Journal of Applied Mathematics and Physics (ZAMP)
Vol. 42, March 1991

0044-2275/91/020169-14 $ 1.50 + 0.20


# Mass-orthogonal formulation of equations of motion for multibody systems

By Hans Brauchli, Institut für Mechanik, ETH-Zentrum, CH-8092 Zürich

## 1. Introduction

The idea to describe complex multibody systems by redundant coordinates, introducing position and velocity variables for each body separately, is now generally accepted (see e.g. [6, 10, 13]). The formulation of the dynamical laws and the constraint conditions results in a set of coupled differential and algebraic equations, which presents difficulties for either the computation of the reactive forces or for applying a direct numerical integration scheme. Various methods for reducing this conglomerate of equations have been proposed [5, 7, 8, 11].

A new formulation of the equations of motion for multibody systems is presented here, where this problem does not arise. It is based on a projection method proposed previously [1], combined with the concept of mass-orthogonality. The multibody system is first described as a system of free bodies. All constraints of the system are written as kinematical constraints and used to construct two projections splitting the space of velocities into the space of admissible velocities of the constrained system and a complementary space of velocities vanishing as a result of the constraints. The projections are uniquely determined by an orthogonality condition in the mass metric of the free system. A corresponding splitting will then also hold in the space of the momenta. For the computation of the projections a positive definite and symmetric matrix, the fundamental matrix, has to be inverted. An explicit formula for the non-working constraint reactions as a function of the position and velocity variables is then obtained together with a system of explicit first order differential equations describing the motion of the constrained system.

The method presented is well suited for the simulation of complex multibody systems as well as for the further development of the theory of dynamics [2, 3].

## 2. The free system

We assume that the multibody system to be investigated, which will be addressed as the *constrained system*, can be obtained from a *free system* by introducing constraint equations. The free system may simply be the system of free bodies of the multibody system, but it could as well be any system easily described by the classical equations of mechanics. We will assume the constrained system to be scleronomic, so that all constraint conditions will be time-independent. We will not assume, however, the constrained system to be either holonomic or conservative. As will be seen, nonholonomic constraints do not present special difficulties. Also, nonconservative and time dependent forces are easily included in the description used.

Let us then assume that the free system is described by a set of generalized coordinates $q^k$ and velocities $u^k$, where $k$ runs from 1 to $n$, and $n$ denotes the degree of freedom of the free system. If holonomic velocities are used, we have

$$\dot{q}^k = u^k. \tag{1}$$

In the case of nonholonomic velocity parameters, they will be linearly related to the $\dot{q}^k$ by

$$u^k = B^k_h(q^j)\dot{q}^h \tag{2}$$

or, denoting the inverse of the matrix $B^k_h$ by $\tilde{B}^k_h$,

$$\dot{q}^k = \tilde{B}^k_h u^h \tag{3}$$

respectively. Here, the summation convention for repeated indices has been adopted.

Let us furthermore assume that the kinetic energy of the free system is a homogenous quadratic form of the velocities,

$$T(q^k, u^h) = \tfrac{1}{2}M_{ij}(q^k)u^i u^j. \tag{4}$$

The coefficient matrix $M_{ij}$ represents the mass matrix of the free system, it is symmetric and positive definite. Then

$$p_k = M_{kj}u^j \tag{5}$$

are the canonical momenta, and with the inverse mass matrix $A^{ij}$ we also have

$$u^k = A^{kj}p_j. \tag{6}$$

Notice that the two matrices $B^k_h$ and $M_{ij}$ will typically be block diagonal matrices, so that their inverses can be easily computed.

For the definition of the generalized forces we introduce the quantities

$$\delta\tilde{q}^k = B^k_h \delta q^h, \tag{7}$$

which in the case of holonomic velocity parameters coincide with the variations of the coordinates $\delta q^k$. The generalized forces $Q_k$ are then defined through the relation

$$\delta W = Q_k \delta \tilde{q}^k \tag{8}$$

for the virtual work. We will also need the further notation

$$\frac{\partial T}{\partial \tilde{q}^k} = \frac{\partial T}{\partial q^h} \tilde{B}^h_k = \frac{1}{2} \frac{\partial M_{ij}}{\partial q^h} \tilde{B}^h_k u^i u^j \tag{9}$$

and introduce the matrices

$$C^h_{ki} = \left\{ \frac{\partial B^h_r}{\partial q^s} - \frac{\partial B^h_s}{\partial q^r} \right\} \tilde{B}^r_k \tilde{B}^s_i . \tag{10}$$

We are now ready to formulate the classical equations of dynamics. In the case of holonomic velocities we write the equations of Lagrange in the form

$$\dot{p}_k = \frac{\partial T}{\partial q^k} + Q_k, \tag{11}$$

$$\dot{q}^k = u^k = A^{kj} p_j . \tag{12}$$

These equations describe the motion of the free system in the canonical variables $p_k$ and $q^k$, while the $u^k$ are used as abbreviations to be inserted into the expression $\partial T / \partial q^k$ in (11).

