymbol{N}_i, \tag{6}$$

with (6) implying the supplementary relation

$$2\,\delta\boldsymbol{\Phi}=\boldsymbol{T}\times\delta\boldsymbol{T}+\boldsymbol{N}_i\times\delta\boldsymbol{N}_i. \tag{7}$$

Introduction of (4a), (5) *and* (6) into (3a) leaves, after appropriate cancellations,

$$\boldsymbol{T}\,\delta\gamma_t+\boldsymbol{N}_i\,\delta\gamma_i=\boldsymbol{T}\,\delta a_t+\boldsymbol{N}_i\,\delta a_i, \tag{8}$$

and therewith,

$$\delta a_t=\delta\gamma_t,\qquad \delta a_i=\delta\gamma_i. \tag{9a}$$

In order to reduce equation (3b) in a corresponding manner we first deduce from (7) the relation

$$(\delta \boldsymbol{\Phi})' = \frac{1}{2} [\boldsymbol{T}' \times \delta \boldsymbol{T} + \boldsymbol{N}_i' \times \delta \boldsymbol{N}_i + \boldsymbol{T} \times \delta(\boldsymbol{T}') + \boldsymbol{N}_i \times \delta(\boldsymbol{N}_i')] . \tag{10}$$

To proceed further we make use of Frenet-type differentiation formulas

$$\boldsymbol{T}' = \frac{\boldsymbol{N}_1}{r_1} + \frac{\boldsymbol{N}_2}{r_2}, \quad \boldsymbol{N}_1' = -\frac{\boldsymbol{T}}{r_1} + \frac{\boldsymbol{N}_2}{r_t}, \quad \boldsymbol{N}_2' = -\frac{\boldsymbol{T}}{r_2} - \frac{\boldsymbol{N}_1}{r_t} . \tag{11}$$

Introduction of (4b), (10) and (11), with

$$\delta \boldsymbol{T}' = \frac{\delta \boldsymbol{N}_1}{r_1} + \frac{\delta \boldsymbol{N}_2}{r_2} + \boldsymbol{N}_1 \, \delta\left(\frac{1}{r_1}\right) + \boldsymbol{N}_2 \, \delta\left(\frac{1}{r_2}\right), \quad \text{etc.} \tag{12}$$

into equation (3b) leaves, after some cancellations, three scalar relations of the form

$$\delta(1/r_t) = \delta \varkappa_t , \qquad \delta(1/r_i) = \delta \varkappa_i . \tag{9b}$$

Having equations (9a, b), in conjunction with the relations $\boldsymbol{t} = \boldsymbol{r}'(s)$, and

$$\boldsymbol{t}' = \frac{\boldsymbol{n}_1}{\varrho_1} + \frac{\boldsymbol{n}_2}{\varrho_2}, \quad \boldsymbol{n}_1' = -\frac{\boldsymbol{t}}{\varrho_1} + \frac{\boldsymbol{n}_2}{\varrho_t}, \quad \boldsymbol{n}_2' = -\frac{\boldsymbol{t}}{\varrho_2} - \frac{\boldsymbol{n}_1}{\varrho_t} \tag{13}$$

we conclude, on the basis of the fact that $a_t = 1$ and $a_i = 0$ when $\boldsymbol{R} = \boldsymbol{r}$, and $\varkappa_t = \varkappa_i = 0$ when $\boldsymbol{T} = \boldsymbol{t}$ and $\boldsymbol{N}_i = \boldsymbol{n}_i$, that we will have, as expressions for the coefficients a_t and a_i in (5) and for the coefficients $1/r_t$ and $1/r_i$ in (11),

$$a_t = 1 + \gamma_t , \qquad a_i = \gamma_i , \tag{14a}$$

and

$$\frac{1}{r_t} = \varkappa_t + \frac{1}{\varrho_t}, \qquad \frac{1}{r_i} = \varkappa_i + \frac{1}{\varrho_i} . \tag{14b}$$

