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On the steady flow of a Newtonian fluid between two parallel disk

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1. Introduction

Steady-state solutions of the Navier-Stokes equations are often obtained by numerical means, either by a Finite Difference, a Finite element or a Boundary Element scheme. These numerical methods are difficult to implement on a computer partly due to the non-linearity of the Navier-Stokes equations and partly due to the need to satisfy the boundary conditions. Even when the solution is separable (in the sense that one or more independent coordinates can be eliminated from the governing equations) one is usually left with a difficult two-point boundary-value problem. Standard packages to solve boundary-value problems are available (e.g. NAGFLIB) but are computer-time consuming if accurate solutions are sought for. In this Note we report a simple method for generating steady state solutions to the flow of a viscous Newtonian fluid between two parallel plates. We will single out two flow configurations which are of interest to the lubrication engineers: the porous squeeze-film flow (sometimes called continuous squeeze-film flow by rheologists) and the coaxial-disk flow. The former flow configuration has been suggested as a basis for a Rheometer [1] and the latter one is commonly known as torsional flow, a flow partially controllable in Pipkin's sense [2]. Both flows have been well studied. Asymptotic solutions for the porous squeeze-film flow have been provided by Terrill and Cornish [3], Rasmussen [4] and Wang [5]. Rasmussen [4] and Wang [5, 6] also provided a numerical solution to this two-point boundary-value problem. The coaxial-disk flow has a more distinguished history dated back to the momentum integral solution of von Kármán [7] and the later works of Batchelor [8] and Stewartson [9]. This flow is fully three-dimensional and has a very complicated structure: it has been known that there are at least 19 non-unique solutions at high enough Reynolds number [10]. In this paper we are only interested in solution at moderate Reynolds number where the uniqueness of von Kármán's solution is guaranteed.

2. Continuous squeeze-film flow

In this section we considered the flow of a Newtonian fluid confined between two parallel circular disks of infinite extent. The bottom plate is rigid and stationary while the upper plate is porous. It is assumed that the flow is generated by the injection of the fluid through the upper plate. In the cylindrical coordinate system depicted in Fig. 1 the velocity field is $\boldsymbol{u} = (u, 0, v)$ where

$$u = \frac{1}{2} V \frac{r}{d} f'(\xi), \qquad v = - V f(\xi), \tag{1}$$

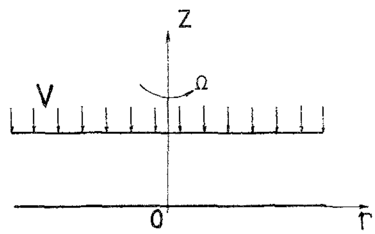


Figure 1
Coordinate system.

in which $-V$ is the vertical velocity of the fluid at the top plate, d is the distance between the plates, $f(\xi)$ is a function to be determined, the prime denotes a derivative with respect to ξ where

$$\xi = z/d$$

is the dimensionless vertical coordinate; $\xi \varepsilon [0, 1]$. Note that (1) satisfies the conservation of mass identically. The boundary conditions on f are

$$f(0) = f'(0) = f'(1) = 0, \quad f(1) = 1. \tag{2}$$

The conservation of linear momentum requires that the steady-state pressure field be given by

$$P = \frac{\eta V}{2d^3} p(r^2 - a^2) - \frac{\eta V}{d}(f' + \tfrac{1}{2}\mathrm{Re}\, f^2) + P_0, \tag{3}$$

where η is the fluid viscosity, a, P_0 are some constants, Re is Reynolds number,

$$\mathrm{Re} = \frac{\varrho V d}{\eta},$$

and p is a constant given by (compatibility between $\partial P/\partial r$ and $\partial P/\partial z$)

$$P = \tfrac{1}{2} f''' - \tfrac{1}{2}\mathrm{Re}\,(\tfrac{1}{2} f'^2 - f f''). \tag{4}$$

If the plates are not of infinite extent but are large compared to the film thickness so that edge effects may be neglected then the above solution will be valid. In that case a may be identified with the common radius of the plates and P_0 may be found by requiring the net radial traction to be zero. Then one can show that the normal force exerted on the top or bottom plate is given by

$$N = -\frac{\pi \eta V a^4}{4 d^3}\left(p + 0\left(\frac{d^2}{a^2}\right)\right). \tag{5}$$