In the case of nonholonomic velocities we formulate the equations of Euler-Lagrange [4, 12]

$$\dot{p}_k + C^h_{ki} p_h u^i = \frac{\partial T}{\partial \tilde{q}^k} + Q_k, \tag{13}$$

$$u^k = A^{kj} p_j . \tag{14}$$

Together with (3) they describe the dynamics of the free system in terms of the canonical variables $p_k$ and $q^k$. For later reference it is convenient to write (11) and (13) in a common form,

$$\dot{p}_k = \hat{Q}_k, \tag{15}$$

where $\hat{Q}_k$ denotes either the right hand side of (11) or, in the case of nonholonomic velocities,

$$\hat{Q}_k = \frac{\partial T}{\partial \tilde{q}^k} + Q_k - C^h_{ki} p_h u^i . \tag{16}$$

The relevant dynamical equations can then be written in matrix form as

$$\dot{\boldsymbol{p}} = \hat{\boldsymbol{Q}}, \tag{17}$$

$$\boldsymbol{u} = \boldsymbol{A} \boldsymbol{p}, \tag{18}$$

and

$$\dot{q} = \tilde{B}u. \tag{19}$$

## 3. Constraints and projections

We will now introduce constraints. They are time independent, since we assume the constrained system to be scleronomic. *Geometric constraints* are of the form

$$f(q^k) = 0, \tag{20}$$

while *kinematical constraints* read

$$e_k(q^i)u^k = 0. \tag{21}$$

Since differentiating a geometric constraint results in a kinematical constraint—the reverse is true only for holonomic constraints—we may assume that all constraints of the system are kinematical. We then arrive at a set of constraint equations of the form

$$e_{\mu k}(q^i)u^k = 0. \tag{22}$$

Let $n_c$ denote the number of constraints, which we assume to be linearly independent. Then the index $\mu$ will run from 1 to $n_c$ and the index $k$ from 1 to $n$. In matrix notation (22) reads

$$\boldsymbol{E}^T\boldsymbol{u} = 0. \tag{23}$$

This represents a homogenous linear system with a coefficient matrix of rank

$$r(\boldsymbol{E}^T) = n_c. \tag{24}$$

The solution space consists of the admissible velocities of the constrained system. At a given point, this is a subspace of the tangent space of the free system and has dimension $n_a$, i.e., the degree of freedom of the constrained system. Obviously,

$$n_a + n_c = n. \tag{25}$$

We now want to construct two complementary projections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ splitting the tangent space of the free system into the admissible velocities

$$\boldsymbol{u}_\alpha = \boldsymbol{\alpha}\boldsymbol{u} \tag{26}$$

of the constrained system and a complement of velocities

$$\boldsymbol{u}_\beta = \boldsymbol{\beta}\boldsymbol{u} \tag{27}$$

vanishing due to the constraint equations. As complementary projections, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the relations

$$\boldsymbol{\alpha}^2 = \boldsymbol{\alpha}, \quad \boldsymbol{\beta}^2 = \boldsymbol{\beta} \tag{28}$$

and

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{I}, \quad \boldsymbol{\alpha}\boldsymbol{\beta} = 0, \quad \boldsymbol{\beta}\boldsymbol{\alpha} = 0, \tag{29}$$

where $\boldsymbol{I}$ denotes the unit matrix. Furthermore, since the image of $\boldsymbol{\alpha}$ is the solution space of (23), we have

$$\boldsymbol{E}^T\boldsymbol{\alpha} = 0 \quad \text{and} \quad \boldsymbol{E}^T\boldsymbol{\beta} = \boldsymbol{E}^T. \tag{30}$$

Also, the rank of $\boldsymbol{\alpha}$ has to be $n_a$, the rank of $\boldsymbol{\beta}$ $n_c$. One verifies easily that a possible choice of projections is

$$\boldsymbol{\beta} = \boldsymbol{KE}(\boldsymbol{E}^T\boldsymbol{KE})^{-1}\boldsymbol{E}^T \tag{31}$$

and

$$\boldsymbol{\alpha} = \boldsymbol{I} - \boldsymbol{\beta}. \tag{32}$$

In these formulas an arbitrary regular symmetric $n$ by $n$ matrix $\boldsymbol{K}$ can be substituted. Choosing for $\boldsymbol{K}$ the unit matrix makes the two projections orthogonal in the usual metric. We will see, however, that an *orthogonality condition in the natural metric of the free system, i.e. the mass metric*, is more appropriate.

