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Bifurcation of subharmonic solutions in time-reversible systems

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1. Introduction

In this paper we study the bifurcation of subharmonic solutions for periodic time-reversible systems depending on a real parameter, of the form

$$\dot{x} = f(t, x, \lambda). \tag{1.1}$$

Here we take $x \in \mathbb{R}^n$, while $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth mapping satisfying the following assumptions:

(H1) (i) $f(t + 2\pi, x, \lambda) = f(t, x, \lambda), \quad \forall (t, x, \lambda);$
(ii) $f(t, 0, \lambda) = 0, \quad \forall (t, \lambda);$
(iii) $f(-t, Sx, \lambda) = -Sf(t, x, \Lambda), \quad \forall (t, x, \lambda),$
where $S \in \mathscr{L}(\mathbb{R}^n)$ is such that $S^2 = I$.

Of course we can replace the period 2π in (i) by any $T > 0$. The condition (iii) is called a *time-reversibility* condition, since it implies that if $x(t)$ is a solution of (1.1), then so is $\tilde{x}(t) := Sx(-t)$. We may without loss of generality assume that S is orthogonal; then it is also necessarily symmetric. An example of a system satisfying (H1) comes from second order scalar equations of the form

$$\ddot{y} + g(t, x, \lambda) = 0, \tag{1.2}$$

where $g: \mathbb{R}^3 \to \mathbb{R}$ is smooth, even and 2π-periodic in t, with $g(t, 0, \lambda) = 0$ for all (t, λ). Bringing (1.2) in the form of a first order system it is easy to see that (H1) is satisfied with $S \in \mathscr{L}(\mathbb{R}^2)$ given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

A particular case of (1.2) was recently studied by Loud [3]; he considers scalar equations of the form

$$\ddot{z} + h(z) = \lambda p(t), \tag{1.4}$$

where $h: \mathbb{R} \to \mathbb{R}$ and $p: \mathbb{R} \to \mathbb{R}$ are smooth, with p even and 2π-periodic. He makes the assumption that (1.4) has for each $\lambda \in \mathbb{R}$ an even 2π-periodic solution $z_0(t, \lambda)$, depending smoothly on λ. Putting $z = z_0(t, \lambda) + y$ one obtains an equation of the form (1.2) for y, with $g(t, y, \lambda)$ given by

$$g(t, y, \lambda) = h(z_0(t, \lambda) + y) - h(z_0(t, \lambda)).$$

The results which we will present in this paper generalize the results of Loud, although the method which we use is quite different: Loud uses normal form theory, while we will use a Liapunov-Schmidt reduction.

Now consider the variational equation of (1.1) at the equilibrium solution $x = 0$; we get

$$\dot{x} = A(t, \lambda)\, x, \qquad (1.5)_\lambda$$

with $A(t, \lambda) := D_x f(t, 0, \lambda)$ 2π-periodic in t; moreover, the time-reversibility (H1) (iii) implies that

$$A(-t, \lambda) S = -S A(t, \lambda), \qquad \forall (t, \lambda). \qquad (1.6)$$

As we will see in Sect. 2 it follows from (1.6) that if $\mu \in \mathbb{C}$ is a characteristic multiplier for the 2π-periodic linear system $(1.5)_\lambda$, then so is μ^{-1}. Consequently the characteristic multipliers of $(1.5)_\lambda$ come in pairs: the two elements of such pair are either both real, with one inside and one outside the unit circle, or they are complex conjugate and both on the unit circle. In particular the second situation will interest us further on; more precisely we will assume:

(H2) The equation (1.5) has for $\lambda = 0$:

(i) a pair of simple characteristic multipliers $(\mu_0, \bar{\mu}_0)$ with,

$$\mu_0 = \exp(2\pi i p/q),$$

$p, q \in \mathbb{N}$, $p \neq 0$, $q \geqq 3$, and such that p and q have no common divisors;

(ii) no other characteristic multipliers $\mu \in \mathbb{C}$ for which $\mu^q = 1$ (*non-resonance*).

Under the hypothesis (H2) the equation $(1.5)_0$ has a two-dimensional space of $2\pi q$-periodic solutions. It is then natural to ask whether the equation (1.1) has for small λ some $2\pi q$-periodic solutions. We will show that there is indeed bifurcation of such subharmonic solutions at $\lambda = 0$ when a further condition is satisfied; this condition can be formulated as follows. It follows from (H2) and our remarks concerning the characteristic multipliers of $(1.5)_\lambda$ that $(1.5)_\lambda$ has for all sufficiently small λ a pair of simple characteristic multipliers $(\mu^*(\lambda), \bar{\mu}^*(\lambda))$, with $\mu^*(\lambda) = \exp(i\psi(\lambda))$ for some smooth $\psi: \mathbb{R} \to \mathbb{R}$ satisfying $\psi(0) = 2\pi p/q$. Our third condition then takes the form

(H3) $\dfrac{d\psi}{d\lambda}(0) \neq 0.$

We will show that under the hypotheses (H1)–(H3) at least $2q$ branches of $2\pi q$-periodic solutions of (1.1) bifurcate at $\lambda = 0$ from the trivial solution $x = 0$. These $2q$ branches of subharmonic split into two groups, each consisting of q branches, and such that two branches in the same group can be obtained one from the other by a phase shift over an appropriate multiple of 2π. Also, the subharmonic solutions from one group are unstable, while those from the other group are stable in a very weak sense (see further). The following remarks are intended to put this result in perspective.

Remark 1. In the usual treatments of bifurcation of subharmonic solutions (see e.g. Iooss and Joseph [2], Vanderbauwhede [4]) it is assumed that a pair of simple characteristic multipliers of $(1.5)_\lambda$ crosses the unit circle transversally at $\exp(\pm 2\pi i p/q)$. Here the time-reversibility prevents such transversal crossing: the characteristic multipliers stay on the unit circle. The hypothesis (H3) gives transversality along the unit circle; this condition is generically satisfied for one-parameter families of time-reversible systems.

Remark 2. Our result states that for generic one-parameter families of time-reversible systems there is bifurcation of subharmonic solutions. This is quite different from what happens when there is no time-reversibility: then there is generically no bifurcation of subharmonic solutions when $q \geqq 5$.

Remark 3. When (H2) and (H3) are satisfied at $\lambda = 0$, then similar hypotheses (with different values of p and q) are satisfied at an infinite number of parameter values near $\lambda = 0$. Hence there will be an infinite number of bifurcations of subharmonic solutions (with increasing period) near $\lambda = 0$. This complex bifurcation behaviour for subharmonic solutions will be reflected in the dynamics of (1.1), which can be described by the period map for (1.1). For a discussion of what happens in the case $n = 2$ we refer to Sect. 5 of Loud's paper [3].