To $0(d^2/a^2)$ p is thus the dimensionless lift force. The velocity field $f(\xi)$ is found from (by taking the derivative of (4))

$$f^{\mathrm{iv}} + \mathrm{Re}\, f f''' = 0. \tag{6}$$

The asymptotic solutions of (6) with the boundary conditions (2) have been well studied [3–6]. For low Reynolds number one has [5]

$$f = 3\xi^2 - 2\xi^3 + \mathrm{Re}\left(\frac{13}{70}\xi^2 - \frac{9}{35}\xi^3 + \frac{1}{10}\xi^6 - \frac{1}{35}\xi^7\right) + 0(\mathrm{Re}^2)$$

and

$$p = -6\left(1 + \frac{9}{70}\mathrm{Re} + 0(\mathrm{Re}^2)\right). \tag{7}$$

At high Reynolds number one has [5]

$$f = 2\xi - \xi^2 + \sqrt{\mathrm{Re}}\left[C(1-\xi)^2 + f_0\left(\frac{\xi}{\sqrt{\mathrm{Re}}}\right) - 2\frac{\xi}{\sqrt{\mathrm{Re}}} - C\right] + 0(\mathrm{Re}^{-1})$$

and

$$p = -6\left(\frac{1}{6}\mathrm{Re} + 0.19\sqrt{\mathrm{Re}} + 0(1)\right), \tag{8}$$

where $C = -1.14$ and $f_0(\cdot)$ is the axi-symmetric stagnation velocity profile, see Wang [5].

We now show that (6) and (2) admit the following power series solution

$$f(\xi) = \sum_{n=1}^{\infty} a_n \xi^{n+1}, \tag{9}$$

where

$$\sum_{n=1}^{\infty} a_n = 1, \quad \sum_{n=1}^{\infty} n a_n = -1, \quad a_3 = 0 \tag{10}$$

and

$$a_{n+3} = -\frac{\mathrm{Re}}{(n+1)(n+2)(n+3)(n+4)} \sum_{m=1}^{n} m(m^2-1) a_m a_{n-m+1}, \quad n > 0. \tag{11}$$

Equations (10a, b) arise due to the need of satisfying the boundary conditions at $\xi = 1$ (the boundary conditions at $\xi = 0$ are satisfied identically) and (10c) and (11) are derived from the equation of motion, (6). The proof of the assertion follows if we can show that the series $\sum_{n=1} n^k a_n$ converges, where $k = 0, 1, 2, \ldots$

First, we observed that $\{a_k\}$ is an alternating sequence. This is easily shown by induction. For, if $\{a_k\}$ is alternating up to $k = n + 2$ then if $n + 3 = 2N$ is even the product $a_m a_{2N-m-2}$ is positive because there are an odd number of terms $(= 2N - 2m - 1)$ between a_m and a_{2N-m+2}. Thus the convolution sum in (11) is positive and a_{2N} is negative. If $n + 3 = 2N + 1$ is odd then there are an even number of terms between a_m and a_{2N-m-1} and therefore a_{2N+1} is positive. The possibility of a_k of the same sign cannot be accepted since it contradicts equation (11). Second, a_n decreases in absolute value and $\lim a_n = 0$ as $n \to \infty$. This is because the main contribution of the convolution sum in (11) comes from the upper limit where $m \sim n$. The absolute value of a_{n+3} decreases faster than any power of n, provided n is large enough. Thus according to a theorem in Calculus the series $\sum_{n=1} n^k a_n$ converges for a fixed k. In particular the three series (9), (10a, b) converge (since $\xi \in [0, 1]$). Thus (9) is indeed the unique solution to the governing equations (6) and (2). We note that if a_1 and a_2 are known then the rest of the sequence $\{a_k\}$ can be computed via the recursive relation (11).