In the space of the velocities of the free system a scalar product is introduced with the mass matrix $\boldsymbol{M}$ of the free system by

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^T\boldsymbol{M}\boldsymbol{v}. \tag{33}$$

Two velocities will be called orthogonal if their scalar product vanishes. A projection $\boldsymbol{\beta}$ will be called orthogonal if its image and its kernel are orthogonal subspaces. We then have

**Theorem 1:** The following conditions are equivalent:

i) the projection $\boldsymbol{\beta}$ is orthogonal,

ii) $\boldsymbol{\alpha}^T\boldsymbol{M}\boldsymbol{\beta} = 0,$ (34)

where $\boldsymbol{\alpha}$ is the complementary projection of $\boldsymbol{\beta}$,

iii) $\boldsymbol{M}\boldsymbol{\beta} = (\boldsymbol{M}\boldsymbol{\beta})^T.$ (35)

**Proof:** Let us first show that i) and ii) are equivalent. Let $\boldsymbol{v}_\beta$ be an element of the image of $\boldsymbol{\beta}$ and $\boldsymbol{u}_\alpha$ an element of the kernel. Then the definition of orthogonality demands that

$$\boldsymbol{u}_\alpha^T\boldsymbol{M}\boldsymbol{v}_\beta = \boldsymbol{u}^T\boldsymbol{\alpha}^T\boldsymbol{M}\boldsymbol{\beta}\boldsymbol{v} = 0 \tag{36}$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$. But this holds if and only if (34) is satisfied.

Now let us assume that (34) holds. Then

$$\beta^T M\beta = M\beta - \alpha^T M\beta = M\beta. \tag{37}$$

Adding the transpose of (34) then results in (35). On the other hand, if (35) holds, multiplication by $\alpha^T$ from the left gives (34) since $\alpha^T\beta^T = 0$.

We are now in a position to state

**Theorem 2:** The following conditions uniquely define the projection $\beta$:

i) $\beta^2 = \beta$, (38)

ii) $E^T\beta = E^T$, $r(\beta) = r(E^T) = n_c$. (39)

iii) $M\beta = (M\beta)^T$. (40)

**Proof:** The solution is unique, since a projection is uniquely determined by its image and kernel. Condition ii) defines the kernel of $\beta$. In fact, the first part of condition ii) implies that the kernel of $\beta$ is included in the kernel of $E^T$. Since $\beta$ and $E^T$ have the same rank, the two kernels coincide. Condition iii) then, by mass-orthogonality, defines the image of $\beta$.

The solution is obtained by setting

$$K = A \tag{41}$$

in (31).

We present the formulas for the two projections by introducing the matrix

$$G = E^T A E, \tag{42}$$

which we call the *fundamental matrix*. It is a symmetric $n_c$ by $n_c$ matrix and because of (24) it is positive definite. Hence

$$\pi = M\beta = EG^{-1}E^T \tag{43}$$

is well defined and the two projections are

$$\alpha = I - A\pi, \quad \beta = A\pi. \tag{44}$$

In the space of forces the transpose projections have to be used, since it is the dual of the space of velocities, therefore we have

$$Q_\alpha = \alpha^T \mathbf{Q} \quad \text{and} \quad Q_\beta = \beta^T Q. \tag{45}$$

$Q_\alpha$ represent the external forces acting on the constrained system, while $Q_\beta$ consists of contributions from loads and from constraint reactions.

## 4. Mass and influence matrices of the constrained system

If, in the expression for the kinetic energy of the free system

$$T = \tfrac{1}{2}\boldsymbol{u}^T\boldsymbol{M}\boldsymbol{u} \tag{46}$$

we insert

$$\boldsymbol{u} = \boldsymbol{u}_\alpha + \boldsymbol{u}_\beta, \tag{47}$$

we arrive at

$$T = \tfrac{1}{2}\boldsymbol{u}_\alpha^T\boldsymbol{M}_{\alpha\alpha}\boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha^T\boldsymbol{M}_{\alpha\beta}\boldsymbol{u}_\beta + \tfrac{1}{2}\boldsymbol{u}_\beta^T\boldsymbol{M}_{\beta\beta}\boldsymbol{u}_\beta, \tag{48}$$

where the matrices

$$\boldsymbol{M}_{\alpha\alpha} = \boldsymbol{\alpha}^T\boldsymbol{M}\boldsymbol{\alpha}, \quad \boldsymbol{M}_{\alpha\beta} = \boldsymbol{\alpha}^T\boldsymbol{M}\boldsymbol{\beta} = \boldsymbol{M}_{\beta\alpha}^T, \quad \boldsymbol{M}_{\beta\beta} = \boldsymbol{\beta}^T\boldsymbol{M}\boldsymbol{\beta}, \tag{49}$$

have been introduced. The most important of these is the mass matrix $\boldsymbol{M}_{\alpha\alpha}$ of the constrained system. It is an $n$ by $n$ matrix of rank $n_a$.

