Journal of Applied Mathematics and Physics (ZAMP)
Vol. 38, July 1987

0044-2275/87/004630-12 $ 3.90/0

Moving Dugdale model *)

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Introduction

It is well-known that the analysis of dynamic fracture for materials with nonlinear behaviour presents fundamental difficulty. The complete solution of this kind of problems is not available at present. The use of some simplified models or approximate analyses for this purpose is beneficial.

The author reports briefly a simplified model named moving Dugdale model for mode I in a letter [1]. This paper gives calculation for the problem for mode I, II and III in detail. The dynamic crack opening, sliding and tearing displacements $\delta_{\mathrm{I}}^{\mathrm{dynamic}}$, $\delta_{\mathrm{II}}^{\mathrm{dynamic}}$ and $\delta_{\mathrm{III}}^{\mathrm{dynamic}}$ are obtained and shown to be significant for describing the dynamic fracture process of materials with nonlinear behaviour.

§ 1. Fundamental assumption

It is assumed that there is a Griffith crack (or a pileup group of dislocations, inclusion or other displacement discontinuity surface) with length $2a$ in an infinite medium. The crack is moving with constant speed v along the direction of axis ox_1 and the body is subjected to a tension $\sigma^{(\infty)}$ along the direction of axis oy far from the crack (see the schematic Fig. 1). In the front of the crack tip, that is, in the zone $y = 0$, $a < |x| < a + R$, the atomic cohesive forces are effective. Suppose further that the cohesive force per unit surface is equal to the yielding stress σ_s of the material along the whole length R which is unknown temporarily. In the above statement there are two coordinate systems, one is the fixed coordinate system (x_1, y, t), the other represents the moving one (x, y). The following relationship holds:

$$x = x_1 - v\,t, \qquad y = y\,. \tag{1}$$

The following analysis will indicate that if $\sigma^{(\infty)} \ll \sigma_s$, then R will tend zero and this model will reduce to the Yoffé problem [2]; when $v \to 0$, the model will reduce to the static Dugdale [3]/Bilby, Cottrell, Swinden [4] problem. With this model the nonlinear dynamic fracture problem can be linearized. The method used either by Yoffé or by Dugdale/Bilby, Cottrell, Swinden can not be directly used to solve this problem. The solving procedure is the one proposed by the author [1].

*) This work is supported by the Research Fellowship of Alexander von Humboldt Foundation, Federal Republic of Germany.

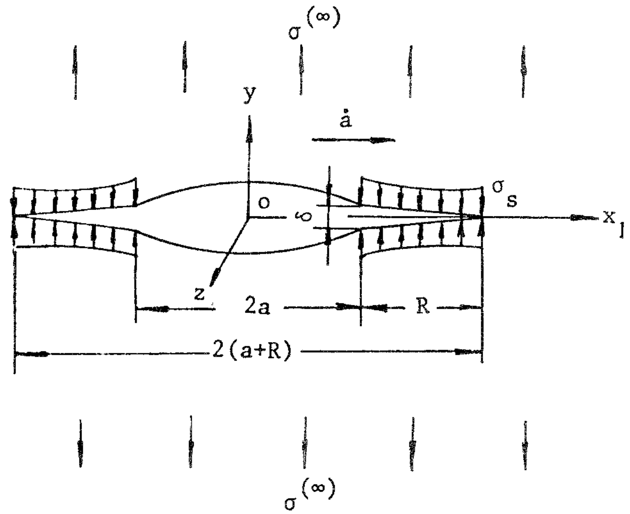

Figure 1

The above description deals only with the problem of mode I, but the problems of mode II and III may be solved with a similar procedure.

The mathematic analysis concerning the above problems is given in detail as follows.

§2. Mathematic analysis for the problem of mode I

1. Basic relations and boundary value problem

Introduce wave functions $\varphi(x_1, y, t)$ and $\psi(x_1, y, t)$ in the fixed coordinate system, that is

$$u_{x_1} = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial y}, \quad u_y = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x_1} \tag{2}$$

where u_{x_1} and u_y are displacements in directions ox_1 and oy respectively, and φ and ψ satisfy the following wave equations

$$\nabla^2 \varphi = \frac{1}{c_1^2}\frac{\partial^2 \varphi}{\partial t^2}, \quad \nabla^2 \psi = \frac{1}{c_2^2}\frac{\partial^2 \psi}{\partial t^2} \tag{3}$$

in which $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial y^2$ is Laplace operator for two dimension, c_1 and c_2 are two velocities of longitudinal and transverse elastic waves, respectively, i.e.

$$c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2} \tag{4}$$

and λ, μ and ϱ are the two Lamé constants and mass density respectively.

