Journal of Applied Mathematics and Physics (ZAMP)
Vol. 39, January 1988

0044-2275/88/001084-12 $ 3.90/0


# A criterion for shape control robustness of space structures

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## Introduction

In a comment [1] the author of this paper argued that for the control of static deformations of space structures ones does not need to take into account the mass of the structure [2]. In the above comment [1] a method for the control of the static deformations of space structures without involving the mass of the structure was also proposed. A somewhat different method in which the mass of the structure is involved was proposed earlier in Refs. 3, 4.

To find the optimal placement of controls for static deformations of space structures one must know in advance the expected distortion from the desired shape of the structure [3]. However, usually this is not the case. Hence, a different approach is needed [8].

Here, a criterion for the robustness of the shape control of space structures is proposed. The criterion is based on the characteristic number of a certain matrix which plays a major role in the process of the shape control of space structures. The criterion is independent of the possible distortions from the desired shape of the structure. It is proposed to define the placement of the control points by optimization of the measure of robustness.

## Static shape control of space structures

In what follows, the method for static shape control of space structures, without involvement of the mass of the structure, proposed in Ref. 1, is rederived with emphasis and explanations of special points connected with the purpose of this paper.

The static behaviour of a free space structure is represented by its stiffness matrix $K(n \times n)$, usually obtained by using finite element methods, where $n$ is the number of coordinates of the free space structure.

The rank of $K$ is

$$\operatorname{rank}(K) = n - r \tag{1}$$

where $r$ represents the number of independent rigid body motions of the structure. (For space structures, $r = 6$.) The stiffness matrix, $K$, is needed to calculate the deformations of the structure constrained in a statically definite way. In this case the constrained stiffness matrix, $k[(n-r) \times (n-r)]$, becomes nonsingular and the deformations of the structure are well defined.

The rigid body displacements of the space structure are characterized by the matrix $R(n \times r)$. For convenience, the matrix $R$ can be orthonormalized to obtain

$$R^t R = 1. \tag{2}$$

It is clear that the rigid body shapes, $R_i$, are linearily independent. Hence,

$$\operatorname{rank}(R) = r. \tag{3}$$

The distortion from the desired shape of the structure is represented by the deformation vector $\psi(n \times 1)$. The vector $\psi$ is assumed to represent only shape distortions. Hence, it is orthogonal to $R$,

$$R^t \psi = 0. \tag{4}$$

Thermal or force controls are applied on the structure to minimize the distortion. The vector $\Delta T(m \times 1)$ represents the temperature of $m$ heating control points on the structure. In the case of force control the forces applied on the free structure must be in equilibrium. This means that any particular group of control forces can be characterized by one parameter $P_i$. The control forces are represented by the vector $P(m \times 1)$ where $m$ is now the number of independent force parameters. Hence, for the force control, one has to replace $\Delta T$ by $P$.

By constraining the structure in a statically definite way, one can calculate the deformations caused by applying a separate unit temperature at any one of the control points. For force control the independent unit temperature must be replaced by the independent unit force parameters. Clearly a mixed temperature-force control is possible in principle. The result of the independently applied unit temperatures will be a field of displacements represented by the matrix $u_0(n \times m)$. Note that the displacements of the $r$ constrained points are zero.

Now, a deeper examination of the displacement matrix, $u_0$, is needed. Any one of the columns of this matrix is obtained by applying mutually independent parameters – unit temperature or unit force parameters. The mathematical meaning of this physical observation is that the columns are linearly independent. Hence,

$$\operatorname{rank}(u_0) = m. \tag{5}$$

As usual, the angle between any two deformation vectors $u_{0i}$ and $u_{0j}$ is defined by

$$\cos \alpha_{ij} = \frac{u_{0i}^T u_{0j}}{\sqrt{u_{0i}^T u_{0i}} \cdot \sqrt{u_{0j}^T u_{0j}}}. \tag{6}$$

Clearly, the angle $\alpha_{ij}$ and hence the independence between the deformation vectors is a function of the placement of the control points. Intuitively, we would like to have independent deformation vectors, hence, to have $|\cos \alpha_{ij}|$ (for $i \neq j$) as small as possible. As explained later the proposed criterion of robustness is based on the above observation.