Remark 4. The subharmonic solutions bifurcating at $\lambda = 0$ will have a pair of characteristic multipliers near $+1$, with product equal to 1. Our earlier statements about stability or instability of the bifurcating subharmonic solutions refer to this pair of characteristic multipliers. If they are real (and not both equal to 1), then the subharmonic solution is unstable, with an instability of saddle-type. The situation is much less clear in the other case, when both multipliers near 1 are nonreal and on the unit circle. The discussion given by Loud [3] indicates that the subharmonic solution is then stable in some weak sense. In this paper we will not analyze the stability question any further; when we say that the subharmonic solutions are stable we just mean that its characteristic multipliers near 1 are nonreal and on the unit circle.

Our analysis will be based on an abstract equation which we define as follows. Let Y_q be the Banach space of $2\pi q$-periodic continuous mappings $y: \mathbb{R} \to \mathbb{R}^n$, equipped with the supremum norm. By X_q we denote the subspace of all $x \in X_q$ which are of class C^1; X_q is a Banach space when equipped with the C^1-supremum norm. We define $M: X_q \times \mathbb{R} \to Y_q$ by

$$M(x, \lambda)(t) := -\dot{x}(t) + f(t, x, (t), \lambda), \quad \forall t \in \mathbb{R}, \ \forall (x, \lambda) \in X_q \times \mathbb{R}. \tag{1.7}$$

In order to find the $2\pi q$-periodic solutions of (1.1) we have to solve the equation

$$M(x, \lambda) = 0 \tag{1.8}$$

for $(x, \lambda) \in X_q \times \mathbb{R}$.

The rest of the paper is divided as follows. In Sect. 2 we discuss the linear variational equation (1.5); in Sect. 3 we apply a Liapunov-Schmidt reduction to (1.8) and obtain a normal form for the bifurcation equation. The main results are proved in Sect. 4, while in Sect. 5 we consider certain perturbations of the equation (1.1).

2. The variational equation

Fix some $\lambda \in \mathbb{R}$ and denote by $\Phi(t, \lambda) \in \mathscr{L}(\mathbb{R}^n)$ the transition matrix for $(1.5)_\lambda$, i.e. $x(t) = \Phi(t, \lambda) x_0$ is the unique solution of $(1.5)_\lambda$ satisfying $x(0) = x_0$. Then $C(\lambda) := \Phi(2\pi, \lambda)$ is a monodromy matrix for the 2π-periodic equation $(1.5)_\lambda$, the eigenvalues of $C(\lambda)$ are the characteristic multipliers of $(1.5)_\lambda$, and we have

$$\Phi(t + 2\pi, \lambda) = \Phi(t, \lambda) C(\lambda), \quad \forall (t, \lambda) \in \mathbb{R}^2. \tag{2.1}$$

Remark that $\det \Phi(t, \lambda) > 0$ for all (t, λ), and hence also $\det C(\lambda) > 0$.

Lemma 1. Assume (H1). Then we have

$$\det C(\lambda) = 1, \quad \forall \lambda \in \mathbb{R}. \tag{2.2}$$

Moreover, if $\mu \in \mathbb{C}$ is an eigenvalue of $C(\lambda)$ and $\xi \in \mathbb{C}^n$ a corresponding eigenvector, then also μ^{-1} is an eigenvalue of $C(\lambda)$, and $S\xi$ a corresponding eigenvector.

Proof. From (1.16) and the uniqeness of solutions of $(1.5)_\lambda$ with given initial value one easily obtains

$$S\Phi(-t, \lambda) = \Phi(t, \lambda) S, \quad \forall (t, \lambda) \in \mathbb{R}^2. \tag{2.3}$$

Taking $t = -\pi$ in (2.1) gives then

$$C(\lambda) = S\Phi(\pi, \lambda)^{-1} S\Phi(\pi, \lambda), \quad \forall \lambda \in \mathbb{R} \tag{2.4}$$

and

$$C(\lambda)^{-1} = SC(\lambda)S, \qquad \forall \lambda \in \mathbb{R}. \tag{2.5}$$

The lemma now follows immediately from (2.5), using $(\det S)^2 = 1$ (since $S^2 = I$) and $\det C(\lambda) > 0$.

In the statement of the next result we use the inner product on $\mathbb{C}^n$ defined by

$$(a, b) := \sum_{i=1}^{n} \bar{a}_i b_i, \qquad \forall a, b \in \mathbb{C}^n. \tag{2.6}$$

Lemma 2. Assume (H1) an (H2) (i), and let $C_0 := C(0)$.Then we can find $\xi_0 \in \mathbb{C}^n \backslash \{0\}$ and $\xi_0^* \in \mathbb{C}^n \backslash \{0\}$ such that:

(i) $N(C_0 - \mu_0 I) = \operatorname{span}\{\xi_0\}$, $N(C_0^T - \bar{\mu}_0 I) = \operatorname{span}\{\xi_0^*\}$;
(ii) $(\xi_0^*, \xi_0) = 2$, $(\xi_0^*, \bar{\xi}_0) = 0$;
(iii) $S\xi_0 = \bar{\xi}_0$, $S\xi_0^* = \bar{\xi}_0^*$.

Proof. (H2) (i) says that $\mu_0 = \exp(2\pi i p/q)$ is a simple eigenvalue of C_0; then $\bar{\mu}_0$ is a simple eigenvalue of C_0^T, and we can find vectors ξ_0 and ξ_0^* satisfying (i). (Remark that all subspaces considered in this proof are complex subspaces of $\mathbb{C}^n$.) We also have

$$\mathbb{C}^n = N(C_0 - \mu_0 I) \oplus R(C_0 - \mu_0 I) \tag{2.7}$$

and

$$R(C_0 - \mu_0 I) = N(C_0^T - \bar{\mu}_0 I)^{\perp} = \{b \in \mathbb{C}^n \mid (\xi_0^*, b) = 0\}, \tag{2.8}$$

by (i). It follows that $(\xi_0^*, \xi_0) \neq 0$, and hence we may normalize ξ_0 and ξ_0^* such that $(\xi_0^*, \xi_0) = 2$. Moreover we have $N(C_0 - \bar{\mu}_0 I) = \operatorname{span}\{\bar{\xi}_0\}$, while (2.7) and $\mu_0 \neq \bar{\mu}_0$ imply that $N(C_0 - \bar{\mu}_0 I) \subset R(C_0 - \mu_0 I)$; it follows then from (2.8) that $(\xi_0^*, \bar{\xi}_0) = 0$. Since $\bar{\mu}_0 = \mu_0^{-1}$ it follows from lemma 1 that also $S\xi_0$ belongs to $N(C_0 - \bar{\mu}_0 I)$; hence we have $S\xi_0 = \alpha\bar{\xi}_0$ for some $\alpha \in \mathbb{C}$. Using the fact that S is real and $S^2 = I$ one easily sees that $|\alpha| = 1$, i.e. $\alpha = \exp(2i\varphi)$ for some $\varphi \in \mathbb{R}$. Replacing ξ_0 by $e^{i\varphi}\xi_0$ we see that we can choose ξ_0 such that $S\xi_0 = \bar{\xi}_0$. A similar argument shows that $S\xi_0^* = \beta\bar{\xi}_0^*$ for some $\beta \in \mathbb{C}$; but then we find from (ii) that

$$2 = (\xi_0^*, S\bar{\xi}_0) = (S\xi_0^*, \bar{\xi}_0) = \bar{\beta}(\bar{\xi}_0^*, \bar{\xi}_0) = 2\bar{\beta},$$

i.e. we have $\beta = 1$. This proves the lemma.