Making the Galileo transformation (1), the wave Eqs. (3) will reduce to the Laplace equations in the moving coordinate system (x, y)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_1^2}\right)\varphi = 0, \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_2^2}\right)\psi = 0 \tag{5}$$

where

$$y_1 = \alpha_1 y, \qquad y_2 = \alpha_2 y,$$
$$\alpha_1 = (1 - v^2/c_1^2)^{1/2}, \quad \alpha_2 = (1 - v^2/c_2^2)^{1/2}. \tag{6}$$

The crack problem stated in Sect. 1 (shown by Fig. 1) may be resolved into two problems. One is a problem of moving crack with length $2(a + R)$, and by using superposition it may be presented as the following equivalent boundary value problem in the sense of fracture mechanics (the difference between the original and the equivalent problems is only a constant term which has no contribution for stress singularity and doesn't produce crack opening displacement):

$$y = 0, \quad |x| < a + R\colon \sigma_{yy} = -\sigma^{(\infty)}, \quad \sigma_{xy} = 0;$$
$$(x^2 + y^2)^{1/2} \to \infty\colon \sigma_{ij} = 0. \tag{7}$$

The other one is the problem of the effect of cohesive force zone and may be reduced to the boundary value problem

$$y = 0, \quad a < |x| < a + R\colon \sigma_{yy} = \sigma_s, \quad \sigma_{xy} = 0;$$
$$y = 0, \quad |x| < a\colon \sigma_{yy} = 0, \quad \sigma_{xy} = 0; \tag{8}$$
$$(x^2 + y^2)^{1/2} \to \infty\colon \sigma_{ij} = 0$$

in which σ_{ij} are stress components and stand for the following expressions

$$\sigma_{xx} = \lambda \nabla^2 \varphi + 2\mu \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right),$$
$$\sigma_{yy} = \lambda \nabla^2 \psi + 2\mu \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right), \tag{9}$$
$$\sigma_{xy} = 2\mu \frac{\partial^2 \varphi}{\partial x \partial y} + \mu \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right).$$

2. *Solution of the boundary value problem – the method of function theory*

The solution of boundary value problem (7) may be easily obtained and will be discussed later. The solution of boundary value problem (8) is given in Ref. [1] in brief, it will be discussed here in detail.

Introducing complex variables

$$z_1 = x_1 + i y_1, \quad z_2 = x + i y_2 \tag{10}$$

then the solution of Eqs. (5) may be expressed as [5]

$$\varphi(x, y_1) = [F_1(z_1) + \overline{F_1(z_1)}], \quad \psi(x, y_2) = i[F_2(z_2) - \overline{F_2(z_2)}] \tag{11}$$

where $F_1(z_1)$ and $F_2(z_2)$ are two analytic functions of complex variables z_1 and z_2, and $\overline{F_1(z_1)}$ and $\overline{F_2(z_2)}$ are their complex conjugates, respectively.

Denoting

$$\Phi(z_1) = \frac{dF_1(z_1)}{dz_1} = F_1'(z_1), \quad \Psi(z_2) = \frac{dF_2(z_2)}{dz_2} = F_2'(z_2) \tag{12}$$

and substituting expressions (11) and (12) into Eqs. (2) and (9) yields

$$u_x + i u_y = (1 - \alpha_1)\Phi(z_1) + (1 + \alpha_1)\overline{\Phi(z_1)} + (1 - \alpha_2)\Psi(z_2) - (1 + \alpha_2)\overline{\Psi(z_2)}, \quad (13)$$

$$\sigma_{xx} + \sigma_{yy} = 2\mu(\alpha_1^2 - \alpha_2^2)[\Phi'(z_1) + \overline{\Phi'(z_1)}],$$
$$\sigma_{xx} - \sigma_{yy} + i2\sigma_{xy} = 2\mu[(1 - \alpha_1)^2\Phi'(z_1) + (1 + \alpha_1)^2\overline{\Phi'(z_1)} + (1 - \alpha_2)^2\Psi'(z_2) - (1 + \alpha_2)^2\overline{\Psi'(z_2)}]. \quad (14)$$

For the problems (7) and (8), the functions $\Phi(z_1)$ and $\Psi(z_2)$ have the following structure (when z_1 and z_2 being sufficient large)

$$\Phi(z_1) = \sum_{n=1}^{\infty} a_n z_1^{-n}, \quad \Psi(z_2) = \sum_{n=1}^{\infty} b_n z_2^{-n} \quad (15)$$

where a_n and b_n are arbitrary complex constants.