At the same time we have from equations (1a, b) in conjunction with the representations

$$\boldsymbol{P} = P_t \boldsymbol{T} + P_i \boldsymbol{N}_i , \qquad \boldsymbol{M} = M_t \boldsymbol{T} + M_i \boldsymbol{T} \times \boldsymbol{N}_t , \tag{15a, b}$$

and the corresponding representations for $\boldsymbol{p}$ and $\boldsymbol{m}$, as scalar equations of equilibrium

$$P_1' + \frac{P_t}{r_1} - \frac{P_2}{r_t} + p_1 = 0 , \qquad P_2' + \frac{P_t}{r_2} + \frac{P_1}{r_t} + p_2 = 0 , \tag{16a, b}$$

$$P_t' - \frac{P_1}{r_1} - \frac{P_2}{r_2} + p_t = 0 , \qquad M_t' + \frac{M_2}{r_1} - \frac{M_1}{r_2} + \gamma_1 P_2 - \gamma_2 P_1 + m_t = 0 , \tag{16c, d}$$

$$M_1' + \frac{M_t}{r_2} - \frac{M_2}{r_t} + (1 + \gamma_t) \, P_1 - \gamma_1 P_t + m_1 = 0 , \tag{16e}$$

$$M_2' - \frac{M_t}{r_1} + \frac{M_1}{r_t} + (1 + \gamma_t) \, P_2 - \gamma_2 P_t + m_2 = 0 , \tag{16f}$$

with these differing from the corresponding equations of Kirchhoff [2] by way of the presence of the force-deformational terms γP.

Strain components in terms of rotational displacement measures

In order to take account of the duality properties of the two unit normal vectors $\boldsymbol{n}_i$ which sets these part from the one unit tangent vector $\boldsymbol{t}$ we proceed as follows to introduce three scalar rotational displacement parameters φ_1, φ_2 and φ_t.

We first introduce two mutually perpendicular unit vectors $\hat{\boldsymbol{N}}_i$, symmetrically in terms of φ_1 and φ_2, by writing

$$\alpha_1 \hat{\boldsymbol{N}}_1 = \boldsymbol{n}_1 - \varphi_1 \boldsymbol{t} - \beta \boldsymbol{n}_2, \qquad \alpha_2 \hat{\boldsymbol{N}}_2 = \boldsymbol{n}_2 - \varphi_2 \boldsymbol{t} - \beta \boldsymbol{n}_1, \tag{17}$$

in conjunction with the defining relations

$$\alpha_i^2 = 1 + \varphi_i^2 + \beta^2, \qquad 2\beta = \varphi_1 \varphi_2. \tag{18}$$

We next take the triad vector $\boldsymbol{T}$ in the form $\boldsymbol{T} = \hat{\boldsymbol{N}}_1 \times \hat{\boldsymbol{N}}_2$, by writing

$$\alpha_1 \alpha_2 \boldsymbol{T} = (1 - \beta^2)\, \boldsymbol{t} + (\varphi_1 + \beta \varphi_2)\, \boldsymbol{n}_1 + (\varphi_2 + \beta \varphi_1)\, \boldsymbol{n}_2. \tag{19}$$

We thereafter introduce the third rotational displacement parameter φ_t by writing as expressions for $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$

$$\alpha_t \boldsymbol{N}_1 = \hat{\boldsymbol{N}}_1 + \varphi_t \hat{\boldsymbol{N}}_2, \qquad \alpha_t \boldsymbol{N}_2 = \hat{\boldsymbol{N}}_2 - \varphi_t \hat{\boldsymbol{N}}_1; \qquad \alpha_t^2 = 1 + \varphi_t^2. \tag{20}$$

Having equations (19) and (20) we may obtain expressions for r_1, r_2 and r_t, for arbitrarily large φ_i and φ_t, by comparing the expressions for $\boldsymbol{T}'$ and $\boldsymbol{N}_i'$ which follow from (19) and (20) with the corresponding expressions in (11), in conjunction with equations (13). We are limiting ourselves in this account for simplicity's sake to the case of *small finite* φ_i and φ_t by stating the appropriate results including all first and second degree terms but neglecting third and higher degree terms in φ_1, φ_2 and φ_t and the derivatives of these quantities.