The momentum of the free system is

$$\boldsymbol{p} = \frac{\partial T}{\partial \boldsymbol{u}} = \boldsymbol{M}\boldsymbol{u}, \tag{50}$$

and if we differentiate the kinetic energy (48) with respect to $\boldsymbol{u}_\alpha$ and $\boldsymbol{u}_\beta$ we find

$$\boldsymbol{p}_\alpha = \frac{\partial T}{\partial \boldsymbol{u}_\alpha} = \boldsymbol{M}_{\alpha\alpha}\boldsymbol{u}_\alpha + \boldsymbol{M}_{\alpha\beta}\boldsymbol{u}_\beta = \boldsymbol{\alpha}^T\boldsymbol{p}, \tag{51}$$

$$\boldsymbol{p}_\beta = \frac{\partial T}{\partial \boldsymbol{u}_\beta} = \boldsymbol{M}_{\beta\alpha}\boldsymbol{u}_\alpha + \boldsymbol{M}_{\beta\beta}\boldsymbol{u}_\beta = \boldsymbol{\beta}^T\boldsymbol{p}. \tag{52}$$

Obviously, by duality, the transpose projections $\boldsymbol{\alpha}^T$ and $\boldsymbol{\beta}^T$ have to be used in the space of momenta and forces. In the constrained system, since $\boldsymbol{u}_\beta$ vanishes, we have the two momenta

$$\boldsymbol{p}_\alpha = \boldsymbol{M}_{\alpha\alpha}\boldsymbol{u}_\alpha \tag{53}$$

and

$$\boldsymbol{p}_\beta = \boldsymbol{M}_{\beta\alpha}\boldsymbol{u}_\alpha. \tag{54}$$

From (54) together with (34) and (49) follows

**Theorem 3:** In the constrained system the momentum $\boldsymbol{p}_\beta$ vanishes for an arbitrary $\boldsymbol{u}_\alpha$ if and only if the projections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are mass-orthogonal.

This theorem will be the basis for deriving the formula for the constraint reactions.

We have already seen that in the spaces of forces and momenta the projections $\boldsymbol{\alpha}^T$ and $\boldsymbol{\beta}^T$ have to be used. An immediate consequence is that the formulas for the influence matrices corresponding to (49) are of the form

$$\boldsymbol{A}_{\alpha\alpha} = \boldsymbol{\alpha A \alpha}^T, \quad \boldsymbol{A}_{\alpha\beta} = \boldsymbol{\alpha A \beta}^T = \boldsymbol{A}_{\beta\alpha}^T, \quad \boldsymbol{A}_{\beta\beta} = \boldsymbol{\beta A \beta}^T. \tag{55}$$

Inserting the expressions (44) for the projections in (49) and (55) we have

$$\boldsymbol{M}_{\alpha\alpha} = \boldsymbol{M} - \boldsymbol{\pi}, \quad \boldsymbol{M}_{\alpha\beta} = 0, \quad \boldsymbol{M}_{\beta\alpha} = 0, \quad \boldsymbol{M}_{\beta\beta} = \boldsymbol{\pi} \tag{56}$$

and

$$\boldsymbol{A}_{\alpha\alpha} = \boldsymbol{A} - \boldsymbol{A\pi A}, \quad \boldsymbol{A}_{\alpha\beta} = 0, \quad \boldsymbol{A}_{\beta\alpha} = 0, \quad \boldsymbol{A}_{\beta\beta} = \boldsymbol{A\pi A}. \tag{57}$$

We see that the matrices $\boldsymbol{A}_{\alpha\beta}$ and $\boldsymbol{A}_{\beta\alpha}$ also vanish due to mass-orthogonality. In addition we obtain

**Theorem 4:** The matrices $\boldsymbol{A}_{\alpha\alpha}$, $\boldsymbol{M}_{\alpha\alpha}$ and $\boldsymbol{A}_{\beta\beta}$, $\boldsymbol{M}_{\beta\beta}$ respectively are generalized inverses in the mass metric of the free system, i.e.:

$$\boldsymbol{A}_{\alpha\alpha}\boldsymbol{M}_{\alpha\alpha} = \boldsymbol{\alpha}, \quad \boldsymbol{M}_{\alpha\alpha}\boldsymbol{A}_{\alpha\alpha} = \boldsymbol{\alpha}^T \tag{58}$$

and

$$\boldsymbol{A}_{\beta\beta}\boldsymbol{M}_{\beta\beta} = \boldsymbol{\beta}, \quad \boldsymbol{M}_{\beta\beta}\boldsymbol{A}_{\beta\beta} = \boldsymbol{\beta}^T. \tag{59}$$