With the conformal mapping

$$z_1, z_2 = \omega(\zeta) = \frac{a + R}{2}\left(\zeta + \frac{1}{\zeta}\right) \quad (16)$$

the outer region of the crack in z_1-plane/z_2-plane is mapped into the inner region of the unit circle γ in ζ-plane (Fig. 2 and 3). Denoting the value of ζ on the unit circle γ as $\sigma = e^{i\theta}$, the points $A(y = 0,\ x = a + R)$, $B(y = 0^+, x = a)$, $C(y = 0,\ x = -a)$, $D(y = 0, x = -a - R)$, $E(y = 0^-, x = -a)$ and $F(y = 0^-, x = a)$ in the physical plane are mapped onto the corresponding points $\sigma_A = 1$, $\sigma_1 = e^{i\theta_1}$, $\sigma_2 = e^{i(\pi - \theta_1)}$, $\sigma_D = -1$, $\sigma_3 = e^{i(\pi + \theta_1)}$ and $\sigma_4 = e^{-i\theta_1}$ on the unit circle in ζ-plane after the conformal mapping.

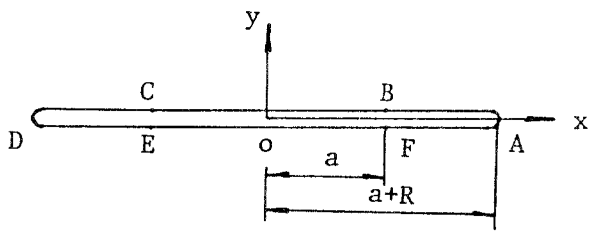

Figure 2

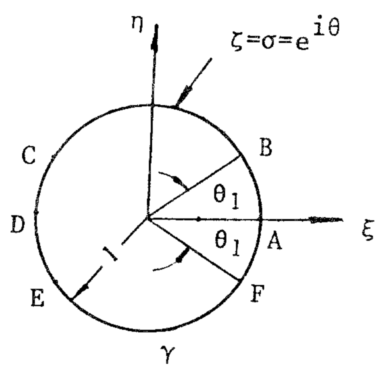

Figure 3

Supposing the solution of problem (8) as $\Phi_1(z_1)$ and $\Psi_1(z_2)$, we have

$$\Phi_1(z_1) = \Phi_1[\omega(\zeta)] = \Phi_*(\zeta), \quad \Psi_1(z_2) = \Psi_1[\omega(\zeta)] = \Psi_*(\zeta) \tag{17}$$

and

$$\Phi_1'(z_1) = \Phi_*'(\zeta)/\omega'(\zeta), \quad \Psi_1'(z_2) = \Psi_*'(\zeta)/\omega'(\zeta) \tag{18}$$

where $\Phi_*(\zeta)$ and $\Psi_*(\zeta)$ are two new unknown functions.

Substituting the relations (14) into boundary conditions (8) leads that

$$\begin{aligned} &\mathrm{Re}\,[(1+\alpha_2^2)\,\Phi_1'(z_1) - 2\alpha_1\,\Psi_1'(z_2)]_{y=0} = \sigma_s f(x), \\ &Im\,[2\alpha_1\,\Phi_1'(z_1) - (1+\alpha_2^2)\,\Psi_1'(z_2)]_{y=0} = 0, \quad |x| < a + R \end{aligned} \tag{19}$$

where the signs Re and Im represent real part and imaginary one of the complex function respectively, and the function $f(x)$ is

$$f(x) = \begin{cases} 1 & a < |x| < a + R \\ 0 & |x| < a \end{cases}. \tag{20}$$

Substituting conformal mapping (16) into formula (19) and considering expressions (17) and (18) yields that

$$\begin{aligned} &G_1(\sigma) - \frac{1}{\sigma^2}\,\overline{G_1(\sigma)} = \frac{\sigma^2 - 1}{\sigma^2}\,\frac{a+R}{2}\,\frac{\sigma_s}{2} f(\theta), \\ &G_2(\sigma) + \frac{1}{\sigma^2}\,\overline{G_2(\sigma)} = 0 \end{aligned} \tag{21}$$

in which

$$\begin{aligned} &G_1(\sigma) = (1+\alpha_2^2)\,\Phi_*'(\sigma) - 2\alpha_2\,\Psi_*'(\sigma), \\ &G_2(\sigma) = 2\alpha_1\,\Phi_*'(\sigma) - (1+\alpha_2^2)\,\Psi_*'(\sigma) \end{aligned} \tag{22}$$

and

$$f(\theta) = \begin{cases} 1 & -\theta_1 < \theta < \theta_1, \quad \pi - \theta_1 < \theta < \pi + \theta_1 \\ 0 & \theta_1 < \theta < \pi - \theta_1, \quad \pi + \theta_1 < \theta < 2\pi - \theta_1 \end{cases}. \tag{23}$$