With this we have then as expressions for $\boldsymbol{T}$, $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$

$$\boldsymbol{T} = (1 - \tfrac{1}{2}\varphi_1^2 - \tfrac{1}{2}\varphi_2^2)\, \boldsymbol{t} + \varphi_1 \boldsymbol{n}_1 + \varphi_2 \boldsymbol{n}_2, \tag{19'}$$

and

$$\boldsymbol{N}_1 = (1 - \tfrac{1}{2}\varphi_1^2 - \tfrac{1}{2}\varphi_t^2)\, \boldsymbol{n}_1 - (\varphi_1 + \varphi_t \varphi_2)\, \boldsymbol{t} + (\varphi_t - \tfrac{1}{2}\varphi_1 \varphi_2)\, \boldsymbol{n}_2,$$

$$\boldsymbol{N}_2 = (1 - \tfrac{1}{2}\varphi_2^2 - \tfrac{1}{2}\varphi_t^2)\, \boldsymbol{n}_2 - (\varphi_2 - \varphi_t \varphi_1)\, \boldsymbol{t} - (\varphi_t + \tfrac{1}{2}\varphi_1 \varphi_2)\, \boldsymbol{n}_1, \tag{20'}$$

and from this follows, upon again omitting all third and higher degree terms

$$\varkappa_1 = \frac{1}{r_1} - \frac{1}{\varrho_1} = \varphi_1' + \varphi_t \varphi_2' - \frac{\varphi_2 - \varphi_1 \varphi_t}{\varrho_t} + \frac{\varphi_t + \frac{1}{2}\varphi_1 \varphi_2}{\varrho_2} - \frac{\varphi_2^2 + \varphi_t^2}{2\,\varrho_1}, \tag{21 a}$$

$$\varkappa_2 = \frac{1}{r_2} - \frac{1}{\varrho_i} = \varphi_2' - \varphi_t \varphi_1' + \frac{\varphi_1 + \varphi_2 \varphi_t}{\varrho_t} - \frac{\varphi_t - \frac{1}{2}\varphi_1 \varphi_2}{\varrho_1} - \frac{\varphi_1^2 + \varphi_t^2}{2\,\varrho_2}, \tag{21 b}$$

$$\varkappa_t = \frac{1}{r_t} - \frac{1}{\varrho_t} = \varphi_t' + \frac{\varphi_2 \varphi_1' - \varphi_1 \varphi_2'}{2} + \frac{\varphi_2}{\varrho_1} - \frac{\varphi_1}{\varrho_2} - \frac{\varphi_1^2 + \varphi_2^2}{2\,\varrho_t}. \tag{21 c}$$

As regards the components of force strain γ_t and γ_i we obtain, on the basis of equations (5), (14a), (19′) and (20′) and with $\boldsymbol{R}' = \boldsymbol{t} + \boldsymbol{u}'$, the vectorial relation

$$\begin{aligned} \boldsymbol{u}' = {} & (\gamma_t - \tfrac{1}{2}\varphi_1^2 - \tfrac{1}{2}\varphi_2^2 - \gamma_1 \varphi_1 - \gamma_2 \varphi_2)\,\boldsymbol{t} \\ & + [(1+\gamma_t)\,\varphi_1 + \gamma_1 - \gamma_2 \varphi_t]\,\boldsymbol{n}_1 + [(1+\gamma_t)\,\varphi_2 + \gamma_2 + \gamma_1 \varphi_t]\,\boldsymbol{n}_2 \end{aligned} \tag{22}$$

for the determination of components of translational displacements in terms of force strains and rotational displacement measures. Alternately, we may obtain force strains in terms of translational and rotational displacement measures in the form

$$\gamma_t = (\boldsymbol{t} + \boldsymbol{u}') \cdot \boldsymbol{T} - 1, \qquad \gamma_i = (\boldsymbol{t} + \boldsymbol{u}') \cdot \boldsymbol{N}_i, \tag{23 a, b}$$

with the form of the final formulas depending on the nature of the component representation for $\boldsymbol{u}$.