The displacement vector of the free structure is given by

$$u = u_0 \Delta T + R\beta \tag{7}$$

where $\beta(r \times 1)$ represents the amount of any of the rigid body shapes. Now, we would like the displacement defined by Eq. (7) to represent only shape changes of the structure. Hence, $u$, like $\psi$, has to be orthogonal to the rigid body shapes, $R$,

$$R^t u = R^t u_0 \Delta T + R^t R \beta = 0. \tag{8}$$

By substitution of Eq. (2) into Eq. (8) one obtains,

$$\beta = - R^t u_0 \Delta T. \tag{9}$$

Equation (7) now becomes

$$u = (I - R R^T)\, u_0 \Delta T. \tag{10}$$

The initial displacement matrix $u_0(n \times m)$ was calculated by constraining the structure in a statically definite way. However, the $r$ constrained points are not uniquely defined. The question that arises is: how the choice of the constrained points influences Eq. (10)? To clarify this point the relation between two displacement vectors caused by the same unit control parameter but with different constrained points will be examined.

Let $u_{0i}^{\alpha}(n \times 1)$ and $u_{0i}^{\gamma}(n \times 1)$ be the displacement vectors due to $\Delta T_i = 1$ (or $P_i = 1$). $\alpha$ and $\gamma$ designate two different configurations of the statically defined constrained structure. It must be emphasized that in both cases the reactions in the $r$ constrained points are zero. Hence, no additional forces are introduced. It is clear that $u_{0i}^{\alpha}$ and $u_{0i}^{\gamma}$ differ only by some amount of rigid body displacements,

$$u_{0i}^{\gamma} = u_{0i}^{\alpha} + R\gamma \tag{11}$$

where $\gamma(r \times 1)$ represents the difference in the amount of rigid body displacements.

By multiplication of Eq. (11) by $R^T$ one obtains,

$$\gamma = R^T (u_{0i}^{\gamma} - u_{0i}^{\alpha}). \tag{12}$$

Substitution of Eq. (12) into Eq. (11) yields,

$$(I - R R^T)\, u_{0i}^{\gamma} = (I - R R^T)\, u_{0i}^{\alpha}. \tag{13}$$

It is clear from Eq. (13) that Eq. (10) is not influenced by the choice of the constrained points. More than this. One can choose different configurations for any of the unit control parameters. The final result will be the same. However, in what follows, for definiteness, we will suppose that the choosen constrained configuration is the same for all the cases. It must be noted that although the choice of the constrained points does not influence Eq. (10), the displacement matrix $u_0$, by itself, strongly depends on the choice of the constrained configuration. It will be interesting to show that the proper choice of the constrained configuration depends on a criterion similar to the criterion of robustness of control proposed here.

The total displacements vector caused by the distortion and the controls will be given by

$$u_T = \psi + u. \tag{14}$$

To calculate the control parameters it is appropriate to minimize the Euclidian norm of $u_T$,

$$u_T^2 = (\psi^t + u^t)(\psi + u) = \psi^t \psi + 2u^t \psi + u^t u. \tag{14}$$

Now

$$\begin{aligned} \frac{\partial u_T^2}{\partial \Delta T} = 0 &= 2\frac{\partial u^t}{\partial \Delta T}\psi + 2\frac{\partial u^t}{\partial \Delta T}u \\ &= 2u_0^t[I - RR^T]\psi + 2u_0^t[I - RR^T][I - RR^T]u_0 \Delta T \\ &= 2u_0^t\psi + 2u_0^t[I - RR^T]u_0\Delta T \\ &= 2u_0^t(\psi + u) = 2u_0^t u_T \end{aligned} \tag{15}$$

or

$$A\Delta T = q \tag{16}$$

where

$$A = u_0^t[I - RR^T]u_0 \tag{17}$$

and

$$q = -u_0^t \psi. \tag{18}$$

Note that the solution of Eq. (16) depends strongly on the behaviour of the matrix $A(m \times m)$ of Eq. (17) which has to be inverted. Clearly, all the discussed process is controlled by the basic matrix $A$. An important additional observation is that due to Eq. (18), distortions orthogonal to $u_0$ can not be corrected by the proposed method. This can be done only by properly chosen control points. However, to do this one must know in advance the distortion vector $\psi$, which is usually not the case.