Multiplying by $d\sigma/2\pi i(\sigma - \zeta)$ both sides of Eq. (21), then integrating along γ with respect to σ one gets

$$\begin{aligned} &\frac{1}{2\pi i}\int_\gamma \frac{G_1(\sigma)}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_\gamma \frac{1}{\sigma^2}\,\overline{G_1(\sigma)}\,\frac{d\sigma}{\sigma - \zeta} = \frac{a+R}{4\pi i}\,\frac{\sigma_s}{\mu}\left[\int_{\sigma_4}^{\sigma_1} + \int_{\sigma_2}^{\sigma_3}\right]\frac{\sigma^2 - 1}{\sigma^2}\,\frac{d\sigma}{\sigma - \zeta}, \\ &\frac{1}{2\pi i}\int_\gamma \frac{G_2(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{1}{\sigma^2}\,\frac{\overline{G_2(\sigma)}}{\sigma - \zeta}\,d\sigma = 0 \end{aligned} \tag{24}$$

In the above formulas $|\zeta| < 1$, that is, it takes its values in the interior of the unit circle.

Because $G_1(\zeta)$ and $G_2(\zeta)$ are the analytic functions in the inner region of the unit circle, based on the Cauchy integral formula, both first integrals on the lefthand side of Eqs. (24) result as

$$\frac{1}{2\pi i}\int_\gamma \frac{G_1(\sigma)}{\sigma - \zeta}\,d\sigma = G_1(\zeta), \quad \frac{1}{2\pi i}\int_\gamma \frac{G_2(\sigma)}{\sigma - \zeta}\,d\sigma = G_2(\zeta)$$

Also, because of the same reason, they may be expressed by the Taylor series

$$G_1(\zeta) = \sum_{n=1}^{\infty} g_n \zeta^n, \quad G_2(\zeta) = \sum_{n=1}^{\infty} h_n \zeta^n, \quad |\zeta| < 1$$

in which g_n and h_n are arbitrary complex constants.

Defining two new functions

$$\bar{G}_1\left(\frac{1}{\zeta}\right) = \sum_{n=1}^{\infty} \bar{g}_n \left(\frac{1}{\zeta}\right)^n \quad \text{and} \quad \bar{G}_2\left(\frac{1}{\zeta}\right) = \sum_{n=1}^{\infty} \bar{h}_n \left(\frac{1}{\zeta}\right)^n, \quad |\zeta| > 1$$

these two functions are analytic in the exterior of the unit circle, therefore the functions in both second integrals on the lefthand side of Eq. (24)

$$\frac{1}{\sigma^2} \overline{G_1(\sigma)} \quad \text{and} \quad \frac{1}{\sigma^2} \overline{G_2(\sigma)}$$

may be considered as the boundary values of the functions (the values of the functions on γ)

$$\frac{1}{\zeta^2} \bar{G}_1\left(\frac{1}{\zeta}\right) \quad \text{and} \quad \frac{1}{\zeta^2} \bar{G}_2\left(\frac{1}{\zeta}\right)$$

respectively, and these two functions are analytic in the outer region of the unit circle as mentioned above. According to the Cauchy integral formula, both second integrals on the lefthand side of the Eqs. (24) must vanish, so that

$$G_1(\zeta) = \frac{a+R}{2} \frac{\sigma_s}{\mu} \left[\int_{\sigma_4}^{\sigma_1} + \int_{\sigma_2}^{\sigma_3}\right] \frac{\sigma^2 - 1}{\sigma^2} \frac{d\sigma}{\sigma - \zeta}$$
$$G_2(\zeta) = 0. \tag{25}$$

After some calculations from formulas (25) and combining with formulas (22)

$$\Phi'_*(\zeta) = \frac{a+R}{2} \frac{\sigma_s}{\mu} \frac{1+\alpha_2^2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2} \left[\frac{2}{\pi} \frac{1}{\zeta^2} \theta_1 + \frac{1}{2\pi i} \frac{1-\zeta^2}{\zeta^2} \ln \frac{e^{2i\theta_1} - \zeta^2}{e^{-2i\theta_1} - \zeta^2}\right]$$
$$\Psi'_*(\zeta) = \frac{a+R}{2} \frac{\sigma_s}{\mu} \frac{2\alpha_1}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2} \left[\frac{2}{\pi} \frac{1}{\zeta^2} \theta_1 + \frac{1}{2\pi i} \frac{1-\zeta^2}{\zeta^2} \ln \frac{e^{2i\theta_1} - \zeta^2}{e^{-2i\theta_1} - \zeta^2}\right]. \tag{26}$$

The problem (8) is now solved completely.