Helical deformations of a helical rod

We now consider a slightly generalized version of the classical problem of a helical rod, acted upon by forces P the line of action of which coincides with the axis of helix, and by moments M turning about this axis, with this condition of loading sometimes being designated as a "wrench", with the axis of the wrench coinciding with the axis of the helix. We observe that for this system of loading we have as explicit solution of the equilibrium equations (1 a, b) the expressions

$$\boldsymbol{P} = P\,\boldsymbol{e}_3, \qquad \boldsymbol{M} = M\,\boldsymbol{e}_3 - P\,\boldsymbol{R} \times \boldsymbol{e}_3, \tag{24 a, b}$$

where it remains to determine the shape of the deformed rod, in terms of the geometrical parameters describing the undeformed rod, and in terms of the loads P and M.

With r and θ being polar coordinates in the plane perpendicular to the axis of the helix and with a and b indicating radius and rise of the center line curve, we have as vector equation of the center line of the undeformed rod

$$\boldsymbol{r} = a\, \boldsymbol{e}_r + b\, \theta\, \boldsymbol{e}_3, \qquad \boldsymbol{e}_r = \boldsymbol{e}_1 \cos\theta + \boldsymbol{e}_2 \sin\theta \tag{25}$$

and from this we deduce as expression for the tangent unit vector $\boldsymbol{t}$, with the help of the relation $\mathrm{d}s = (a^2 + b^2)^{1/2}\, \mathrm{d}\theta = c\, \mathrm{d}\theta$, and with $a = c\cos\varphi$, $b = c\sin\varphi$,

$$\boldsymbol{t} = \boldsymbol{r}' = \cos\varphi\, \boldsymbol{e}_\theta + \sin\varphi\, \boldsymbol{e}_3, \qquad \boldsymbol{e}_\theta = -\,\boldsymbol{e}_1 \sin\theta + \boldsymbol{e}_2 \cos\theta\,. \tag{26}$$

As regards the normal unit vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ we first introduce a special set $\hat{n}_1$ and $\hat{n}_2$ of unit vectors by writing

$$\hat{n}_1 = \boldsymbol{e}_r, \qquad \hat{n}_2 = \boldsymbol{e}_\theta \sin\varphi - \boldsymbol{e}_3 \cos\varphi\,, \tag{27}$$

with $\boldsymbol{t}$, $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$ evidently being mutually perpendicular, and by then writing in in terms of an angle ψ,

$$\boldsymbol{n}_1 = \hat{\boldsymbol{n}}_1 \cos\psi + \hat{\boldsymbol{n}}_2 \sin\psi, \qquad \boldsymbol{n}_2 = \hat{\boldsymbol{n}}_2 \cos\psi - \hat{\boldsymbol{n}}_1 \sin\psi\,, \tag{28}$$

with the directions of $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ coinciding with the principal axes in the plane of the cross section of the rod.

Given equations (26) and (28) as defining relations for the triad $\boldsymbol{t}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ it is then readily established that the coefficients in the differentiation formulas (13) are

$$\frac{1}{\varrho_1} = -\frac{\cos\varphi\cos\psi}{c}, \qquad \frac{1}{\varrho_2} = \frac{\cos\varphi\sin\psi}{c}, \qquad \frac{1}{\varrho_t} = \frac{\sin\varphi}{c}\,. \tag{29}$$

Given equation (25) for the undeformed center line we now write the corresponding relation for the deformed center line as

$$\boldsymbol{R} = A\, \boldsymbol{e}_R + B\, \Theta\, \boldsymbol{e}_3, \qquad \boldsymbol{e}_R = \boldsymbol{e}_1 \cos\Theta + \boldsymbol{e}_2 \sin\Theta\,, \tag{30}$$

with Θ being given in terms of θ in the form $\Theta = k\,\theta$. We then have from this

$$\boldsymbol{R}' = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}s} = \frac{k}{c}\,(A\, \boldsymbol{e}_\Theta + B\, \boldsymbol{e}_3)\,, \qquad \boldsymbol{e}_\Theta = \boldsymbol{e}_2 \cos\Theta - \boldsymbol{e}_1 \sin\Theta\,, \tag{31}$$

and it now remains to define unit vectors $\boldsymbol{T}$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ associated with the radius vector $\boldsymbol{R}$ as given by (30).