**Proof:** The definition of the generalized inverse [9] demands that the product of the two matrices be an orthogonal projection acting as unity on each factor. In our case orthogonality will hold in the mass metric. In order to prove the first part of (58) we use the relation

$$\boldsymbol{\pi A \pi} = \boldsymbol{M\beta A M \beta} = \boldsymbol{M\beta}^2 = \boldsymbol{M\beta} = \boldsymbol{\pi} \tag{60}$$

to find

$$\begin{aligned} \boldsymbol{A}_{\alpha\alpha}\boldsymbol{M}_{\alpha\alpha} &= (\boldsymbol{A} - \boldsymbol{A\pi A})(\boldsymbol{M} - \boldsymbol{\pi}) = \boldsymbol{AM} - \boldsymbol{A\pi} - \boldsymbol{A\pi AM} + \boldsymbol{A\pi A\pi} \\ &= \boldsymbol{I} - \boldsymbol{A\pi} = \boldsymbol{\alpha}. \end{aligned} \tag{61}$$

and

$$\boldsymbol{M}_{\alpha\alpha}\boldsymbol{A}_{\alpha\alpha} = (\boldsymbol{A}_{\alpha\alpha}\boldsymbol{M}_{\alpha\alpha})^T = \boldsymbol{\alpha}^T. \tag{62}$$

The formulas (59) follow in a similar way.

As a result of Theorem 4 the relation (53) can be inverted to yield

$$\boldsymbol{u}_\alpha = \boldsymbol{A}_{\alpha\alpha}\boldsymbol{p}_\alpha. \tag{63}$$

## 5. Dual formulation of constraints

In section 3 we have assumed kinematical constraint equations. The rows of the matrix $\boldsymbol{E}^T$ in (23) can be interpreted as a virtual constraint force, and this equation then simply states that the power of such forces has to vanish for an admissible motion.

In a dual formulation one can assume a matrix $\boldsymbol{F}^T$ of virtual admissible velocities and formulate the constraint condition as

$$\boldsymbol{F}^T\boldsymbol{Q} = 0. \tag{64}$$

The linear space of forces satisfying (64) then coincides with the image of $\boldsymbol{\beta}^T$. By duality, the two projection operators $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be constructed from $\boldsymbol{F}^T$, uniqueness being guaranteed again by an orthogonality condition. Notice that the metric in the force space is given by the matrix $\boldsymbol{A}$. The dual formulation of Theorem 2 yields

**Theorem 2′:** The following conditions uniquely define the projection $\boldsymbol{\alpha}$:

i) $\boldsymbol{\alpha}^2 = \boldsymbol{\alpha}$, (65)

ii) $\boldsymbol{\alpha F} = \boldsymbol{F}, \quad r(\boldsymbol{\alpha}) = r(\boldsymbol{F}) = n_a$, (66)

iii) $\boldsymbol{\alpha A} = (\boldsymbol{\alpha A})^T$. (67)

The *fundamental matrix* now is

$$\boldsymbol{H} = \boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F}. \tag{68}$$

It is a symmetric positive definite matrix of order $n_a$. With

$$\boldsymbol{\psi} = \boldsymbol{\alpha A} = \boldsymbol{F}\boldsymbol{H}^{-1}\boldsymbol{F}^T, \tag{69}$$

the two projections are

$$\boldsymbol{\alpha} = \boldsymbol{\psi M}, \quad \boldsymbol{\beta} = \boldsymbol{I} - \boldsymbol{\psi M}. \tag{70}$$

The expressions for the mass and influence matrices (56) and (57) now read

$$\boldsymbol{M}_{\alpha\alpha} = \boldsymbol{M\psi M}, \quad \boldsymbol{M}_{\alpha\beta} = 0, \quad \boldsymbol{M}_{\beta\alpha} = 0, \quad \boldsymbol{M}_{\beta\beta} = \boldsymbol{M} - \boldsymbol{M\psi M} \tag{71}$$

$$\boldsymbol{A}_{\alpha\alpha} = \boldsymbol{\psi}, \quad \boldsymbol{A}_{\alpha\beta} = 0, \quad \boldsymbol{A}_{\beta\alpha} = 0, \quad \boldsymbol{A}_{\beta\beta} = \boldsymbol{A} - \boldsymbol{\psi}. \tag{72}$$

The dual approach is convenient for mechanisms and systems with $n_a$ small in comparison to $n_c$. One has to realize, however, that there is a basic difference between the two descriptions. While it is possible to choose matrices $\boldsymbol{E}^T$ of full rank globally, this will not in general be possible for the matrices $\boldsymbol{F}^T$. In fact, a global regular $\boldsymbol{F}^T$ would imply the existence of $n_a$ linearly independent vector fields tangent to the configuration manifold of the constrained system. This is only possible if this manifold is a torus, an Euclidean space or a product of the two.

## 6. Derivatives of projections

In order to derive the dynamical equations of the constrained system, the time-derivative of the two projections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will be needed. This time-derivative, which we denote by a dot, is understood to be the convected derivative as the point representing the system moves along the configuration manifold.