Now we return to the operator M defined in the introduction; M is a smooth operator, and $M(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. The operator $L := D_x M(0, 0) \in \mathscr{L}(X_q, Y_q)$ is explicitly given by

$$(Lx)(t) = -\dot{x}(t) + A(t, 0)x(t), \qquad \forall t \in \mathbb{R}. \tag{2.9}$$

We know from classical Floquet theory that L is a Fredholm operator with zero index; we want to describe $N(L)$ and $R(L)$. To do so we will need an inner product on the complexification Y_q^c of Y_q defined by

$$\langle u, v\rangle := \frac{1}{2\pi q}\int_0^{2\pi q} (u(t), v(t))\,dt, \qquad \forall u, v \in Y_q^c. \tag{2.10}$$

With respect to this inner product the operator L has a formal adjoint $L^* \in \mathscr{L}(X_q, Y_q)$ given by

$$(L^* x)(t) = \dot{x}(t) + A(t,0)^T x(t), \qquad \forall t \in \mathbb{R},\ \forall x \in X_q; \tag{2.11}$$

L^* is associated with the adjoint equation

$$\dot{x} = -A(t,0)^T x. \tag{2.12}$$

This is again a linear 2π-periodic equation; its transition matrix is $(\Phi(t,0)^T)^{-1}$, and its monodromy matrix $(C_0^T)^{-1}$ has the same eigenvalues as C_0, by lemma 1. Finally we define $\zeta\colon \mathbb{R} \to \mathbb{C}^n$ and $\zeta^*\colon \mathbb{R} \to \mathbb{C}^n$ by

$$\zeta(t) := \Phi(t,0)\,\xi_0, \qquad \zeta^*(t) := (\Phi(t,0)^T)^{-1}\,\xi_0^*, \qquad \forall t \in \mathbb{R}. \tag{2.13}$$

The space $U := N(L)$ consists of all $2\pi q$-periodic solutions of $(1.5)_0$. It is easy to see from $\mu_0^q = 1$ and from the nonresonance condition in (H2) that U is two-dimensional, and spanned by $\zeta_1 := \operatorname{Re}\zeta$ and $\zeta_2 := \operatorname{Im}\zeta$. The mapping $\chi\colon \mathbb{C} \to U$ defined by

$$\chi(z)(t) := \operatorname{Re}(z\,\zeta(t)), \qquad \forall t \in \mathbb{R},\ \forall z \in \mathbb{C} \tag{2.14}$$

is a linear isomorphism when we consider $\mathbb{C}$ as a two-dimensional *real* vector-space. We will use the complex number z to parametrize the elements of U.

It follows in a similar way that $N(L^*)$ is two-dimensional, and spanned by $\zeta_1^* := \operatorname{Re}\zeta^*$ and $\zeta_2^* := \operatorname{Im}\zeta^*$. Then we know from Floquet theory that

$$R(L) = \{ y \in Y_q \mid \langle u^*, y\rangle = 0,\ \forall u^* \in N(L^*)\}$$

and hence

$$R(L) = \{ y \in Y_q \mid \langle \zeta^*, y\rangle = 0\}. \tag{2.15}$$

We can summarize our result as follows.

Lemma 3. Assume (H1) and (H2), let $L := D_x M(0,0) \in \mathscr{L}(X_q, Y_q)$ and define $P \in \mathscr{L}(Y_q)$ by

$$P y := \operatorname{Re}(\langle \zeta^*, y\rangle\,\zeta) = \chi(\langle \zeta^*; y\rangle), \qquad \forall y \in Y_q. \tag{2.16}$$

Then P is a projection operator, with

$$R(P) = U := N(L) \quad \text{and} \quad N(P) = R(L). \tag{2.17}$$

Proof. The fact that P is a projection follows easily from lemma 2 (ii), while (2.17) follows from $\chi(\mathbb{C}) = U$ and from (2.15).

3. The Liapunov-Schmidt reduction

We want now to solve the equation (1.8) for (x, λ) near the origin in $X_q \times \mathbb{R}$. However, before doing so it is important to realize that the operator M, as defined by (1.7), has some symmetry properties, which we can describe as follows. We define linear operators $\gamma \in \mathscr{L}(Y_q)$ and $\sigma \in \mathscr{L}(Y_q)$ by

$$(\gamma y)(t) := y(t + 2\pi), \qquad (\sigma y)(t) := S y(-t), \qquad \forall t \in \mathbb{R}, \qquad \forall y \in Y_q. \tag{3.1}$$

It is then easy to verify that

$$M(\gamma x, \lambda) = \gamma M(x, \lambda), \qquad \forall (x, \lambda) \in X_q \times \mathbb{R} \tag{3.2}$$

and

$$M(\sigma x, \lambda) = -\sigma M(x, \lambda), \qquad \forall (x, \lambda) \in X_q \times \mathbb{R}. \tag{3.3}$$

We have $\gamma^q = \sigma^2 =$ identity and $\sigma\gamma = \gamma^{-1}\sigma$; it follows that the operators γ and σ generate a finite group $\Gamma \subset \mathscr{L}(Y_q)$, having $2q$ elements and isomorphic to the dihedral group D_q (the symmetry group of a regular q-polygon). So (3.2) and (3.3) imply that M is *D_q-equivariant.*

In order to solve (1.8) we will use an equivariant Liapunov-Schmidt method, as explained for example in [4]; the main tool needed in the application of this method is the projection operator P given by lemma 3. In order to preserve the equivariance when making the reduction P needs also to be D_q-equivariant.