In the numerical scheme described later we found that an excessive number of terms need to be taken to ensure convergence at high Reynolds number ($\mathrm{Re} \geqq 5$). This is because there is a boundary layer formed at the bottom plate. To overcome this we rewrite (9) as

$$f(\xi) = 1 + \sum_{n=1}^{\infty} b_n (1-\xi)^{n+1}. \tag{12}$$

This series, by virtue of the convergence of (9), also converges to a unique solution. The b_n satisfy the following

$$\sum_{n=1}^{\infty} b_n = -1, \quad \sum_{n=1}^{\infty} n\, b_n = 1, \tag{13}$$

$$b_3 = \tfrac{1}{4} \operatorname{Re} b_2, \tag{14}$$

and

$$b_{n+3} = \operatorname{Re}\left\{\frac{1}{n+4} b_{n+2} + \frac{1}{(n+1)(n+2)(n+3)(n+4)} \cdot \sum_{m=1}^{\infty} m(m^2-1)\, b_m b_{n-m+1}\right\}, \quad n > 0. \tag{15}$$

3. Coaxial-disk flow

Next, we consider the flow of a Newtonian fluid between two rigid circular disks of infinite extent. The lower disk is stationary and the upper disk is rotating at an angular velocity of Ω, see Fig. 1. For this flow the velocity field is (von Kármán's solution [7]) $\boldsymbol{u} = \{u, v, w\}$ where

$$\begin{aligned} u &= r\, d\, \Omega\, h'(\xi), \\ v &= r\, \Omega\, g(\xi), \\ w &= -2\Omega\, d\, h(\xi). \end{aligned} \tag{16}$$

The no-slip boundary conditions read

$$h(0) = h'(0) = h(1) = h'(1) = 0, \quad g(0) = 0, \quad g(1) = 1. \tag{17}$$

The conservation of linear momentum requires

$$g'' - 2 \operatorname{Re}(h' g - h g') = 0, \tag{18}$$

and

$$h^{iv} + 2 \operatorname{Re}(g' g + h h''') = 0. \tag{19}$$

Equation (18) arises from the axi-symmetry of the problem and (19), from the compatibility of $\partial P/\partial r$ and $\partial P/\partial z$.

A power series solution to (17–19) is

$$g = \sum_{n=1}^{\infty} c_n \xi^n, \quad h = \sum_{n=1}^{\infty} d_n \xi^{n+1}, \tag{20}$$

where

$$\sum_{n=1}^{\infty} c_n = 1, \quad \sum_{n=1}^{\infty} d_n = 0, \quad \sum_{n=1}^{\infty} n\, d_n = 0 \tag{21}$$

and

$$c_2 = c_3 = 0 = d_3 \tag{22}$$

$$c_{n+2} = \frac{2 \operatorname{Re}}{(n+1)(n+2)} \sum_{m=1}^{n-1} (2m - n + 1)\, d_m c_{n-m}, \quad n \geqq 2, \tag{23}$$

$$d_{n+3} = \frac{-2 \operatorname{Re}}{(n+1)(n+2)(n+3)(n+4)} \sum_{m=1}^{n} \{m\, c_m c_{n-m+1} + m(m^2-1)\, d_m d_{n-m+1}\}, \quad n \geqq 1. \tag{24}$$

Note that (21) comes from the need to satisfy the boundary conditions at $\xi = 1$ and (22–24) are derived from (18–19).

Again in this flow problem if c_1, d_1 and d_2 are known then the rest of $\{c_k\}$ and $\{d_k\}$ can be found from the recursive relation (23–24).

4. Numerical results

Effectively we have transformed a two-point boundary-value problem into solving a non-linear equation, viz. in the continuous squeeze film flow we wish to find a_1 and a_2 – or b_1 and b_2 – so that (10a, b) – or (13) – are satisfied. Next, the series are truncated and in the case of the continuous squeeze-film problem we seek to minimize

$$\left(\sum_{n=1}^{N} a_n - 1\right)^2 + \left(\sum_{n=1}^{N} n\,a_n + 1\right)^2$$

with respect to a_1 and a_2.

This unconstraint minimization problem has received a great deal of attention and has been efficiently coded [11]. We used a standard routine (EO4EDF) in the NAG Fortran Library.