We will in what follows restrict attention to cases for which $\boldsymbol{T}$ is tangent to the $\boldsymbol{R}$-curve by writing

$$\boldsymbol{T} = \cos\Phi\, \boldsymbol{e}_\Theta + \sin\Phi\, \boldsymbol{e}_3\,, \tag{32}$$

where $\cos\Phi = kA/C$ and $\sin\Phi = kB/C$, with $C = k(A^2 + B^2)^{1/2}$, and therewith

$$\boldsymbol{R}' = \frac{C}{c}\,\boldsymbol{T}\,, \tag{33}$$

so that, in accordance with (5) and (14a),

$$\gamma_t = \frac{C}{c} - 1, \qquad \gamma_1 = \gamma_2 = 0. \tag{34}$$

Having $\boldsymbol{T}$ as in (32) we now define vectors $\hat{\boldsymbol{N}}_i$ and $\boldsymbol{N}_i$, consistent with (27) and (28), in the form

$$\hat{\boldsymbol{N}}_1 = \boldsymbol{e}_R, \qquad \hat{\boldsymbol{N}}_2 = \boldsymbol{e}_\Theta \sin \Phi - \boldsymbol{e}_3 \cos \Phi, \tag{35}$$

and

$$\boldsymbol{N}_1 = \hat{\boldsymbol{N}}_1 \cos \Psi + \hat{\boldsymbol{N}}_2 \sin \Psi, \qquad \boldsymbol{N}_2 = \hat{\boldsymbol{N}}_2 \cos \Psi - \hat{\boldsymbol{N}}_1 \sin \Psi. \tag{36}$$

With $\boldsymbol{T}$ as in (32) and $\boldsymbol{N}_i$ as in (36) we obtain as expressions for the coefficients in the differentiation formulas (11),

$$\frac{1}{r_1} = -k \frac{\cos \Phi \cos \Psi}{c}, \qquad \frac{1}{r_2} = k \frac{\cos \Phi \sin \Psi}{c}, \qquad \frac{1}{r_t} = k \frac{\sin \Phi}{c}, \tag{37}$$

and therewith, in accordance with equations (14b), as expression for bending strains $\varkappa_i$ and twisting strain $\varkappa_t$

$$\varkappa_1 = -c^{-1}(k \cos \Phi \cos \Psi - \cos \varphi \cos \psi), \tag{38a}$$

$$\varkappa_2 = c^{-1}(k \cos \Phi \sin \Psi - \cos \varphi \sin \psi), \tag{38b}$$

$$\varkappa_t = c^{-1}(k \sin \Phi - \sin \varphi). \tag{38c}$$

Formulation of stress strain relations

We assume that the cross section of the rod in its deformed state will be symmetric with respect to an axis parallel to $\boldsymbol{N}_1$ and we will designate cross sectional coordinates in the directions of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ by x_1 and x_2. We further assume that the origin of the x_1, x_2-system defines the center line of the rod which is taken to be the line of shear centers of the cross sections, with the centroids of the cross sections being on the x_1-axis, at a distance x_c from the shear center. Limiting attention to the case of linear stress strain relations we may then immediately write two of the stress strain relations in the form

$$M_t = D_t \varkappa_t, \qquad M_2 = D_2 \varkappa_2. \tag{39a, b}$$

Two additional one-dimensional stress strain relations follow from the integral relations

$$P_t = \int E\, t(\gamma_t + x_1 \varkappa_1)\, dx_1, \qquad M_1 = \int E\, t(\gamma_t + x_1 \varkappa_1)\, x_1\, dx_1. \tag{40a, b}$$

in conjunction with the defining relations

$$\int E\, t\, dx_1 = S, \qquad \int E\, t\, x_1\, dx_1 = x_c S, \qquad \int E\, t\, x_1^2\, dx_1 = x_c^2 S + D_1, \tag{41}$$

where $D_1 = \int E\, t\, (x_1 - x_c)^2 \, \mathrm{d}x_1$, in the form

$$P_t = S\,\gamma_t + x_c\, S\, \varkappa_1 , \qquad M_1 = x_c\, S\, \gamma_t + (x_c^2\, S + D_1)\, \varkappa_1 . \tag{39c, d}$$