Lemma 4. The projection P defined by (2.16) is D_q-equivariant, i.e. we have

$$P\gamma = \gamma P \quad \text{and} \quad P\sigma = \sigma P. \tag{3.4}$$

Proof. It follows easily from the definitions of γ and σ that

$$\langle u, \gamma v\rangle = \langle \gamma^{-1} u, v\rangle, \qquad \forall u, v \in Y_q^c \tag{3.5}$$

and

$$\langle u, \sigma v\rangle = \langle \sigma u, v\rangle, \qquad \forall u, v \in Y_q^c. \tag{3.6}$$

We have also from (2.1), (2.3) and lemma 3 that

$$\gamma\chi = \mu_0\zeta, \qquad \gamma\zeta^* = \mu_0\zeta^* \tag{3.7}$$

and

$$\sigma\zeta = \bar{\zeta}, \qquad \sigma\zeta^* = \bar{\zeta}^*. \tag{3.8}$$

From this (3.4) follows directly. We remark that (3.7) and (3.8) also imply that

$$\gamma \chi(z) = \chi(\mu_0 z) \quad \text{and} \quad \sigma \chi(z) = \chi(\bar{z}), \quad \forall z \in \mathbb{C}. \tag{3.9}$$

We now return to the equation (1.8) in which we write $x \in X_q$ as

$$x = u + v, \quad u = Px \in U, \quad v = (I - P)x \in V := X_q \cap \mathbb{R}(L).$$

Then we can rewrite (1.8) as the system:

$$(I - P)\, M(u + v, \lambda) = 0, \tag{3.10.a}$$

$$P M(u + v, \lambda) = 0. \tag{3.10.b}$$

The equation (3.10.a) can be solved for $v = v^*(u, \lambda)$, by the implicit function theorem; bringing this solution into (3.10.b) we obtain the *bifurcation equation*

$$F(u, \lambda) := P M(u + v^*(u, \lambda), \lambda) = 0. \tag{3.11}$$

Using the isomorphism $\chi: \mathbb{C} \to U$ this equation is equivalent to the complex equation

$$G(z, \lambda) := \chi^{-1} F(\chi(z), \lambda) = \langle \zeta^*, M(\chi(z) + v^*(\chi(z), \lambda), \lambda) \rangle = 0. \tag{3.12}$$

To each solution $(z, \lambda) \in \mathbb{C} \times \mathbb{R}$ of (3.12) there corresponds the solution $(x, \lambda) = (\chi(z) + v^*(\chi(z), \lambda), \lambda)$ of (1.8), and conversely: if $(x, \lambda) \in X_q \times \mathbb{R}$ is a solution of (1.8) and sufficiently near (0, 0), then $(z, \lambda) = (\langle \zeta^*, x \rangle, \lambda)$ is a solution of (3.12).

In the next lemma we summarize the properties of the mappings v^*, F and G; the proof is easy and is essentially based on the uniqueness part of the implicit function theorem (see [4] for the details). We also emphasize again that $\mathbb{C}$ is considered as a two-dimensional real vectorspace, and that smoothness of a complex function of a complex variable means smoothness as a mapping between real Banach spaces.

Lemma 5. The mappings $v^*: U \times \mathbb{R} \to V$, $F: U \times \mathbb{R} \to U$ and $G: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ are defined and smooth in a neighborhood of the origin, and satisfy there the following properties:

(i) $v^*(0, \lambda) = 0$, $F(0, \lambda) = 0$, $G(0, \lambda) = 0, \forall \lambda$;
(ii) $D_u v^*(0, 0) = 0$, $D_u F(0, 0) = 0$, $D_z G(0, 0) = 0$;
(iii) $v^*(\gamma u, \lambda) = \gamma v^*(u, \lambda)$, $v^*(\sigma u, \lambda) = \sigma v^*(u, \lambda)$;
(iv) $F(\gamma u, \lambda) = \gamma F(u, \lambda)$, $F(\sigma u, \lambda) = -\sigma F(u, \lambda)$;
(v) $G(\mu_0 z, \lambda) = \mu_0 G(z, \lambda)$, $G(\bar{z}, \lambda) = -\overline{G(z, \lambda)}$.

Our further analysis of the bifurcation equation (3.12) will be based on the property (v), which expresses the D_q-equivariance of the bifurcation function G.

We remark that since $\mu_0 = \exp(2\pi i p/q)$ with p and q having no common divisors, (v) implies that

$$G(\delta_q z, \lambda) = \delta_q G(z, \lambda),$$

where $\delta_q := \exp(2\pi i/q)$. The following lemma gives a normal form for smooth D_q-equivariant functions.

Lemma 6. Let Λ be Banach space, and let $G: \mathbb{C} \times \Lambda \to \mathbb{C}$ be a smooth mapping such that

$$G(\delta_q z, \lambda) = \delta_q G(z, \lambda) \quad \text{and} \quad G(\bar{z}, \lambda) = -\overline{G(z, \lambda)}, \quad \forall (z, \lambda). \tag{3.13}$$

Then there exist unique smooth mappings $g_i: \mathbb{C} \times \Lambda \to \mathbb{R}$ $(i = 1, 2)$ such that:

(i) $G(z, \lambda) = i g_1(z, \lambda) z + i g_2(z, \lambda) \bar{z}^{q-1}$

and

(ii) $g_i(\delta_q z, \lambda) = g_i(\bar{z}, \lambda) = g_i(z, \lambda), \quad i = 1, 2.$

Proof. Define $H: \mathbb{C} \times \Lambda \to \mathbb{R}$ by

$$H(z, \lambda) := \operatorname{Re}(G(z, \lambda) \bar{z}), \quad \forall (z, \lambda) \in \mathbb{C} \times \Lambda. \tag{3.14}$$

Then H is a smooth mapping, satisfying

$$H(\delta_q z, \lambda) = H(z, \lambda), \quad H(\bar{z}, \lambda) = -H(z, \lambda), \quad \forall (z, \lambda). \tag{3.15}$$

It follows that $H(z, \lambda) = 0$ if $\operatorname{Im} \bar{z}^q = 0$. The set $\{z = x + i y \in \mathbb{C} \mid \operatorname{Im} \bar{z}^q = 0\}$ consists of q distinct lines through the origin, given in cartesian coordinates (x, y) by the equations

$$\varphi_j(x, y) := x \sin\left(j \frac{\pi}{q}\right) - y \cos\left(j \frac{\pi}{q}\right) = 0, \quad j = 0, 1, \ldots, q-1. \tag{3.16}$$

Now, if $\varphi: \mathbb{R}^2 \to \mathbb{R}$ is a linear functional and $h: \mathbb{R}^2 \to \mathbb{R}$ a smooth function such that $h(x, y) = 0$ if $\varphi(x, y) = 0$, then there exists a smooth function $\tilde{h}: \mathbb{R}^2 \to \mathbb{R}$ such that $h(x, y) = \varphi(x, y) \tilde{h}(x, y)$; if φ is nontrivial then $\tilde{h}$ is even uniquely determined. Applying this result a finite number of times on the function H we see that there is a unique smooth function $\tilde{H}: \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ such that

$$H(x + i y), \lambda) = \left(\prod_{j=0}^{q-1} \varphi_j(x, y)\right) \tilde{H}(x, y, \lambda). \tag{3.17}$$

A similar result holds for the function $Q(x, y) := \operatorname{Im}(x - i y)^q$; but since Q is a homogeneous polynomial of degree q the factor $\tilde{Q}(x, y)$ must be constant, i.e. we have

$$\operatorname{Im}(x - i y)^q = c_q \prod_{j=0}^{q-1} \varphi_j(x, y), \tag{3.18}$$

with $c_q \in \mathbb{R}\backslash\{0\}$ depending only on q. Combining (3.17) and (3.18) we conclude that there is a uniquely determined smooth function $g_2 : \mathbb{C} \times \Lambda \to \mathbb{R}$ such that

$$H(z, \lambda) = -g_2(z, \lambda) \operatorname{Im} \bar{z}^q, \qquad \forall (z, \lambda); \tag{3.19}$$

it then follows from (3.15) that g_2 satisfies the condition (ii) of the lemma.