The problem (7) may be solved in a similar manner but the calculation is simpler. Denoting the solution of problem (7) by $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$, we have (the solving approach here is different from that adopted by Yoffé [2]

$$\Phi_0(\zeta) = -\frac{a+R}{2} \frac{\sigma^{(\infty)}}{\mu} \frac{1+\alpha_2^2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2} \zeta$$
$$\Psi_0(\zeta) = -\frac{a+R}{2} \frac{\sigma^{(\infty)}}{\mu} \frac{2\alpha_1}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2} \zeta. \tag{27}$$

By superposing solutions (26) and (27) one may obtain the total solution $\Phi(z_1)$ and $\Psi(z_2)$ of the problem, that is

$$\Phi'(z_1) = \frac{\Phi_0'(\zeta) + \Phi_*'(\zeta)}{\omega'(\zeta)}, \qquad \Psi'(z_2) = \frac{\Psi_0'(\zeta) + \Psi_*'(\zeta)}{\omega'(\zeta)} \tag{28}$$

Substituting formulas (26) and (27) into (28) and integrating, the expressions of functions $\Phi(z_1)$ and $\Psi(z_2)$ may be given directly.

Based on these expressions and formulas (13) and (14) the displacement and stress fields of the problem are determined completely.

3. Determination of size R of the cohesive force zone

At the end of the cohesive force zone, that is at $y = 0$, $x = \pm(a + R)$, or at $\zeta = \pm 1$ in ζ-plane, the stresses must be finite. From formulas (26), (27) and (28), this condition leads

$$\frac{2\theta_1 \sigma_s}{\pi} - \sigma^{(\infty)} = 0 \quad \text{or} \quad \theta_1 = \frac{\pi \sigma^{(\infty)}}{2\sigma_s} \tag{29}$$

because of

$$\frac{a}{a+R} = \cos\theta_1 = \cos\left(\frac{\pi\sigma^{(\infty)}}{2\sigma_s}\right)$$

so that

$$R = a\left[\sec\left(\frac{\pi\sigma^{(\infty)}}{2\sigma_s}\right) - 1\right] \tag{30}$$

this is identical to the result of the Dugdale model in nonlinear fracture statics [3] as well as the result of the dislocation model proposed by Bilby, Cottrell and Swinden [4] for the static case.

4. Dynamic crack opening displacement

From formula (13) one can find that

$$u_y = -2\,\mathrm{Im}\,[\alpha_1 \Phi(z_1) - \Psi(z_2)] \tag{31}$$

and by using formulas (26), (27) and (28) and after some calculations, one gets

$$u_y = \frac{1}{\pi}(a+R)\frac{\sigma_s}{\mu}\frac{\alpha_1(\alpha_2^2 - 1)}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2}\left[\mathrm{Im}\left\{\frac{2\theta_1\sigma_s}{\zeta} + \pi\sigma^{(\alpha)}\zeta\right\}\right.$$
$$\left. + \mathrm{Re}\left\{(e^{i\theta} + e^{-i\theta})\ln\frac{\zeta^2 - e^{2i\theta_1}}{\zeta^2 - e^{-2i\theta_1}} - (e^{i\theta_1} - e^{-i\theta_1})\ln\frac{(\zeta + e^{i\theta_1})(\zeta - e^{-i\theta_1})}{(\zeta + e^{-i\theta_1})(\zeta - e^{i\theta_1})}\right\}\right]. \tag{32}$$

Let ζ take its value on the unit circle (that is setting $\zeta = \sigma = e^{i\theta}$), making the limitation according to the following formula, the dynamic crack opening displacement is obtained as

$$\delta_{\mathrm{I}}^{\mathrm{dynamic}} = \lim_{\theta \to \theta_1} 2u_y = \frac{4}{\pi} a \frac{\sigma_s}{\mu}\frac{\alpha_1(1-\alpha_2^2)}{4\alpha_1\alpha_2 - (1+\alpha_2^2)^2}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\sigma_s}\right). \tag{33}$$

