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The flow of a visco-elastic fluid near a point of re-attachment

By J. M. Dorrepaal, Dept of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529, USA and O. P. Chandna and F. Labropulu, Dept of Mathematics and Statistics, University of Windsor, Windsor, Ontario, Canada N9B 3P4

1. Introduction

In a paper by Beard and Walters [1], the equations of motion for a visco-elastic fluid are derived and solved for the case of two-dimensional flow near a stagnation point. It has also been shown by three authors independently [2, 3, 4] that stagnation point flow can be modified to describe the flow of a Newtonian fluid near a point of re-attachment. In this modification the dividing streamline approaches a flat rigid boundary at an arbitrary angle of incidence. Dorrepaal [4] has shown that the slope of this streamline at the point of re-attachment divided by the slope of the same streamline far from the wall is a constant independent of the angle of incidence.

In the present paper the same visco-elastic fluid considered in [1] will be studied near a point of re-attachment. In particular, the relationship discovered by Dorrepaal for a Newtonian fluid will be investigated for a visco-elastic fluid.

2. Equations of motion

Following Beard and Walters [1] we assume the fluid occupies the upper half plane $y > 0$. Since the fluid is incompressible and the flow is two-dimensional, a stream function $\psi(x, y)$ exists from which the velocity components $u(x, y)$ and $v(x, y)$ can be derived:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

The equation of motion, after scaling, is as follows:

$$\nabla^4 \psi + \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} + k\left\{\frac{\partial \psi}{\partial x}\frac{\partial \nabla^4 \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \nabla^4 \psi}{\partial x}\right\} = 0 \tag{2.2}$$

where k, the Weissenberg number, is a ratio of elastic effects to viscous effects and is always positive. The equation is only valid to $O(k)$ and so any visco-elastic solution obtained from this equation must be regarded as a perturbation of the corresponding Newtonian flow.

Consider first the stagnation point flow investigated by Beard and Walters [1]. For future reference we sketch here a modified version of their solution.

Consider a hyperbolic flow impinging on a flat wall at $y = 0$. The no-slip conditions apply at the wall and the stream function far from the wall has the form $\psi(x, y) \sim xy$ which suggests a similarity solution of the form $\psi(x, y) = xF(y; k)$. After substitution into (2.2), the o.d.e. satisfied by $F(y; k)$ is found to be

$$F^{iv} + FF''' - F'F'' + k\{FF^{v} - F'F^{iv}\} = 0. \tag{2.3}$$

The boundary conditions are $F(0) = F'(0) = 0$ and $F'(\infty) = 1$. Equation (2.3) can be integrated once and after the condition at infinity is invoked, we have

$$F''' + FF'' - F'^2 + 1 + k\{FF^{iv} - 2F'F''' + F''^2\} = 0. \tag{2.4}$$

Despite the fact that the equation is fourth order, it can be solved as a third order equation by using a perturbation approach. We assume

$$F(y; k) = F_0(y) + kF_1(y) + O(k^2) \tag{2.5}$$

where

$$F_0''' + F_0F_0'' - F_0'^2 + 1 = 0 \tag{2.6}$$

$$F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = 1$$

defines the well-known Hiemenz function [5] describing the stagnation point flow of a Newtonian fluid.

From a numerical solution recorded by Goldstein [6], we have

$$\begin{aligned} F_0(y) &= \tfrac{1}{2}Cy^2 - \tfrac{1}{6}y^3 + O(y^5), \\ F_0(y) &\sim y - A + O\{(y - A)^{-4} \exp[-\tfrac{1}{2}(y - A)^2]\}, \end{aligned} \tag{2.7}$$

where $C = 1.232588$, $A = 0.647900$.

The term of $O(k)$ is, from Beard and Walters [1],

$$F_1''' + F_0F_1'' - 2F_0'F_1' + F_0''F_1 = 2F_0'F_0''' - F_0F_0^{iv} - F_0''^2 \tag{2.8}$$

$$F_1(0) = F_1'(0) = 0, \quad F_1'(\infty) = 0.$$

This equation is linear in $F_1(y)$ and can be numerically integrated in its present form. However, a similar equation with a more compact right side can be obtained by making the substitution

$$F_1(y) = f(y) - F_0''(y). \tag{2.9}$$

The boundary-value problem for $f(y)$ is as follows:

$$\begin{aligned} &f''' + F_0 f'' - 2F_0' f' + F_0'' f = F_0^{v} \\ &f(0) = C \\ &f'(0) = -1 \quad f'(\infty) = 0. \end{aligned} \tag{2.10}$$

The numerical integration of (2.10) yields the following results which are in agreement with Beard and Walters:

$$\begin{aligned} &f''(0) = D = 1.139019 \\ &f(\infty) = A_1 = 0.825547. \end{aligned} \tag{2.11}$$