Next we define $\tilde{G} : \mathbb{C} \times \Lambda \to \mathbb{C}$ by

$$\tilde{G}(z, \lambda) = \tilde{G}_1(z, \lambda) - i\tilde{G}_2(z, \lambda) := G(z, \lambda) - i g_2(z, \lambda) \bar{z}^{q-1};$$

the function $\tilde{G}$ has the same symmetry properties (3.13) as G, and it follows from (3.14) and (3.19) that

$$\operatorname{Re}(\tilde{G}(z, \lambda)\bar{z}) = 0, \qquad \forall (z, \lambda). \tag{3.20}$$

Expressing (3.20) in cartesian coordinates we see that $\tilde{G}_1(x + iy, \lambda) = 0$ if $y = 0$; hence we have $\tilde{G}_1(x + iy, \lambda) = -g_1(x + iy, \lambda) y$ for some smooth $g_1 : \mathbb{C} \times \Lambda \to \mathbb{R}$. This function g_1 is uniquely determined, and (3.20) implies then that

$$\tilde{G}(z, \lambda) = i g_1(z, \lambda) z, \qquad \forall (z, \lambda). \tag{3.21}$$

This proves (i), while (3.21) and the symmetry properties of $\tilde{G}$ imply that g_1 also satisfies (ii).

Remark 5. A result similar to lemma 6 is valid when the mapping G is only C_q-equivariant, that is when we have only $G(\delta_q z, \lambda) = \delta_q G(z, \lambda)$. In that case one has to allow the functions g_1 and g_2 in the formulation of the lemma to be complex-valued. One can prove this by applying lemma 6 to the mappings $G_1(z, \lambda) := G(z, \lambda) - \overline{G(\bar{z}, \lambda)}$ and $G_2(z, \lambda) := i(G(z, \lambda) + \overline{G(\bar{z}, \lambda)})$.

Using lemma 6 the bifurcation equation (3.12) takes the form

$$g_1(z, \lambda) z + g_2(z, \lambda) \bar{z}^{q-1} = 0, \tag{3.22}$$

where $g_1 : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ are smooth functions, satisfying the condition (ii) of lemma 6, and uniquely determined by the relation

$$\langle \zeta^*, M(\chi(z) + v^*(\chi(z), \lambda), \lambda) \rangle = i g_1(z, \lambda) z + i g_2(z, \lambda) \bar{z}^{q-1}, \quad \forall (z, \lambda) \in \mathbb{C} \times \mathbb{R}. \tag{3.23}$$

The following lemma describes the lower order terms in the Taylor expansion of $g_1(z, \lambda)$.

Lemma 7. Let $g_1(z, \lambda)$ be defined by (3.23); let $\psi(\lambda)$ be as in (H3) and let $r := \left[\frac{q-1}{2}\right]$. Then there exist a neighborhood ω of $\lambda = 0$ in $\mathbb{R}$ and smooth functions $A_l\colon \omega \to \mathbb{R}$ $(l = 0, 1, \ldots, r)$ such that

$$g_1(z, \lambda) = \sum_{l=0}^{r} A_l(\lambda)|z|^{2l} + 0(|z|^q) \quad \text{as} \quad z \to 0, \tag{3.24}$$

uniformly for $\lambda \in \omega$. Moreover we have

$$A_0(0) = g_1(0, 0) = 0 \quad \text{and} \quad D_\lambda A_0(0) = D_\lambda g_1(0, 0) = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\lambda}(0). \tag{3.25}$$

Proof. Since $g_1(z, \lambda)$ is smooth we have

$$g_1(z, \lambda) = \sum_{0 \leqq l+m<q} B_{l,m}(\lambda) z^l \bar{z}^m + 0(|z|^q) \quad \text{as} \quad z \to 0,$$

uniformly for λ in a sufficiently small neighborhood ω of $\lambda = 0$; the functions $B_{l,m}(\lambda)$ are smooth and complex-valued. Since g_1 is real-valued it follows that

$$B_{l,m}(\lambda) = \overline{B_{m,l}(\lambda)}. \tag{3.26}$$

Also, the symmetry properties of g_1 given by lemma 6(ii) imply that

$$B_{l,m}(\lambda) \exp\left[i(l-m)\frac{2\pi}{q}\right] = B_{l,m}(\lambda) \tag{3.27}$$

and

$$B_{l,m}(\lambda) = B_{m,l}(\lambda). \tag{3.28}$$

Since we restrict to values of l and m for which $0 \leqq l + m < q$ it follows that $B_{l,m}(\lambda) = 0$ except when $l = m$ and $0 \leqq l \leqq r$; in that case $A_l(\lambda) := B_{l,l}(\lambda)$ is real-valued. This proves (3.24). The fact that $g_1(0, 0) = 0$ follows from lemma 5(ii). It remains to calculate $D_\lambda g_1(0, 0)$.

We take $z = \varrho \in \mathbb{R}$ in (3.23) and differentiate at $\varrho = 0$; since $q \geqq 3$ this gives

$$i g_1(0, \lambda) = \langle \zeta^*, D_x M(0, \lambda) \cdot (\zeta_1 + D_u v^*(0, \lambda) \cdot \zeta_1) \rangle, \tag{3.29}$$

with $\zeta_1 = \operatorname{Re} \zeta$. Differentiating again and using (2.15) gives

$$i D_\lambda g_1(0, 0) = \langle \zeta^*, D_x D_\lambda M(0, 0) \cdot \zeta_1 \rangle. \tag{3.30}$$

Next we use some of our results from Sect. 2. For each λ near 0 let $\xi(\lambda) \in \mathbb{C}^n$ be an eigenvector of the monodromy matrix $C(\lambda)$, corresponding to the eigenvalue $\mu^*(\lambda) = \exp(i\psi(\lambda))$; we can choose $\xi(\lambda)$ to depend smoothly on λ and such that $\xi(0) = \xi_0$. We define $\tilde{\zeta}(\lambda)\colon \mathbb{R} \to \mathbb{C}^n$ by

$$\tilde{\zeta}(\lambda)(t) := \exp\left[-i\left(\frac{1}{2\pi}\psi(\lambda) - \frac{p}{q}\right)t\right] \Phi(t, \lambda)\,\xi(\lambda), \quad \forall t \in \mathbb{R}. \tag{3.31}$$