The result is calculated for plane strain state, while the result for plane stress state can easily be obtained from formula (33) only by making a substitution of material constants. The latter is more significant physically. The dynamic crack opening displacement for these cases may be written uniformly as below

$$\delta_{\mathrm{I}}^{\mathrm{dynamic}} = A'(v)\,\delta^{\mathrm{static}} \tag{34}$$

where

$$A'(v) = \begin{cases} A(v)(1+\nu) & \text{(plane stress state)} \\ A(v)/(1-\nu) & \text{(plane strain state)} \end{cases} \tag{35}$$

$$A(v) = \frac{\alpha_1(1-\alpha_2^2)}{4\alpha_1\alpha_2 - (1+\alpha_2^2)^2} = \frac{(v/c_2)^2(1-v^2/c_1^2)^{1/2}}{4(1-v^2/c_1^2)^{1/2}(1-v^2/c_2^2)^{1/2} - (2-v^2/c_2^2)^2} \tag{36}$$

$$\delta_{\mathrm{I}}^{\mathrm{static}} = \frac{8\,a\,\sigma_s}{\pi E'} \ln \sec\left(\frac{\pi\,\sigma^{(\infty)}}{2\,\sigma_s}\right) \tag{37}$$

$$E' = \begin{cases} E & \text{(plane stress state)} \\ E/(1-\nu^2) & \text{(plane strain state)} \end{cases} \tag{38}$$

In the above formulas the expression δ^{static} is identical to the result obtained by Goodier and Field [6] in which E and ν are Young's modulus and Poisson's ratio respectively. The formula (34) is mentioned in Ref. [1].

§ 3. The solution of moving Dugdale model for mode II

By using superposition, this problem may be reduced to the following two boundary value problems of set (5):

$$\begin{aligned} &y = 0, \quad |x| < a + R{:}\ \sigma_{xy} = -\,\tau^{(\infty)}, \quad \sigma_{yy} = 0; \\ &\quad (x^2 + y^2)^{1/2} \to \infty{:}\ \sigma_{ij} = 0 \end{aligned} \tag{39}$$

and

$$\begin{aligned} &y = 0, \quad a < |x| < a + R{:}\ \sigma_{xy} = \tau_s, \quad \sigma_{yy} = 0; \\ &y = 0, \quad |x| < a{:}\ \sigma_{yy} = 0, \quad \sigma_{xy} = 0; \\ &(x^2 + y^2)^{1/2} \to \infty{:}\ \sigma_{ij} = 0. \end{aligned} \tag{40}$$

Similarly, introduce two analytic functions $\Phi(z_1)$ and $\Psi(z_2)$ and

$$\begin{aligned} \Phi(z_1) &= \Phi_1(z_1) + \Phi_2(z_1) = \Phi_1[\omega(\zeta)] + \Phi_2[\omega(\zeta)] = \Phi_*(\zeta) + \Phi_0(\zeta) \\ \Psi(z_2) &= \Psi_1(z_2) + \Psi_2(z_2) = \Psi_1[\omega(\zeta)] + \Psi_2[\omega(\zeta)] = \Psi_*(\zeta) + \Psi_0(\zeta) \end{aligned} \tag{41}$$

in which the functions $\Phi_1(z_1)$ and $\Psi_1(z_2)$ or $\Phi_*(\zeta)$ and $\Psi_*(\zeta)$ in the ζ-plane are the solution of the problem (40) and the functions $\Phi_2(z_1)$ and $\Psi_2(z_2)$ or $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ in the ζ-plane are the solution of problem (39), respectively. In a similar procedure to §2 we obtain the functions (or their derivatives) as below

$$\Phi'_*(\zeta) = i\,\frac{a+R}{2}\,\frac{\tau_s}{\mu}\,\frac{2\alpha_2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2}\left[\frac{2}{\pi}\frac{1}{\zeta^2}\theta_1 + \frac{1}{2\pi i}\,\frac{1-\zeta^2}{\zeta^2}\,\frac{e^{2i\theta_1}-\zeta^2}{e^{-2i\theta_1}-\zeta^2}\right]$$

$$\Psi'_*(\zeta) = i\,\frac{a+R}{2}\,\frac{\tau_s}{\mu}\,\frac{1+\alpha_2^2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2}\left[\frac{2}{\pi}\frac{1}{\zeta^2}\theta_1 + \frac{1}{2\pi i}\,\frac{1-\zeta^2}{\zeta^2}\,\frac{e^{2i\theta_1}-\zeta^2}{e^{-2i\theta_1}-\zeta^2}\right] \tag{42}$$

and

$$\Phi_0(\zeta) = -\,i\,\frac{a+R}{2}\,\frac{\tau^{(\infty)}}{\mu}\,\frac{2\alpha_2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2}\,\zeta$$

$$\Psi_0(\zeta) = -\,i\,\frac{a+R}{2}\,\frac{\tau^{(\infty)}}{\mu}\,\frac{1+\alpha_2^2}{(1+\alpha_2^2)^2 - 4\alpha_1\alpha_2}\,\zeta \tag{43}$$

The notations here are the same as those in the preceding paragraph.