Following Ref. 3, one can define an efficient coefficient

$$g^2 = u_T^2/\psi^2. \tag{19}$$

It is easy to show that

$$g^2 = \frac{u_T^t u_T}{\psi^t \psi} = 1 - \frac{\Delta T^t q}{\psi^t \psi} = 1 - \frac{\psi^t u_0 A^{-1} u_0^t \psi}{\psi^t \psi}. \tag{20}$$

Again, to calculate the efficient coefficient defined by Eq. (20) one must assume that the matrix $A$ is invertible and that the distortion vector $\psi$ is known in advance.

## Analysis of the basic matrix $A$

In the realization of the proposed shape control method the matrix $A$ from Eq. (17) was assumed tacitly to be invertible. Is it? Clearly, an analysis of this matrix is needed. To do so we will begin with the rigid body shape matrix $R(n \times r)$.

Following Ref. 5, one obtains from Eq. (3),

$$\operatorname{rank}(R) = \operatorname{rank}(RR^T) = \operatorname{rank}(R^T R) = \operatorname{rank}(R^T) = r. \tag{21}$$

The matrix $RR^T$ is symmetric and therefore [6] one can find an orthogonal matrix $U(n \times n)$ so that,

$$U^T R R^T U = v, \qquad U^T U = I \tag{22}$$

where $v$ is a real diagonal matrix.

However, it is easy to show that the matrix $RR^T$ has the following eigenvalues

$$\begin{aligned} v_i &= 1 \qquad i = 1 \div r \\ v_j &= 0 \qquad j = r + 1 \div n. \end{aligned} \tag{22a}$$

It is also easy to show that the eigenvectors connected with the eigenvalues $v_i = 1$ of the matrix $RR^t$ are the rigid body shapes $R_i$.

Now, we will try to diagonalize the matrix $[I - RR^T]$ by using the same orthogonal matrix $U$,

$$U^T[I - RR^T] U = I - U^T R R^T U = 1 - v = \lambda \tag{23}$$

or

$$\begin{aligned} \lambda_i &= 1 - v_i = 0 \qquad i = 1 \div r \\ \lambda_j &= 1 - v_j = 1 \qquad j = r + 1 \div n. \end{aligned} \tag{24}$$

For the matrix $[I - RR^T]$ the zero eigenvalues are connected to the rigid body shapes.

From Eqs. (23) and (24) it is clear that,

$$\operatorname{rank}[I - RR^T] = n - r. \tag{25}$$

The initial deformation matrix $u_0(n \times m)$, was obtained by constraining the structure in a statically definite way and by applying independent unit temperature or force loads. Hence, the columns of the matrix $u_0$ can not be obtained by any linear combination of the rigid body mode shapes,

$$u_0 \alpha \neq R\phi \tag{26}$$

for any $\alpha(m \times 1)$ and $\phi(r \times 1)$ different than zero.

It will be shown now that the rank of the matrix $[I - RR^T]\, u_0$ is the same as the matrix $u_0$ (Eq. 5). To prove this we will assume the opposite, that is, we will assume that the vectors $[I - RR^T]\, u_{0i}$ are linearily dependent,

$$[1 - RR^T]\, u_0 \alpha = 0 \tag{27}$$

where $\alpha(m \times 1)$ represents the coefficients of linear dependence. But, as noted before, it is easy to show that any combination of rigid body shapes is a eigenvector of the matrix $[I - RR^T]$ with zero eigenvalue. Indeed,

$$[I - RR^T]\, R\phi = R\phi - R\phi = 0\,. \tag{28}$$

To satisfy Eq. (27) the vector $u_0 \alpha$ must be an eigenvector of the matrix $[I - RR^T]$ with zero eigenvalue. Then,

$$u_0 \alpha \stackrel{?}{=} R\phi \tag{29}$$

but Eq. (29) contradicts Eq. (26), hence

$$\alpha = 0; \qquad \phi = 0; \tag{30}$$

Equations (27) and (30) show that the vectors $[I - RR^T]\, u_{0i}$ are linearly independent, or

$$\operatorname{rank}([I - RR^T]\, u_0) = \operatorname{rank}(u_0) = m\,. \tag{31}$$