One can easily check that $\tilde{\zeta}(\lambda)$ is $2\pi q$-periodic (i.e. we have $\tilde{\zeta}(\lambda) \in X_q^c$), that $\tilde{\zeta}(0) = \zeta$ and that

$$D_x M(0, \lambda)\, \tilde{\zeta}(\lambda) := i\left(\frac{1}{2\pi}\psi(\lambda) - \frac{p}{q}\right)\tilde{\zeta}(\lambda), \quad \forall \lambda. \tag{3.32}$$

Putting $\tilde{\zeta}_1(\lambda) := \operatorname{Re}\tilde{\zeta}(\lambda)$ and $\tilde{\zeta}_2(\lambda) := \operatorname{Im}\tilde{\zeta}(\lambda)$, and taking the real part of (3.32) one obtains

$$D_x M(0, \lambda)\, \tilde{\zeta}_1(\lambda) = -\left(\frac{1}{2\pi}\psi(\lambda) - \frac{p}{q}\right)\tilde{\zeta}_2(\lambda), \quad \forall \lambda. \tag{3.33}$$

Now differentiate at $\lambda = 0$; since $\psi(0) = 2\pi p/q$ this gives

$$D_x D_\lambda M(0, 0) \cdot \zeta_1 + L D_\lambda \tilde{\zeta}_1(0) = -\frac{1}{2\pi}\frac{\mathrm{d}\psi}{\mathrm{d}\lambda}(0)\,\zeta_2\,. \tag{3.34}$$

Taking the inner product with ζ^* and comparing with (3.30) gives the desired expression for $D_\lambda g_1(0, 0)$.

4. Bifurcating subharmonics

We now return to the task of solving the bifurcation equation (3.22); to do so we will use polar coordinates, i.e. we write $z \in \mathbb{C}$ in the form $z = \varrho \exp(i\theta)$. We also write $\tilde{g}_i(\varrho, \theta, \lambda)$ for $g_i(\varrho \exp(i\theta), \lambda)$, $(i = 1, 2)$. For nontrivial solutions we can multiply (3.22) by $\bar{z}$ and then divide by ϱ^2; the result is the equation

$$\tilde{g}_1(\varrho, \theta, \lambda) + \varrho^{q-2}\tilde{g}_2(\varrho, \theta, \lambda)\, e^{-iq\theta} = 0\,. \tag{4.1}$$

Splitting into real and imaginary part we obtain

$$h(\varrho, \theta, \lambda) := \tilde{g}_1(\varrho, \theta, \lambda) + \varrho^{q-2}\tilde{g}_2(\varrho, \theta, \lambda)\cos q\theta = 0 \tag{4.2}$$

and

$$\tilde{g}_2(\varrho, \theta, \lambda)\sin q\theta = 0\,. \tag{4.3}$$

For arbitrary (ϱ, λ) the equation (4.3) has the solutions

$$\theta = \theta_{q,j}^{(1)} := j\frac{2\pi}{q}, \qquad j = 0, 1, \ldots, q-1 \tag{4.4}$$

and

$$\theta = \theta_{q,j}^{(2)} := \frac{\pi}{q} + j\frac{2\pi}{q}, \quad j = 0, 1, \ldots, q-1\,. \tag{4.5}$$

If we assume that $g_2(0, \theta, 0) = g_2(0, 0) \neq 0$ these are the only solutions of (4.3) for sufficiently small (ϱ, λ).

For $\theta = \theta_{q,j}^{(i)}$ $(i = 1, 2; j = 0, 1, \ldots, q-1)$ the equation (4.2) reduces to

$$h^{(i)}(\varrho, \lambda) := h(\varrho, \theta_{q,j}^{(i)}, \lambda) = 0; \tag{4.6}$$

the fact that this equation does not depend on the value of j is a consequence of lemma 6 (ii), which implies that

$$h\left(\varrho, \theta + \frac{2\pi}{q}, \lambda\right) = h(\varrho, \theta, \lambda) = h(\varrho, -\theta, \lambda). \tag{4.7}$$

We also have

$$h(-\varrho, \theta + \pi, \lambda) = h(\varrho, \theta, \lambda), \tag{4.8}$$

while (3.25), (H3) and $q > 2$ imply that

$$h(0, \theta, 0) = g_1(0, 0) = 0 \quad \text{and} \quad D_\lambda h(0, \theta, 0) = \frac{1}{2\pi} \frac{d\psi}{d\lambda}(0) \neq 0. \tag{4.9}$$

Using (4.9) and the implicit function theorem the equation (4.6) has a unique solution branch $\lambda = \tilde{\lambda}^{(i)}(\varrho)$ $(i = 1, 2)$, with $\tilde{\lambda}^{(i)}: \mathbb{R} \to \mathbb{R}$ defined and smooth near the origin, and with $\tilde{\lambda}^{(i)}(0) = 0$. By (4.8) we have

$$\tilde{\lambda}^{(i)}(-\varrho) = \tilde{\lambda}^{(i)}(\varrho) \quad (i = 1, 2), \quad \text{if} \quad q = \text{even},$$

and

$$\tilde{\lambda}^{(1)}(-\varrho) = \tilde{\lambda}^{(2)}(\varrho) \quad \text{if} \quad q = \text{odd}.$$

We can use lemma 7 to get more details on the functions $\tilde{\lambda}^{(1)}(\varrho)$ and $\tilde{\lambda}^{(2)}(\varrho)$. It follows from this lemma that $h^{(1)}(\varrho, \lambda) = h^{(2)}(\varrho, \lambda) + 0(|\varrho|^{q-2})$, and hence

$$\tilde{\lambda}^{(1)}(\varrho) = \tilde{\lambda}^{(2)}(\varrho) + 0(|\varrho|^{q-2}) \quad \text{as} \quad \varrho \to 0. \tag{4.10}$$

We also have:

(a) for $q = 3$:

$$\begin{aligned} \tilde{\lambda}^{(1)}(\varrho) &= -\frac{g_2(0,0)}{A_0'(0)} \varrho + 0(\varrho^2), \\ \tilde{\lambda}^{(2)}(\varrho) &= \frac{g_2(0,0)}{A_0'(0)} \varrho + 0(\varrho^2); \end{aligned} \tag{4.11}$$

(b) for $q = 4$:

$$\begin{aligned} \tilde{\lambda}^{(1)}(\varrho) &= -\frac{A_1(0) + g_2(0,0)}{A_0'(0)} \varrho^2 + 0(\varrho^3), \\ \tilde{\lambda}^{(2)}(\varrho) &= -\frac{A_1(0) - g_2(0,0)}{A_0'(0)} \varrho^2 + 0(\varrho^3); \end{aligned} \tag{4.12}$$

(c) for $q \geqq 5$:

$$\tilde{\lambda}^{(i)}(\varrho) = -\frac{A_1(0)}{A_0'(0)}\varrho^2 + 0(\varrho^3), \qquad i = 1, 2. \tag{4.13}$$

Further on we will make the following non-degeneracy assumptions, which are inspired by (4.11)–(4.13).