Requiring the stresses to be finite at $y = 0$, $x = \pm(a+R)$ and considering the formulas (41), (42) and (43) yields that

$$R = a\left[\sec\left(\frac{\pi\,\tau^{(\infty)}}{2\tau_s}\right) - 1\right] \tag{44}$$

This determines the size of the cohesive force zone.

The formula (13) leads

$$u_x = 2\,\mathrm{Re}\,[\Phi(z_1) - \alpha_2\,\Psi(z_2)] \tag{45}$$

and based on formulas (45), (41), (42), (44) and the definition

$$\delta_{\mathrm{II}}^{\mathrm{dynamic}} = \lim_{\theta\to\theta_1} 2u_x$$

we obtain the dynamic crack sliding displacement as follows

$$\delta_{\mathrm{II}}^{\mathrm{dynamic}} = B'(v)\,\delta_{\mathrm{II}}^{\mathrm{static}} \tag{46}$$

in which

$$B'(v) = \begin{cases} B(v)\,(1+\nu) & \text{(plane stress state)} \\ B(v)/(1-\nu) & \text{(plane strain state)} \end{cases} \tag{47}$$

$$B(v) = \frac{\alpha_2(1-\alpha_2^2)}{4\alpha_1\alpha_2 - (1+\alpha_2^2)^2} = \frac{(v/c_2)^2(1-v^2/c_2^2)^{1/2}}{4(1-v^2/c_1^2)^{1/2}(1-v^2/c_2^2)^{1/2} - (2-v^2/c_2^2)^2} \tag{48}$$

$$\delta_{\mathrm{II}}^{\mathrm{static}} = \frac{8a\tau_s}{\pi E'}\ln\sec\left(\frac{\pi\,\tau^{(\infty)}}{2\tau_s}\right) \tag{49}$$

E' is the same as in definition (38).

§4. The solution of moving Dugdale model for mode III

The governing equation of mode III is

$$\nabla^2 w = \frac{1}{c_2^2} \frac{\partial^2 w}{\partial t^2} \tag{50}$$

where w represents the displacement in direction of axis oz and c_2 is defined by formula (4).

In terms of transformation (1), Eq. (50) is reduced to the Laplace equation in the z_2-plane, that is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_2^2}\right) w = 0 \tag{51}$$

in which the definition of y_2 is given in formulas (6).

In this case only two stress components σ_{xz} and σ_{yz} do not vanish. They satisfy the relations

$$\sigma_{xz} = \mu \frac{\partial w}{\partial x}, \qquad \sigma_{yz} = \mu \frac{\partial w}{\partial y} \tag{52}$$

the meaning of μ is as the same as before.

The Dugdale model of mode III for a moving crack may be found from the following two boundary value problems by using superposition and neglecting a constant term

$$\begin{aligned} &y = 0, \quad |x| < a + R\colon \sigma_{yz} = -\tau^{(\infty)}; \\ &\quad (x^2 + y^2)^{1/2} \to \infty\colon \sigma_{ij} = 0 \end{aligned} \tag{53}$$

and

$$\begin{aligned} &y = 0, \quad a < |x| < a + R\colon \sigma_{yz} = \tau_s; \\ &y = 0, \quad |x| < a\colon \sigma_{yz} = 0; \\ &(x^2 + y^2)^{1/2} \to \infty\colon \sigma_{ij} = 0. \end{aligned} \tag{54}$$

The solution of boundary value problem (53) is easy to obtain and will be given later. The solution of problem (54) is considered first.

Because w is a harmonic function of variables (x, y_2), it may be expressed as a real or imaginary part of an analytic function $\chi(z_2)$ where $z_2 = x + i y_2$. For problem (54) setting

$$\chi(z_2) = \chi_1(z_2) \quad \text{and} \quad w(x, y_2) = \operatorname{Re} \chi_1(z_2) \tag{55}$$

from formulas (52) and (55) one has

$$\sigma_{xz} - i\sigma_{yz} = \mu \chi_1'(z_2) = \mu \frac{\chi_*'(\zeta)}{\omega'(\zeta)} \tag{56}$$

where $\chi_*(\zeta) = \chi_1[\omega(\zeta)]$ and $\omega(\zeta)$ is given by formula (16).