From

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{I} \tag{73}$$

we find

$$\dot{\boldsymbol{\alpha}} = -\dot{\boldsymbol{\beta}}. \tag{74}$$

Let us first assume that $A$ is constant. Differentiating

$$\boldsymbol{\pi} = \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{E}^T = \boldsymbol{E}(\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E})^{-1}\boldsymbol{E}^T \tag{75}$$

and introducing

$$\dot{\boldsymbol{E}} = \boldsymbol{D}, \tag{76}$$

we find

$$\begin{aligned} \dot{\boldsymbol{\pi}} := \boldsymbol{\omega} &= \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{D}^T - \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{D}^T\boldsymbol{A}\boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{E}^T \\ &\quad + \boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{E}^T - \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{E}^T \\ &= \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{D}^T(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{\pi}) + (\boldsymbol{I} - \boldsymbol{A}\boldsymbol{\pi})\boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{E}^T, \end{aligned} \tag{77}$$

or, with the notation

$$\boldsymbol{\Omega} = \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{D}^T, \tag{78}$$

finally,

$$\boldsymbol{\omega} = \boldsymbol{\Omega}\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\boldsymbol{\Omega}^T. \tag{79}$$

In a second step, we assume $\boldsymbol{E}^T$ constant and compute the influence of $\dot{\boldsymbol{M}}$. From

$$\boldsymbol{\beta} = \boldsymbol{A}\boldsymbol{\pi} = \boldsymbol{A}\boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{E}^T = \boldsymbol{A}\boldsymbol{E}(\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E})^{-1}\boldsymbol{E}^T \tag{80}$$

we get

$$\begin{aligned} \dot{\boldsymbol{\beta}} &= \dot{\boldsymbol{A}}\boldsymbol{\pi} - \boldsymbol{A}\boldsymbol{\pi}\dot{\boldsymbol{A}}\boldsymbol{\pi} \\ &= (\boldsymbol{I} - \boldsymbol{A}\boldsymbol{\pi})\dot{\boldsymbol{A}}\boldsymbol{M}\boldsymbol{\beta} = \boldsymbol{\alpha}\dot{\boldsymbol{A}}\boldsymbol{M}\boldsymbol{\beta} \\ &= -\boldsymbol{\alpha}\boldsymbol{A}\dot{\boldsymbol{M}}\boldsymbol{\beta} = -\boldsymbol{A}\boldsymbol{\alpha}^T\dot{\boldsymbol{M}}\boldsymbol{\beta} = -\boldsymbol{A}\dot{\boldsymbol{M}}_{\alpha\beta}. \end{aligned} \tag{81}$$

As a result we obtain the formula

$$\dot{\boldsymbol{\beta}} = -\dot{\boldsymbol{\alpha}} = \boldsymbol{A}\boldsymbol{\omega} - \boldsymbol{A}\dot{\boldsymbol{M}}_{\alpha\beta}. \tag{82}$$

In the dual formulation the corresponding formulas read

$$\boldsymbol{\Theta} = \dot{\boldsymbol{F}}\boldsymbol{H}^{-1}\boldsymbol{F}^T, \tag{83}$$

$$\boldsymbol{\theta} = \boldsymbol{\beta}\boldsymbol{\Theta} + \boldsymbol{\Theta}^T\boldsymbol{\beta}^T, \tag{84}$$

$$\dot{\boldsymbol{\alpha}} = -\dot{\boldsymbol{\beta}} = \boldsymbol{\theta}\boldsymbol{M} - \dot{\boldsymbol{A}}_{\alpha\beta}\boldsymbol{M}. \tag{85}$$

## 7. Dynamics of the constrained system

In Theorem 3 we have seen that the momentum $\boldsymbol{p}_\beta$ of the constrained system vanishes as a consequence of mass-orthogonality. Differentiating the equation

$$\boldsymbol{p}_\beta = \boldsymbol{\beta}^T\boldsymbol{p} = 0 \tag{86}$$

we find

$$\begin{aligned}\dot{\boldsymbol{p}}_\beta &= \boldsymbol{\beta}^T\dot{\boldsymbol{p}} + \dot{\boldsymbol{\beta}}^T\boldsymbol{p} \\ &= \boldsymbol{\beta}^T\hat{\boldsymbol{Q}} + \boldsymbol{\omega}\boldsymbol{A}\boldsymbol{p} - \dot{\boldsymbol{M}}_{\beta\alpha}\boldsymbol{A}\boldsymbol{p} = 0,\end{aligned} \tag{87}$$

or

$$\hat{\boldsymbol{Q}}_\beta = -\boldsymbol{\omega}\boldsymbol{A}\boldsymbol{p} + \dot{\boldsymbol{M}}_{\beta\alpha}\boldsymbol{A}\boldsymbol{p}. \tag{88}$$

Let us now assume that the forces $\boldsymbol{Q}$ are composed of loads $\boldsymbol{F}$ and non-working constraint reactions $\boldsymbol{R} = \boldsymbol{R}_\beta$. Then

$$\boldsymbol{Q}_\alpha = \boldsymbol{F}_\alpha, \quad \boldsymbol{Q}_\beta = \boldsymbol{F}_\beta + \boldsymbol{R}_\beta \tag{89}$$

and

$$\hat{\boldsymbol{Q}}_\alpha = \hat{\boldsymbol{F}}_\alpha, \quad \hat{\boldsymbol{Q}}_\beta = \hat{\boldsymbol{F}}_\beta + \boldsymbol{R}_\beta. \tag{90}$$