From Eqs. (25) and (27) it follows

$$m \leqq n - r\,. \tag{32}$$

Now [5]

$$\begin{aligned} \operatorname{rank}(A) &= \operatorname{rank}(u_0^T [I - RR^t]\, u_0) = \operatorname{rank}(u_0^t [I - RR^t]\, [I - RR^t]\, u_0) \\ &= \operatorname{rank}(([I - RR^t]\, u_0)^t\, ([I - RR^T)\, u_0)) = \operatorname{rank}([I - RR^T)\, u_0) = m\,. \end{aligned} \tag{33}$$

Equation (33) shows that the basic matrix $A(m \times m)$ is invertible. More than that. The matrix $A$ is positive definite. To prove this we will introduce the auxiliary vectors $V(n \times 1)$ and $X(m \times 1)$,

$$V = [I - RR^t]\, u_0 x\,. \tag{34}$$

Due to Eqs. (27) and (30)

$$V \neq 0 \qquad \text{for any } x_i \neq 0\,. \tag{35}$$

Now,

$$\begin{aligned} V^t V &= X^t u_0 [I - R R^t] [I - R R^t] u_0 X = X^t u_0^t [I - R R^T] u_0 x \\ &= X^t A X > 0 \qquad \text{for any } x_i \neq 0 . \end{aligned} \tag{36}$$

From Eq. (36) it follows that the matrix $A$ is positive definite and then, all its eigenvalues are positive.

## Robustness

It was shown that the matrix $A$ is nonsingular and the control parameters can be calculated by applying Eq. (16). However, they will depend on the ability to invert the basic matrix $A$. An accepted measure for this ability is called: "the spectral condition number of $A$ with respect to inversion" [6]. For a positive definite matrix this condition number is defined by [6],

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}} \tag{37}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $A$.

It is proposed here to define the shape control robustness of space structures by utilization of the characteristic of the spectral condition number,

$$\mu_{rb} = \frac{1}{\kappa(A)} = \frac{\lambda_{\min}}{\lambda_{\max}} \leqq 1 . \tag{38}$$

$\mu_{rb}$ is the criterion for shape control robustness of space structures, or measure of robustness. One can see that in contrast to the efficient coefficient $g^2$ defined by Eq. (19) the measure of robustness does not depend on the usually unknown in advance distortion vector $\psi$.

For an efficient shape control system of a space structure we propose to find the optimal placement of the controls by maximization of the criterion for shape control robustness $\mu_{rb}$.

In Ref. 3 it was proposed to find the placement of the control points by minimization of the coefficient $g^2$ which can not be done without knowledge of the distortion vector $\psi$. However, if some knowledge of the possible distortion vectors $\psi_i$ exists in advance some compromise between the maximization of $\mu_{rb}$ and minimization of $g^2$ must be worked out.

For the calculation of the initial deformation matrix $u_0$ one has to invert the constrained stiffness matrix $k[(n-r) \times (n-r)]$. It is clear now that the constrained points must be chosen so that the $\mu_{rb}$ coefficient of the stiffness matrix $k$ be made as large as possible.

## Numerical Example

To demonstrate the usefulness of the proposed criterion for shape control robustness of space structures a very simple example was chosen: a free beam is modeled as a five degrees of freedom discrete structure. In Fig. 1 one can see the structure and two configurations of control forces. The structure is permitted to translate in the $y$ direction and rotate in the $x, y$ plane. Hence, $r = 2$. The orthonormalized rigid body shapes are:

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{bmatrix} \qquad \operatorname{rank}(R) = 2 \tag{39}$$

$$R R^T = \begin{bmatrix} 0.6 & 0.4 & 0.2 & 0 & -0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ -0.2 & 0 & 0.2 & 0.4 & 0.6 \end{bmatrix}. \tag{40}$$

Using the Gram-Schmidt orthogonalization process [7] one possible orthogonal matrix [6] is:

$$U = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{30}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{30}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} & 0 & 0 & \frac{2}{\sqrt{10}} \end{bmatrix}. \tag{41}$$