After some calculations the boundary value problem (54) leads to the following functional equation in the ζ-plane

$$\frac{1}{2\pi i}\int_\gamma \frac{\chi'_*(\zeta)}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{1}{\sigma^2}\frac{\overline{\chi'_*(\zeta)}}{\sigma-\zeta}\,d\sigma = -i\frac{\tau_s}{\mu}(a+R)\frac{1}{2\pi i}\int_\gamma \frac{\sigma^2-1}{\sigma^2} f(\theta)\frac{d\sigma}{\sigma-\zeta} \tag{57}$$

where $\sigma = e^{i\theta}$, $f(\theta)$ are defined by formula (23) and γ represents the unit circle contour.

By using a similar analysis as in §2, the solution of function Eq. (57) is obtained

$$\chi'_*(\zeta) = -(a+R)\frac{\tau_s}{\mu}\left[\frac{2i}{\pi}\frac{1}{\zeta^2}\theta_1 + \frac{1}{2\pi}\frac{1-\zeta^2}{\zeta^2}\ln\frac{e^{2i\theta_1}-\zeta^2}{e^{2i\theta_1}-\zeta^2}\right] \tag{58}$$

Thus, problem (54) is solved.

In similar manner, the solution of boundary value problem (53) may be found as

$$\chi_2(z_2) = \chi_2[\omega(\zeta)] = \chi_0(\zeta) = i(a+R)\frac{\tau^{(\infty)}}{\mu}\zeta. \tag{59}$$

The total solution of the problem consists of $\chi_1(z_2)$ and $\chi_2(z_2)$ (or $\chi_*(\zeta)$ and $\chi_0(\zeta)$ in ζ-plane).

Using the formulas (56), and (59) and requiring the stresses to be finite at $y = 0$, $x = \pm(a+R)$ (or at $\zeta = \pm 1$ in ζ-plane) leads to

$$R = a\left[\sec\left(\frac{\pi\tau^{(\infty)}}{2\tau_s}\right) - 1\right] \tag{60}$$

This is the size of the plastic zone for mode III.

The displacement $w(x, y_2)$ is equal to

$$w(x, y_2) = \mathrm{Re}\,[\chi_*(\zeta) + \chi_0(\zeta)] \tag{61}$$

From this, the dynamic crack tearing displacement is determined

$$\delta_{\mathrm{III}}^{\mathrm{dynamic}} = \lim_{\theta\to\theta_1} 2w = \frac{4}{\pi}a\frac{\tau_s}{\mu}\ln\sec\left(\frac{\pi\tau^{(\infty)}}{2\tau_s}\right) = \frac{8a\tau_s(1+\nu)}{\pi E}\ln\sec\left(\frac{\pi\tau^{(\infty)}}{2\tau_s}\right) \tag{62}$$

It is identical to the static one [4].

§5. A crack propagation criterion based on the opening displacement

As in static case, one can propose a criterion of crack opening displacement for the dynamic problem as follows

$$\delta_{\mathrm{I}}^{\mathrm{dynamic}} = \delta_{\mathrm{Ic}}(v) \tag{63}$$

in which $\delta_{\mathrm{I}}^{\mathrm{dynamic}}$ is described by formulas (34)–(38) and $\delta_{\mathrm{Ic}}(v)$ should be a material constant but depending on the crack moving speed v. This criterion may be used to check crack propagation and determines the critical speed or critical length of the crack.

After some modifications it may be used in the analysis of a moving crack for materials with elastic perfectly plastic or with hardening behaviour.

A comparison with experimental results and other aspects will be discussed in another report.

Acknowledgement

The author gratefully acknowledges the Alexander von Humboldt Foundation for a research fellowship, he is also grateful to Prof. H. G. Hahn at the University of Kaiserslautern in the Federal Republic of Germany for his help.

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Abstract

This study proposes a moving Dugdale model for modes I, II and III, presents a fully dynamic analysis of the problem with the help of complex function theory and gives an exact analytical solution. From this solution, the stress and displacement fields and the dynamic crack opening displacement for mode I, II and III are determined. Based on this, the author proposes a criterion to describe dynamic behaviour of a moving defect in solids.

Zusammenfassung

Diese Arbeit schlägt ein bewegliches Dugdale Modell für Mode I, II und III vor. Sie gibt eine vollständige dynamische Analyse des Problems mit der Methode der komplexen Funktionstheorie und liefert die exakte analytische Lösung. Aus dieser Lösung werden die Spannungs- und Verschiebungsfelder und damit die dynamischen Rißöffnungsverschiebungen für Mode I, II und III dieses Problems bestimmt. Auf dieser Grundlage schlägt der Autor ein Kriterium zur Beschreibung des dynamischen Verhaltens eines bewegten Defekts in einem Festkörper vor.

(Received: August 8, 1986)