Here the correction terms (16) have been included in the loads. We thus have

**Theorem 5:** The non-working constraint reactions are given as functions of position and velocities by

$$\boldsymbol{R}_\beta = -\hat{\boldsymbol{F}}_\beta - \boldsymbol{\omega}\boldsymbol{u} + \dot{\boldsymbol{M}}_{\beta\alpha}\boldsymbol{u}. \tag{91}$$

In order to derive the dynamical equations for the constrained system we differentiate

$$\boldsymbol{p}_\alpha = \boldsymbol{\alpha}^T\boldsymbol{p} = \boldsymbol{p} \tag{92}$$

to obtain

$$\dot{\boldsymbol{p}}_\alpha = \dot{\boldsymbol{p}} = \hat{\boldsymbol{Q}}_\alpha + \hat{\boldsymbol{Q}}_\beta. \tag{93}$$

Inserting (88), we get

$$\dot{\boldsymbol{p}}_\alpha = \hat{\boldsymbol{Q}}_\alpha - \boldsymbol{\omega} \boldsymbol{A} \boldsymbol{p}_\alpha + \dot{\boldsymbol{M}}_{\beta\alpha} \boldsymbol{A} \boldsymbol{p}_\alpha, \tag{94}$$

which has to be combined with

$$\boldsymbol{u}_\alpha = \boldsymbol{A}_{\alpha\alpha} \boldsymbol{p}_\alpha = \boldsymbol{\alpha} \boldsymbol{A} \boldsymbol{p}_\alpha \tag{95}$$

and, eventually,

$$\dot{\boldsymbol{q}} = \tilde{\boldsymbol{B}} \boldsymbol{u}. \tag{96}$$

Dropping the indices $\boldsymbol{\alpha}$ in (94) and (95) we get

**Theorem 6:** The equations below correctly describe the dynamics of the constrained systems on its configuration manifold:

$$\dot{\boldsymbol{p}} = \hat{\boldsymbol{Q}}_\alpha - \boldsymbol{\omega} \boldsymbol{A} \boldsymbol{p} + \dot{\boldsymbol{M}}_{\beta\alpha} \boldsymbol{A} \boldsymbol{p}, \tag{97}$$

$$\dot{\boldsymbol{q}} = \tilde{\boldsymbol{B}} \boldsymbol{u}, \quad \boldsymbol{u} = \boldsymbol{\alpha} \boldsymbol{A} \boldsymbol{p}. \tag{98}$$

The extension of the dynamical equations from the constrained system to the phase space of the free system, however, is not unique. Another version of the basic equations is given by

**Theorem 7:** The equations below correctly describe the dynamics of the constrained system on its configuration manifold:

$$\dot{\boldsymbol{p}} = \hat{\boldsymbol{Q}}_\alpha - \boldsymbol{\Omega} \boldsymbol{u} + \boldsymbol{\beta}^T \dot{\boldsymbol{M}} \boldsymbol{u}, \tag{99}$$

$$\dot{\boldsymbol{q}} = \tilde{\boldsymbol{B}} \boldsymbol{u}, \quad \boldsymbol{u} = \boldsymbol{A} \boldsymbol{p}. \tag{100}$$

**Proof:** For the constrained system

$$\boldsymbol{u} = \boldsymbol{u}_\alpha = \boldsymbol{\alpha} \boldsymbol{A} \boldsymbol{p}_\alpha = \boldsymbol{A} \boldsymbol{\alpha}^T \boldsymbol{p}_\alpha = \boldsymbol{A} \boldsymbol{p}_\alpha = \boldsymbol{A} \boldsymbol{p}, \tag{101}$$

which proofs the equivalence of (98) and (100). Also, from (79)

$$\boldsymbol{\omega} \boldsymbol{u} = \boldsymbol{\omega} \boldsymbol{u}_\alpha = \boldsymbol{\Omega} \boldsymbol{u}_\alpha = \boldsymbol{\Omega} \boldsymbol{u}, \tag{102}$$

since

$$\boldsymbol{\Omega}^T \boldsymbol{\alpha} = 0. \tag{103}$$

Furthermore,

$$\dot{\boldsymbol{M}}_{\beta\alpha} \boldsymbol{u} = \boldsymbol{\beta}^T \dot{\boldsymbol{M}} \boldsymbol{\alpha} \boldsymbol{u} = \boldsymbol{\beta}^T \dot{\boldsymbol{M}} \boldsymbol{u} \tag{104}$$

So (99) is equivalent to (97).