(H4) $g_2(0,0) \neq 0$, and

(i) $|A_0(0)| \neq |g_2(0,0)|$ if $q = 4$;
(ii) $A_1(0) \neq 0$ if $q \geqq 5$.

We have then the following result.

Theorem 1. Under the hypotheses (H1)–(H3) there is at $\lambda = 0$ a bifurcation of at least $2q$ branches of $2\pi q$-periodic solutions of (1.1). These solution branches have the form

$$\{(\gamma^j \tilde{x}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)) \mid 0 < \varrho < \varrho_0\}, \quad i = 1, 2, \quad j = 0, 1, \ldots, q-1 \tag{4.14}$$

for some $\varrho_0 > 0$, with $\tilde{\lambda}^{(i)}(\varrho)$ given as the solution of (4.6), and with $\tilde{x}^{(i)}: \mathbb{R} \to X_q$ a smooth function, satisfying $\tilde{x}^{(i)}(0) = 0$ and

$$\langle \zeta^*, \tilde{x}^{(1)}(\varrho) \rangle = \varrho, \quad \langle \zeta^*, \tilde{x}^{(2)}(\varrho) \rangle = \varrho\, e^{i\frac{\pi}{q}}, \quad \forall \varrho. \tag{4.15}$$

If $g_2(0,0) \neq 0$ then there are no further bifurcating $2\pi q$-periodic solutions.

If (H4) holds then:

(a) for $q = 3$ there are 3 subcritical branches and 3 supercritical branches;
(b1) for $q = 4$ and if $|A_1(0)| < |g_2(0,0)|$ there are 4 subcritical branches and 4 supercritical branches;
(b2) for $q = 4$ and if $|A_1(0)| > |g_2(0,0)|$ all 8 branches are subcritical or all supercritical, depending on the sign of $A_1(0) A_0'(0)$;
(c) for $q \geqq 5$ all $2q$ branches are subcritical or all supercritical, depending on the sign of $A_1(0) A_0'(0)$.

Proof. We first remark that multiplication with $\delta_q = \exp(i\,2\pi/q)$ in $\mathbb{C}$ corresponds via χ with the operator γ^m on U, where $m \in \mathbb{N}$ is such that $pm = 1 \pmod q$ (see (3.9)). Since p and q have no common divisors it follows that $\{\gamma^{jm} \mid 0 \leqq j < q\} = \{\gamma^j \mid 0 \leqq j < q\}$. This gives the $2q$ solution branches

$$\{(\gamma^j \tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)) \mid 0 < \varrho < \varrho_0\}, \qquad i = 1, 2, \qquad j = 0, 1, \ldots, q-1,$$

for (3.11), with $\tilde{u}^{(i)}(\varrho) := \chi(\varrho \exp(i\,\theta_{q,0}^{(i)}))$. We take $\varrho > 0$ in order to avoid double counting. Via our equivariant Liapunov-Schmidt reduction we then obtain the $2q$ solution branches (4.14) for the equation (1.8), with

$$\tilde{x}^{(i)}(\varrho) = \tilde{u}^{(i)}(\varrho) + v^*(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)). \tag{4.16}$$

Together with the expressions (4.11)–(4.13) for $\tilde{\lambda}^{(i)}(\varrho)$ this proves the theorem. Remark also that we have

$$\sigma\,\tilde{u}^{(1)}(\varrho) = \tilde{u}^{(1)}(\varrho), \qquad \sigma\,\tilde{x}^{(1)}(\varrho) = \tilde{x}^{(1)}(\varrho) \tag{4.17}$$

and

$$\gamma^m\,\sigma\,\tilde{u}^{(2)}(\varrho) = \tilde{u}^{(2)}(\varrho), \quad \gamma^m\,\sigma\,\tilde{x}^{(2)}(\varrho) = \tilde{x}^{(2)}(\varrho). \tag{4.18}$$

We now turn to the problem of obtaining some information on the stability properties of these bifurcating subharmonic solutions. First consider a general solution $(x_0, \lambda_0) \in X_q \times \mathbb{R}$ of (1.8), i.e. $x_0(t)$ is a $2\pi q$-periodic solution of (1.1) for $\lambda = \lambda_0$. Let $v_0 = \exp(2\pi q\,\alpha_0) \in \mathbb{C}\backslash\{0\}$ be a characteristic multiplier of this solution, which means that v_0 is a characteristic multiplier of the $2\pi q$-periodic linear variational equation

$$\dot{x} = D_x f(t, x_0(t), \lambda_0)\,x. \tag{4.19}$$

Let $\psi_0(t)$ be the transition matrix for (4.19). Then v_0 is an eigenvalue of the monodromy matrix $\psi(2\pi q)$, say with eigenvector $\eta_0 \in \mathbb{C}^n\backslash\{0\}$. It is now easy to see that the mapping $\bar{x}\colon \mathbb{R} \to \mathbb{C}^n$ defined by

$$\bar{x}(t) := e^{-\alpha_0 t}\,\psi_0(t)\,\eta_0, \qquad \forall\, t \in \mathbb{R},$$

belongs to X_q^c, and that

$$D_x M(x_0, \lambda_0)\cdot \bar{x} = \alpha_0\,\bar{x}. \tag{4.20}$$

Conversely, if (4.20) holds for some $\alpha_0 \in \mathbb{C}$ and for some $\bar{x} \in X_q^c\backslash\{0\}$, then $v_0 = \exp(2\pi q\,\alpha_0)$ is a characteristic multiplier for (4.19). We conclude that the characteristic *exponents* of (4.19) are precisely the eigenvalues of $D_x M(x_0, \lambda_0)$.

Using this result we have to study the eigenvalues of the operators $D_x M(\gamma^j \tilde{x}^{(i)}(\varrho),\ \tilde{\lambda}^{(i)}(\varrho))$, for $i = 1, 2$, $j = 0, 1, \ldots, q-1$ and $\varrho > 0$ sufficiently small. It follows from the equivariance properties of M that

$$D_x M(\gamma^j x, \lambda)\cdot \gamma^j = \gamma^j D_x M(x, \lambda),$$

and hence the eigenvalues of $D_x M(\gamma^j x, \lambda)$ are independent of j. So it is sufficient to consider the eigenvalues of the operators

$$\tilde{L}^{(i)}(\varrho) := D_x M(\tilde{x}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)), \qquad i = 1, 2. \tag{4.21}$$

It follows from (4.17) and (4.18) that

$$\tilde{L}^{(1)}(\varrho)\,\sigma = -\,\sigma\,\tilde{L}^{(1)}(\varrho) \quad \text{and} \quad \tilde{L}^{(2)}(\varrho)\,\gamma^m\,\sigma = -\,\gamma^m\,\sigma\,\tilde{L}^{(2)}(\varrho). \tag{4.22}$$

This implies that if $\alpha \in \mathbb{C}$ is an eigenvalue of $\tilde{L}^{(i)}(\varrho)$, then so is $-\alpha$.