It is easy to check that

$$U^T U = I \tag{42}$$

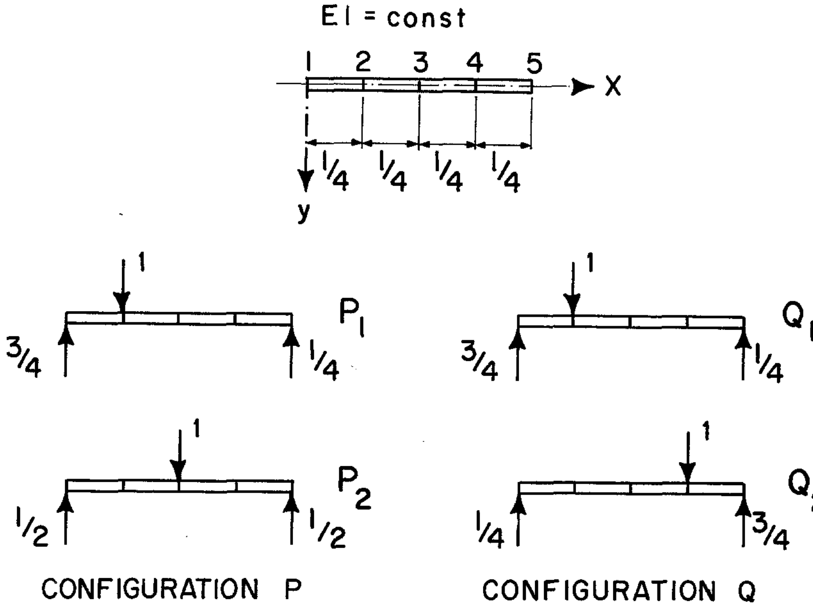


Figure 1
Structure and control forces.

and

$$U^T R R^T U = v = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{43}$$

The initial deformation matrix $u_0$ was obtained by constraining the structure at points 1 and 5 and loading it successively by the different unit control forces (Fig. 1). The so obtained displacements were nondimensionalized by dividing any one of them by the quantity

$$C = \frac{l^3}{76.8\,EI}. \tag{44}$$

For the configurations $P$ and $Q$ one obtains,

$$u_{0P} = \begin{bmatrix} 0 & 0 \\ 0.9 & 1.1 \\ 1.1 & 1.6 \\ 0.7 & 1.1 \\ 0 & 0 \end{bmatrix}; \quad u_{0Q} = \begin{bmatrix} 0 & 0 \\ 0.9 & 0.7 \\ 1.1 & 1.1 \\ 0.7 & 0.9 \\ 0 & 0 \end{bmatrix}. \tag{45}$$

Using (40) and (45) one obtains

$$A_P = u_{0P}^T [I - R R^T]\, u_0^P = \begin{bmatrix} 1.048, & 1.468 \\ 1.468, & 2.092 \end{bmatrix} \tag{46}$$

and

$$A_0 = \begin{bmatrix} 1.048, & 1.016 \\ 1.016, & 1.048 \end{bmatrix}. \tag{47}$$

From (46) and (47) it yields,

$$\mu_{rbP} = \frac{\lambda_{\min}}{\lambda_{\max}} = \frac{0.012}{3.128} = 0.0038; \quad \mu_{rbO} = \frac{0.032}{2.064} = 0.0155\,. \tag{48}$$

From Eqs. (48) it follows that configuration $Q$ has a robustness coefficient larger than configuration $P$ and hence it is a better configuration.