Notice, however, that the two sets of dynamical equations differ in respect to the integrals they admit in the space of the free system. The equations (99) and (100) of Theorem 7 admit the integral

$$J = E^T A p. \tag{105}$$

In fact,

$$\begin{aligned} \dot{J} &= E^T A \dot{p} + E^T \dot{A} p + \dot{E}^T A p \\ &= E^T A \hat{Q}_\alpha - E^T A \Omega u + E^T A \beta^T \dot{M} u \\ &\quad - E^T A \dot{M} A p + \dot{E}^T A p. \end{aligned} \tag{106}$$

But, due to

$$E^T A \hat{Q}_\alpha = E^T A \alpha^T \hat{Q} = E^T \alpha A \hat{Q} = 0, \tag{107}$$

$$-E^T A \Omega u + \dot{E}^T A p = -E^T A E G^{-1} \dot{E}^T u + \dot{E}^T u = 0 \tag{108}$$

and finally

$$E^T A \beta^T \dot{M} u - E^T A \dot{M} A p = E^T \beta A \dot{M} u - E^T A \dot{M} u = 0, \tag{109}$$

the right hand side of (106) vanishes. This allows for the interpretation that in the set of equations (99) and (100) no additional instabilities are introduced by describing the constrained system in the space of the free system.

In using either Theorem 6 or Theorem 7 for computations (see e.g. [2]) one will have to be careful to start with initial conditions satisfying the constraints of the system both for position and momenta.

## References


[1] H. Brauchli and R. W. Weber, *Canonical approach to multibody systems using redundant coordinates*, Dynamics of Multibody Systems, IUTAM/IFToMM Symposium, Udine/Italy 1985, G. Bianchi and W. Schiehlen eds., Springer 1986, pp. 31–41.

[2] H. Brauchli, G. Devaquet, Ch. Schaerer and J. Rohrer, *Ein Baukastensystem zur Beschreibung von Vielkörpersystemen*, Simulationstechnik, Zürich 1987, J. Halin Hrsg., Springer 1987, pp. 136–143.

[3] H. Brauchli and R. Weber, *Dynamical equations in natural coordinates*, submitted for publication.

[4] G. Hamel, *Die Lagrange-Eulerschen Gleichungen der Mechanik*, Teubner, Leipzig 1903.

[5] H. Hemami and F. C. Weimer, *Modelling of nonholonomic dynamic systems with applications*, J. Appl. Mech. *48*, 177–182 (1981).

[6] R. L. Huston, *Useful procedures in multibody dynamics*, Dynamics of Multibody Systems, IUTAM/IFToMM Symposium, Udine/Italy 1985, G. Bianchi and W. Schiehlen eds., Springer, pp. 69–77.

[7] J. W. Kamman and R. L. Huston, *Dynamics of constrained multibody systems*, J. Appl. Mech. *51*, 899–903 (1984).

[8] C. E. Passarello and R. L. Huston, *Another look at nonholonomic systems*, J. Appl. Mech. *40*, 101–104 (1973).

[9] R. Penrose, *A generalized inverse for matrices*, Proc. Cambr. Phil. Soc. *51*, 406–413 (1955).

[10] R. E. Roberson and R. Schwertassek, *Dynamics of multibody systems*, Springer 1988.

[11] R. P. Singh and P. W. Likins, *Singular value decomposition for constrained dynamical systems*, J. Appl. Mech. *52*, 943–948 (1985).

[12] R. W. Weber, *Eine alternative Herleitung der Euler-Hamilton Gleichungen der Mechanik*, Z. ang. Math. Phys. *31*, 780–784 (1980).

[13] J. Wittenburg, *Dynamics of systems of rigid bodies*, Teubner, Stuttgart 1977.

**Abstract**

A projection method based on the classical equations of mechanics for the description of multibody systems is presented. The system is described by arbitrary coordinates and velocity parameters. From the constraint equations two complementary projections splitting the space of velocities into the space of admissible and inadmissible velocities respectively are constructed. They are uniquely determined by a condition of mass-orthogonality. A consistent description of the dynamics of the constrained system results. The constraint reactions are given as functions of position and velocities and an explicit system of differential equations for the motion of the constrained system is derived.

**Zusammenfassung**

Eine Projektionsmethode zur Beschreibung von Vielkörpersystemen, die auf den klassischen Gleichungen der Mechanik basiert, wird vorgestellt. Das System wird mit beliebigen Lagekoordinaten und Geschwindigkeiten beschrieben. Aus den Bindungsgleichungen werden zwei komplementäre Projektionen konstruiert, die den Raum der Geschwindigkeiten des gebundenen Systems in den Raum der zulässigen und unzulässigen Geschwindigkeiten spalten. Sie sind durch eine Bedingung der Massenorthogonalität eindeutig festgelegt. Es resultiert eine konsistente Beschreibung der Dynamik des gebundenen Systems. Die Bindungsreaktionen werden als Funktion von Lage und Geschwindigkeiten angegeben, und ein explizites System von Differentialgleichungen für die Bewegung des gebundenen Systems wird hergeleitet.

(Received: July 5, 1990)