The solution to equation (2.4) to terms of $O(k)$ is therefore

$$F = F_0 + k(f - F_0'') + O(k^2). \tag{2.12}$$

The function $F(y; k)$ has the following properties to $O(k)$:

$$\begin{aligned} &F(y; k) = \tfrac{1}{2}(C + kD)y^2 - \tfrac{1}{6}(1 + kC^2)y^3 + O(y^5) \\ &F(y; k) \sim y - (A - kA_1) + \text{exp.} \quad y \to \infty. \end{aligned} \tag{2.13}$$

3. The flow near a point of re-attachment

We assume the stream function far from the wall to be of the form

$$\psi(x, y) \sim xy + ny^2 \tag{3.1}$$

where n is a constant. The dividing streamline which comes into the wall from infinity is defined by $\psi(x, y) = 0$ and so the slope of the dividing streamline at infinity is $-1/n$. Equation (3.1) suggests that $\psi(x, y)$ has the form

$$\psi(x, y) = xF(y) + G(y) \tag{3.2}$$

where

$$\left.\begin{aligned} F(y) &\sim y \\ G(y) &\sim ny^2 \end{aligned}\right\} y \to \infty. \tag{3.3}$$

When (3.2) is substituted into (2.2), it is found that $F(y)$ satisfies exactly the same boundary-value problem as that considered in §2. The solution given there applies and, to $O(k)$, the small-y and large-y expansions for $F(y)$ are shown in (2.13).

The equation for $G(y)$ is

$$G^{iv} + FG''' - F''G' + k\{FG^{v} - F^{iv}G'\} = 0 \tag{3.4}$$

$$G(0) = G'(0) = 0, \quad G''(\infty) = 2n$$

and after one integration, we have

$$G''' + FG'' - F'G' + k\{FG^{iv} - F'G''' + F''G'' - F'''G'\}$$
$$= 2n(-A + kA_1). \tag{3.5}$$

The right side of (3.5) is the value of the integration constant obtained by examining the behaviors of $F(y)$ and $G(y)$ at infinity. The order of equation (3.5) can be reduced once more by the substitution $G'(y) = 2nH(y)$ which yields

$$H'' + FH' - F'H + k\{FH''' - F'H'' + F''H' - F'''H\} = -A + kA_1 \tag{3.6}$$

$$H(0) = 0, \quad H'(\infty) = 1.$$

Following the pattern of §2, we assume a perturbation solution for $H(y)$ of the form

$$H(y) = H_0(y) + k[h(y) - H_0''(y)] + O(k^2). \tag{3.7}$$

The boundary-value problem for $H_0(y)$ is given by

$$H_0'' + F_0H_0' - F_0'H_0 = -A \tag{3.8}$$

$$H_0(0) = 0, \quad H_0'(\infty) = 1$$

and has been solved by Dorrepaal [4]. The Maclaurin series for $H_0(y)$ has the form

$$H_0(y) = Ey - \tfrac{1}{2}Ay^2 + O(y^4) \tag{3.9}$$

where $E = 1.406544$.

The equation for $h(y)$ is easily obtained by substituting (3.7) and (2.12) into (3.6) and retaining terms of $O(k)$. We have

$$h''(y) + F_0(y)h'(y) - F_0'(y)h(y) = R(y) \tag{3.10}$$

$$h(0) = -A, \quad h(\infty) = 0$$

where $R(y) = A_1 + H_0^{iv}(y) - f(y)H_0'(y) + f'(y)H_0(y)$.

The condition at infinity is a consequence of the fact that $h'(y)$ must vanish as y gets large. This implies that $h(y)$ must approach a constant whose value must be zero in order to maintain consistency with the large-y behaviors of the other terms in equation (3.10).

The problem defined by (3.10) is a linear second order differential equation which can be solved analytically using reduction of order. The reduction is possible because $F_0''(y)$ is a solution of the homogeneous equation. Omitting the details, we have

$$h(y) = -(A/C)F_0''(y) + F_0''(y) \cdot \int_0^y \frac{K(\varrho)}{F_0''(\varrho)I(\varrho)}\, d\varrho \tag{3.11}$$

where

$$\begin{aligned} K(\varrho) &= \int_0^{\varrho} R(r)I(r)\, dr \\ I(\varrho) &= F_0''(\varrho) \cdot \exp\left[\int_0^{\varrho} F_0(s)\, ds\right]. \end{aligned} \tag{3.12}$$

The condition at infinity is satisfied because both terms in (3.11) decay exponentially as $y \to +\infty$.