For the continuous squeeze-film problem we found that the series solution (9) is slowly convergent when the Reynolds number is large (Re $\geqq$ 5). This is due to a thin boundary layer at the lower disk. The series (12) converge much faster and one can obtain highly accurate solution with very little computer time. In Table 1 we report the first two coefficients of the series and the dimensionless force at different Reynolds number. All these figures are accurate to six significant figures. The number of the significant figures was determined by increasing the number of terms in the series by 50 to 100 and comparing the coefficients. The highest Reynolds number that we can achieve in this way is 18. At a Reynolds number of 20 the number of terms required for a solution accurate to 3 significant figures is more than 1000 and we do not feel it is justified to push the method any further. It is noteworthy that both the perturbation solution (when Re $\ll$ 1) and the asymptotic solution (when Re $\gg$ 1) predict the dimensionless load well. Wang [5] has suggested that the dimensionless load be given approximately by

$$-p = 6(1 + 0.176\,\mathrm{Re}).$$

His formula is accurate to within 4% for the whole range of Reynolds number considered in Table 1. All computing was done on a Cyber computer in single precision

Table 1
The first two coefficients in the series (12) and the dimensionless normal load. N is the number of terms required to achieve 6 significant figures. Exponents of 10 are given in parentheses. The *CPU* time ranges from 1 sec to 1000 sec for Re = 1 and Re = 18, respectively.

Re	b_1	b_2	$-p$	N
1	−2.71037	1.36414	6.80278	50
5	−1.96457	0.239885	10.5425	100
10	−1.58270	0.192510(−1)	15.8847	200
15	−1.42820	0.123804(−2)	21.4267	500
18	−1.39339	0.242526(−3)	25.0817	750

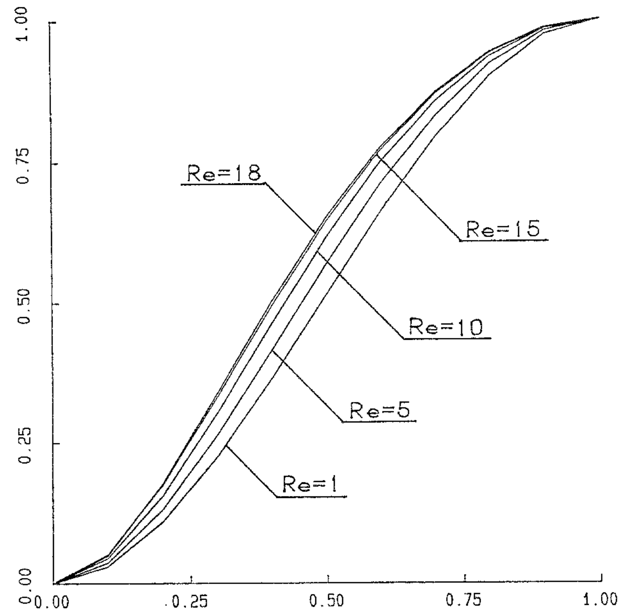


Figure 2
Velocity profile (f) for the continuous squeezing flow.

mode which retains 13 to 14 significant figures. The velocity profile (f) is given in Fig. 2.

For the coaxial-disc flow problem we minimize the unconstraint function

$$\left(\sum_{n=1}^{N} c_n - 1\right)^2 + \left(\sum_{n=1}^{N} d_n\right)^2 + \left(\sum_{n=1}^{N} n\, d_n\right)^2$$

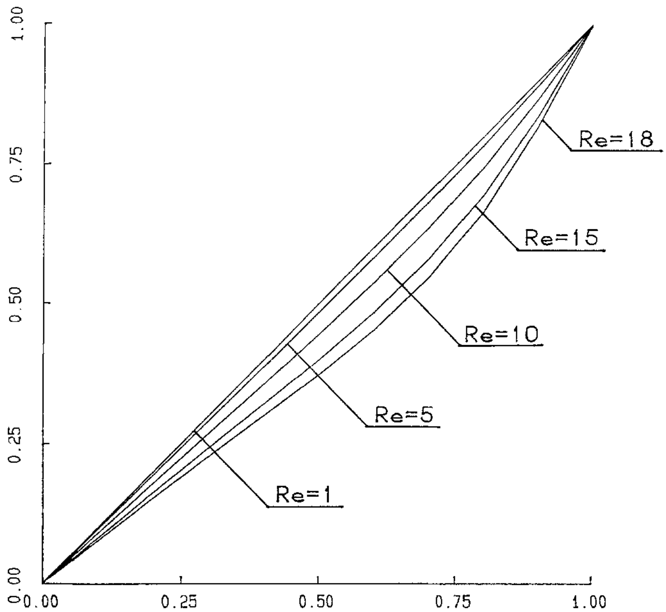
with respect to the first three coefficients c_1, d_1 and d_2. The problem is now three-dimensional but we have no problem obtaining solution accurate to six significant figures for Re $\leqq$ 18. The results are tabulated in Table 2. Of interest to the experimentalists is the dimensionless pressure gradient per radius:

$$p = \frac{d^2}{\eta \Omega r} \frac{\partial P}{\partial r} = h''' - \mathrm{Re}\,(h'^2 - g^2 - 2\,h\,h'') = \text{constant}. \tag{10}$$

At low Reynolds number it can be shown that $p = \frac{3}{10}$ Re + 0(Re^2). This prediction is quite accurate at low Reynolds number (Re $\leqq$ 5). At Re = 18 the perturbation

Table 2
The first three coefficients in the series (20) and the dimensionless gradient/radius. N is the number of terms required to achieve 6 significant figures. Exponents of 10 are given in parentheses. The *CPU* time ranges from 3 sec to over 2000 sec for Re = 1 and Re = 18, respectively.

Re	c_1	d_1	d_2	p	N
1	0.998734	−0.332603(−1)	0.498727(−1)	0.299236	50
5	0.970244	−0.158119	0.235168	1.41101	50
10	0.899309	−0.275956	0.401679	2.41007	100
15	0.818088	−0.346083	0.490256	2.941536	250
18	0.769588	−0.366367	0.514351	3.086106	900



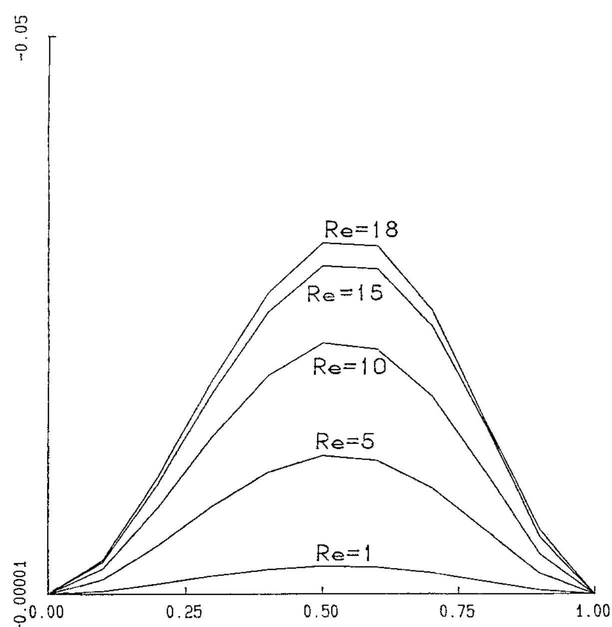


Figure 3a, b
Velocity profile (g and h) for the coaxial-disk flow.

prediction overestimates the true value to 75%. The velocity profiles (g and h) are given in Fig. 3a–b.

Clearly the power series method will work when one has a combination of the above two flows, e. g. coaxial-disk flow with suction or injection, or even coaxial-disk flow with sliding and suction or injection. We are currently investigating the applicability of this method in solving corresponding non-Newtonian flow problems.

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Summary

We show that the steady flow of a viscous fluid of moderate Reynolds number between two parallel disks has an exact solution which takes the form of a power series. Employing this exact solution the two-point boundary value problem for this class of flow is reduced to a nonlinear algebraic system which is then solved by a standard optimization method. Results are given for two particular cases, the continuous squeezing flow and the coaxial-disk flow.

Zusammenfassung

Wir zeigen, daß die ständige Strömung einer viskosen Flüssigkeit von mäßiger Reynoldszahl zwischen zwei parallelen Scheiben eine exakte Lösung hat, die die Form einer Potenzreihe annimmt. Wenn wir diese exakte Lösung anwenden, wird das Zwei-Punkte-Grenzwertproblem für diese Art von Strömung reduziert auf ein nichtlineares algebraisches System, das dann durch eine Optimierungsmethode gelöst werden kann. Angegeben werden Resultate für zwei Einzelfälle: die kontinuierliche Quetschströmung und die Strömung für Koaxialscheiben.

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