The four equations in (39) in conjunction with the four defining relations in (34) and (38) will become a system of four simultaneous equations for the determination of the four quantities Φ, Ψ, k, C in terms of the given geometrical quantities φ, ψ, c and the given loads P and M, upon expressing M_t, M_2, M_1 and P_t in terms of P and M through use of the defining relations which follow from (15), in conjunction with (24), (32) and (36), in the form

$$P_t = \boldsymbol{P} \cdot \boldsymbol{T} = P\, \boldsymbol{e}_3 \cdot \boldsymbol{T} = P \sin \Phi , \tag{42a}$$

$$M_1 = \boldsymbol{M} \cdot \boldsymbol{N}_2 = (M\, \boldsymbol{e}_3 + PA\, \boldsymbol{e}_\Theta) \cdot (\hat{\boldsymbol{N}}_2 \cos \Psi - \hat{\boldsymbol{N}}_1 \sin \Psi)$$
$$= -\, M \cos \Phi \cos \Psi + PA \sin \Phi \cos \Psi , \tag{42b}$$

$$M_2 = -\, \boldsymbol{M} \cdot \boldsymbol{N}_1 = M \cos \Phi \sin \Psi - PA \sin \Phi \sin \Psi , \tag{42c}$$

$$M_t = \boldsymbol{M} \cdot \boldsymbol{T} = M \sin \Phi + PA \cos \Phi . \tag{42d}$$

With $A = k^{-1}\, C \cos \Phi$, in accordance with the defining relations in the text which follow equation (32), we then have altogether as equations for the determination of Φ, Ψ, k and C,

$$P \sin \Phi = S \left[\left(\frac{C}{c} - 1 \right) - \frac{x_c}{c} (k \cos \Phi \cos \Psi - \cos \varphi \cos \psi) \right] , \tag{43a}$$

$$-M \cos \Phi \cos \Psi + \frac{C}{k} P \cos \Phi \sin \Phi \cos \Psi$$
$$= x_c\, S \left(\frac{C}{c} - 1 \right) - \frac{x_c^2\, S + D_1}{c} (k \cos \Phi \cos \Psi - \cos \varphi \cos \psi) , \tag{43b}$$

$$M \cos \Phi \sin \Psi - \frac{C}{k} P \cos \Phi \sin \Phi \sin \Psi$$
$$= \frac{D_2}{c} (k \cos \Phi \sin \Psi - \cos \varphi \sin \psi) , \tag{43c}$$

$$M \sin \Phi + \frac{C}{k} P \cos^2 \Phi = \frac{D_t}{c} (k \sin \Phi - \sin \varphi) . \tag{43d}$$

Among the various special cases of the system (43) we mention the following.

Rod with doubly symmetric cross section

Setting $x_c = 0$ in equations (43a, b) we may use equation (43a) in order to reduce (43b, c, d) to a system of three equations for Φ, Ψ and k, upon setting in these three equations $C = c + S^{-1} P \sin \Phi$. In general, this latter relation will be effectively equivalent to $C = c$, as implied by the original Kirchhoff form of the theory. Strictly speaking, we will have $C = c$ upon stipulating $S = \infty$, with P then being reactive. Aside from the fact that it is possible to imagine cases for which this would not be justified (as for example for a rod with helical spring cross sections) it should be noted that, on the basis of equations (41), the stipulation $S = \infty$ should by rights be associated with a stipulation $D_1 = \infty$, that is with a stipulation of complete circumferential fiber inextensibility. This difficulty, however, may be by-passed by considering the problem with $C = c$ and $D_1 < \infty$ as the first step of a perturbation expansion in powers of P/S.