Having obtained $h(y)$ analytically, it is now possible to obtain Maclaurin series for $H(y)$ and $G(y)$ via back-substitution. Using (3.11), (3.9) and (3.7), we obtain the following results:

$$\left.\begin{aligned} h(y) &= -A + (A/C)y + \tfrac{1}{2}A_1 y^2 + O(y^3) \\ H(y) &= [E + k(A/C)]y - \tfrac{1}{2}[A + k(EC - A_1)]y^2 + O(y^4) \\ G(y) &= n[E + k(A/C)]y^2 - \frac{n}{3}[A + k(EC - A_1)]y^3 + O(y^5) \end{aligned}\right\}. \tag{3.13}$$

4. Behavior of the flow near the wall

By substituting into equation (3.2) the Maclaurin series for $F(y)$ and $G(y)$ found in (2.13) and (3.13) respectively, we obtain a small-y expansion for the stream function $\psi(x, y)$ which has the form

$$\psi(x, y) = N_2(k)y^2 + N_3(k)y^3 + M_2(k)xy^2 + M_3(k)xy^3 + O(y^5) \tag{4.1}$$

where N_i, M_i are the corresponding coefficients.

Expression (4.1) can be rewritten

$$\psi(x, y) = M_2 y^2\{x + N_2' + N_3' y + M_3' xy + O(y^3)\} \tag{4.2}$$

where $N_i' = N_i/M_2$, $M_3' = M_3/M_2$.

If the change of variables $X = x + N_2'$ is made and then terms of $O(Xy^3)$ are neglected, we have

$$\psi(X, y) = M_2 y^2\{X + (N_3' - M_3' N_2')y + O(Xy)\}. \tag{4.3}$$

Thus near the wall, the dividing streamline $\psi = 0$ has the equation

$$X + (N_3' - M_3' N_2')y = 0 \tag{4.4}$$

and its slope m_w at the point of re-attachment ($X = 0, y = 0$) is

$$m_w = \frac{1}{M_3' N_2' - N_3'} = \frac{M_2^2}{M_3 N_2 - N_3 M_2}. \tag{4.5}$$

From §3, the slope of the dividing streamline at infinity is $m_\infty = -1/n$. Thus after substituting for the coefficients N_i, M_i in (4.5) and neglecting terms of $O(k^2)$, we find that the ratio of slopes is

$$\frac{m_w}{m_\infty} = 3.748513\{1 + k(0.523675) + O(k^2)\}. \tag{4.6}$$

The point of re-attachment is located at

$$x = -N_2' = -2.282262n\{1 - k(0.550376) + O(k^2)\}. \tag{4.7}$$

In Fig. 1, the effects of elasticity are illustrated. The streamline on the left is the dividing streamline in a Newtonian fluid ($k = 0$) impinging on a flat rigid boundary. The streamline on the right is the same streamline in the flow of visco-elastic fluid ($k > 0$). The effect of elasticity is to move the point of re-attachment closer to the origin and to steepen the angle at which the dividing streamline meets the wall.

As in the Newtonian case, the slope ratio is constant for all angles of incidence of the dividing streamline at infinity. The constant depends, of course, on the Weissenberg number k and as $k \to 0$, Dorrepaal's result [4] is recovered. Relationship (4.6) means that the Weissenberg number of a slightly visco-elastic fluid can be determined by examining the steady flow of such a fluid near a point of re-attachment. If the slope-ratio of the dividing streamline can be measured from a photograph, say, the value of k can be easily calculated. Once k is known, it is possible using (4.7) to determine the relative strength n of the shear flow which is responsible for the displacement of the stagnation point by the amount $|x|$.

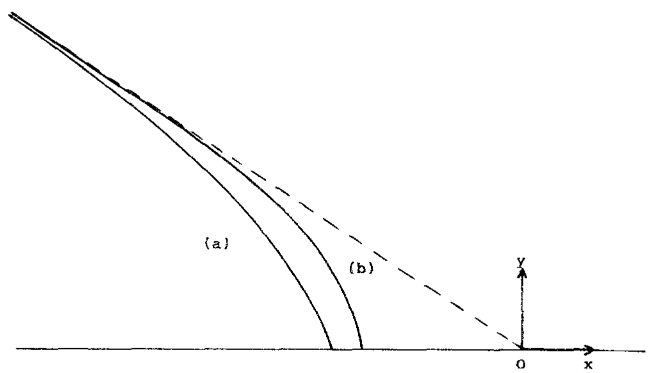

Figure 1
Dividing streamlines near a point of re-attachment for (a) Newtonian fluid; (b) visco-elastic fluid. The streamlines have the same angle of incidence far from the wall.

References

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Abstract

The flow of a visco-elastic fluid near a point of re-attachment is examined to see how elasticity affects the constant slope-ratio relationship which Newtonian fluids exhibit. The slope of the dividing streamline at the point of re-attachment divided by the slope of the same streamline far from the wall is found to be constant in the visco-elastic case, as well, for all angles of incidence. The slope ratio depends solely on the Weissenberg number which measures the extent to which viscous effects dominate elasticity effects.

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