To check this proposition it was decided to generate a distortion vector $\psi_0$ by using a random generator. In this way two distortion vectors were generated.

$$\psi_{10} = \begin{Bmatrix} 6.09 \\ 1.91 \\ 7.16 \\ 8.59 \\ 0.48 \end{Bmatrix}; \quad \psi_{20} = \begin{Bmatrix} 8.29 \\ 6.79 \\ 8.81 \\ 7.76 \\ 4.95 \end{Bmatrix}. \tag{49}$$

The distortion vector orthogonal to the rigid body mode shapes was found by using an expression similar to Eq. (10),

$$\psi = (I - RR^T)\,\psi_0 \tag{50}$$

and then,

$$\psi_1 = \begin{Bmatrix} 0.336 \\ -3.390 \\ 2.314 \\ 4.198 \\ -3.458 \end{Bmatrix}; \quad \psi_2 = \begin{Bmatrix} -0.172 \\ -1.101 \\ +1.490 \\ +1.011 \\ -1.228 \end{Bmatrix}. \tag{51}$$

By using Eqs. (16) and (19) one finds, (see Fig. 2),

$$P_1 = \begin{Bmatrix} 44.120 \\ -33.155 \end{Bmatrix}; \quad g^2_{P_1} = 0.036; \quad Q_1 = \begin{Bmatrix} 22.173 \\ -25.265 \end{Bmatrix}; \quad g^2_{Q_1} = 0.014 \tag{52}$$

and

$$P_2 = \begin{Bmatrix} 13.843 \\ -10.006 \end{Bmatrix}; \quad g^2_{P_2} = 0.012; \quad Q_2 = \begin{Bmatrix} 5.835 \\ -7.352 \end{Bmatrix}; \quad g_{Q_2} = 0.138. \tag{53}$$

The initial and final deformations are shown in Fig. 2.

One can see that in case 1 the control forces and the efficient coefficient $g^2$ are smaller for the $Q$ configuration. For case 2 the distortion vector $\psi$ happened to be almost a linear combination of the initial displacement vectors $u_{0P}$ and hence the efficient coefficient $g^2$ is smaller for the $P$ configuration. However, even in this case the applied control forces $Q$ are smaller than the control forces $P$.

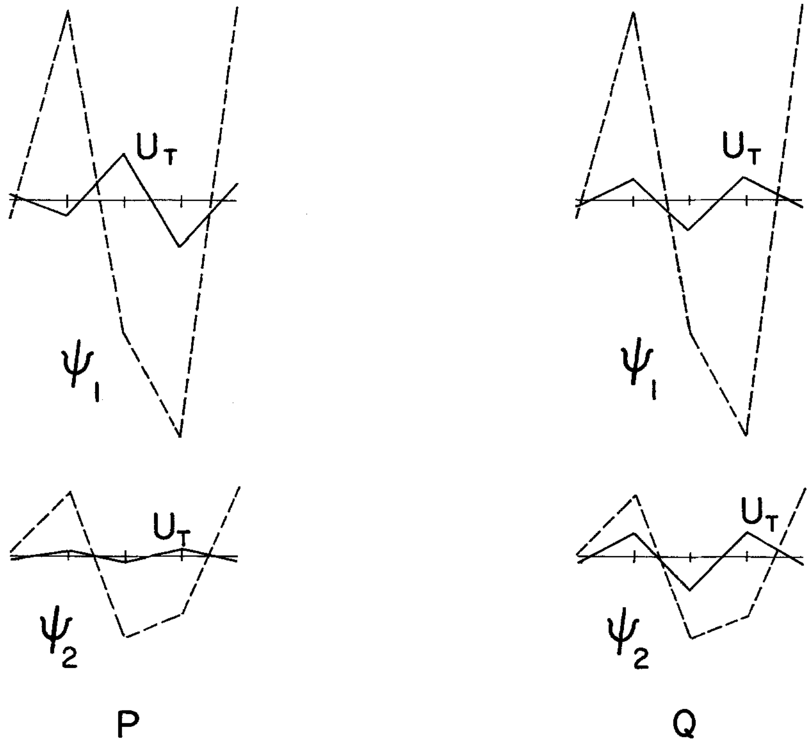


Figure 2
Two different cases of initial distortion.

## Conclusions

A criterion for the robustness of shape control of static deformation was proposed. It was proposed to locate the control points so that the measure of robustness be maximum. A simple numerical example was presented.

## Acknowledgement

The research was supported by the fund for Promotion of Research of the Technion.

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**Abstract**

In the shape control method of space structures a certain matrix plays a major role. It is proposed here that the reciprocal value of the characteristic number of this matrix be taken as criterion for the robustness of the method.

(Received: June 30, 1987)