Rod with kinetically symmetric cross section

By kinetic symmetry we mean, in accordance with Love [3] that, in addition to $x_c = 0$, we have $D_2 = D_1 \equiv D_b$. We obtain the results stated in [3], and there credited to Kelvin and Tait [1], within the present context by recognizing that when $D_2 = D_1$ then part of the solution of the system (43) is given by the relation

$$\Psi = \psi . \tag{44}$$

With this equations (43b) and (43c) are *both* equivalent to the one relation

$$M \cos \Phi - \frac{C}{k} P \cos \Phi \sin \Phi = \frac{D_b}{c} (k \cos \Phi - \cos \varphi) , \tag{45}$$

with (45) and (43d) now being two equations for the determination of Φ and k, in terms of M and P, [with $C = (P/S)\, c \sin \Phi$]. We will limit ourselves here to using (45) and (43d) for the derivation of the set of relations

$$M = \frac{D_t}{c} \sin \Phi (k \sin \Phi - \sin \varphi) + \frac{D_b}{c} \cos \Phi (k \cos \Phi - \cos \varphi) , \tag{46a}$$

$$\frac{C}{k} P \cos \Phi = \frac{D_t}{c} \cos \Phi (k \sin \Phi - \sin \varphi) - \frac{D_b}{c} \sin \Phi (k \cos \Phi - \cos \varphi) , \tag{46b}$$

which may readily be recognized to be equivalent to equations (40) on page 415 in [3], upon setting $C = c$.

Finite pure bending of a circular ring

We obtain equations for this problem upon setting $\varphi = \Phi = 0$, $c = a$ and $P = 0$ in the system (43). We then have (43d) satisfied automatically and equations (43 a, b, c) assume the form

$$\frac{A}{a} - 1 - \frac{x_c}{a}(k \cos \Psi - \cos \psi) = 0 , \tag{47 a}$$

$$M \cos \Psi = \frac{D_1}{a}(k \cos \Psi - \cos \psi), \tag{47 b}$$

$$M \sin \Psi = \frac{D_2}{a}(k \sin \Psi - \sin \psi) . \tag{47 c}$$

Equations (47 b, c) imply as implicit relation for k in terms of M, for given values of a, ψ, D_2 and D_1,

$$\left(\frac{D_2 \sin \psi}{a M - D_2 k}\right)^2 + \left(\frac{D_1 \cos \psi}{a M - D_1 k}\right)^2 = 1 , \tag{48 a}$$

with Ψ given in terms of k and M in the form

$$\frac{\tan \Psi}{\tan \psi} = \frac{D_2}{D_1} \frac{a M - D_1 k}{a M - D_2 k} . \tag{48 b}$$

Bending and twisting of a partially rigid rod

Given a rod with *narrow* cross section, such that $D_2 \ll D_1$, we may consider the problem of bending and twisting approximately, by considering equations (43) subject to the assumptions $D_1 = \infty$ and $S = \infty$. We now have, from (43 a, b), as constraint conditions

$$k \cos \Phi \cos \Psi - \cos \varphi \cos \psi = 0 , \quad C = c, \tag{49 a, b}$$

with k, Φ and Ψ to be determined, in terms of M and P, by means of equations (49 a) and (43 c, d). We note that the special case $\varphi = 0$ of this problem is the one-dimensional analogue of a problem of inextensional bending in two-dimensional shell theory which has been considered in [4].

References

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Summary

A recent generalization of Kirchhoff's equations for the analysis of spacecurved beams, in which account is taken of force-deformational effects in addition to the deformational effects of bending and twisting moments (Studies Appl. Math. *52*, 87–95, 1973), is re-derived more simply, including a new description of rotational displacement states, and including an application to the problem of helical deformations of originally helical beams.

Zusammenfassung

Eine Verallgemeinerung der Kirchhoffschen Stabgleichungen, in welcher der Einfluß der Stabkräfte auf die Verformungen berücksichtigt wird, zusätzlich zu dem Einfluß der Biege- und Torsionsmomente (Studies Appl. Math. *52*, 87–95, 1973) wird neu und vereinfacht abgeleitet, einschließlich einer neuen Beschreibungsweise für rotationelle Verschiebungszustände, und einschließlich einer neuen Anwendung auf das Problem der schraubenförmigen Verformungen von ursprünglich schraubenförmigen Stäben.

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