Lemma 8. Assume (H1)–(H4). Let $\omega \subset \mathbb{C}$ be an open neighborhood of 0, such that

$$\{\alpha \in \omega \mid \exp(2\pi q\,\alpha) \in \sigma(C_0^q)\} = \{0\}. \tag{4.23}$$

Then there exists a $\varrho_0 > 0$ such that for $i = 1, 2$ and for $\varrho \in (0, \varrho_0)$ the operator $\tilde{L}^{(i)}(\varrho)$ has in ω precisely two simple eigenvalues $\alpha^{(i)}(\varrho)$ and $-\alpha^{(i)}(\varrho)$, with

$$\begin{aligned}\alpha^{(i)}(\varrho)^2 &= 3\varrho^2 g_2(0,0)^2 + 0(\varrho^3), \quad \text{if} \quad q = 3;\\ &= 8\varrho^4 g_2(0,0)\cos(q\,\theta_{q,0}^{(i)})(A_1(0) + g_2(0,0)\cos(q\,\theta_{q,0}^{(i)})) + 0(\varrho^5),\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad q = 4;\\ &= 2q\varrho^q g_2(0,0)A_1(0)\cos(q\,\theta_{q,0}^{(i)}) + 0(\varrho^{q+1}), \quad \text{if} \quad q \geqq 5. \end{aligned} \tag{4.24}$$

Proof. We will use the results of the papers [6] and [7]. The eigenvalues of $L = D_x M(0,0)$ are those $\alpha \in \mathbb{C}$ such that $\exp(2\pi q\alpha)$ is a characteristic multiplier of $(1.5)_0$, when we consider this equation as a $2\pi q$-periodic equation. As such $(1.5)_0$ has the monodromy matrix C_0^q, and hence $\exp(2\pi q\alpha)$ is a characteristic multiplier iff it is an eigenvalue of C_0^q. Now (H2) and (4.23) imply that zero is a semi-simple eigenvalue of L, with multiplicity two, and that L has no other eigenvalues in ω. We may also assume that ω is symmetric, i.e. $\alpha \in \omega$ implies $-\alpha \in \omega$.

The general results of [6] and [7] then imply that for each $(u, \lambda) \in U \times \mathbb{R}$ in a sufficiently small neighborhood of the origin the sum of the algebraic multiplicities of the eigenvalues of $D_x M(u + v^*(u,\lambda), \lambda)$ which belong to ω equals two; moreover, these eigenvalues coincide with the eigenvalues of an operator $\Phi(u,\lambda) \in \mathscr{L}(U)$ which has the form

$$\Phi(u,\lambda) = \eta(u,\lambda)\, D_u F(u,\lambda), \tag{4.25}$$

where $F(u,\lambda)$ is the bifurcation mapping given by (3.11), while $\eta\colon U \times \mathbb{R} \to \mathscr{L}(U)$ is a smooth mapping with

$$\eta(0,0) = I_U, \qquad D_u\eta(0,0) = 0, \qquad D_\lambda\eta(0,0) = 0. \tag{4.26}$$

We will apply this result for $(u,\lambda) = (\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho))$. If we can show that

$$\det D_u F(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)) = B^{(i)}\varrho^k + 0(\varrho^{k+1}) \tag{4.27}$$

for some $k \in \mathbb{N}$ and some $B^{(i)} \neq 0$, then (4.25) and (4.26) imply that

$$\det \Phi(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)) = B^{(i)}\varrho^k + 0(\varrho^{k+1}). \tag{4.28}$$

From this it follows that for $\varrho > 0$ sufficiently small zero is not an eigenvalue of $\Phi(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho))$, and hence also not of $\tilde{L}^{(i)}(\varrho)$. Since these eigenvalues come in pairs $\{\alpha, -\alpha\}$, it follows that $\tilde{L}^{(i)}(\varrho)$ has in ω two simple eigenvalues $\alpha^{(i)}(\varrho)$ and $-\alpha^{(i)}(\varrho)$, whose product is given by

$$-\alpha^{(i)}(\varrho)^2 = B^{(i)}\varrho^k + 0(\varrho^{k+1}). \tag{4.29}$$

This proves the lemma, on condition that we show (4.27) and calculate $B^{(i)}$ and k explicitly, in order to get (4.24) from (4.29).

Suppose that $F(u_0, \lambda_0) = 0$ for some $(u_0, \lambda_0) \in U \times \mathbb{R}$; this means that u_0 is, for $\lambda = \lambda_0$, an equilibrium of the autonomous equation

$$\dot{u} = F(u,\lambda). \tag{4.30}$$

Let now $w_0 \in \mathbb{R}^2$, and let $W: \mathbb{R}^2 \to U$ be a mapping with $W(w_0) = u_0$ and such that W is a diffeomorphism from a neighborhood of w_0 in $\mathbb{R}^2$ onto a neighborhood of u_0 in U. The transformation $u = W(w)$ then brings the equation (4.30) in the form

$$\dot{w} = D\,W(w)^{-1}\,F(W(w), \lambda) =: \tilde{F}(w, \lambda). \tag{4.31}$$

This equation has for $\lambda = \lambda_0$ the equilibrium w_0, and we have

$$D_w \tilde{F}(w_0, \lambda_0) = D\,W(w_0)^{-1}\,D_u F(u_0, \lambda_0)\,D\,W(w_0);$$

it follows in particular that

$$\det D_u F(u_0, \lambda_0) = \det D_w \tilde{F}(w_0, \lambda_0). \tag{4.32}$$

We will use this to calculate $\det D_u F(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho))$. We transform (4.30) using the mapping

$$W: \mathbb{R}^2 \to U, \quad (\varrho, \theta) \to W(\varrho, \theta) := \chi(\varrho\, e^{i\theta}), \tag{4.33}$$

which is a diffeomorphism for $\varrho \neq 0$. The easiest way to obtain the transformed equation is by putting $z = \varrho\, e^{i\theta}$ in the complex equation

$$\dot{z} = G(z, \lambda) \tag{4.34}$$

which is equivalent to (4.30) via the linear transformation $\chi: \mathbb{C} \to U$. Using lemma 6 we find

$$\begin{aligned} \dot{\varrho} &= \varrho^{q-1}\,\tilde{g}_2(\varrho, \theta, \lambda)\sin q\theta, \\ \dot{\theta} &= h(\varrho, \theta, \lambda). \end{aligned} \tag{4.35}$$

Since $W^{-1}(\tilde{u}^{(i)}(\varrho)) = (\varrho, \theta^{(i)}_{q,0})$ we have to calculate the Jacobian (in (ϱ, θ)) of the right hand side of (4.35) for $\theta = \theta^{(i)}_{q,0}$ and $\lambda = \tilde{\lambda}^{(i)}(\varrho)$. Using (4.7) and (4.32) we find

$$\begin{aligned} &\det D_u F(\tilde{u}^{(i)}(\varrho), \tilde{\lambda}^{(i)}(\varrho)) \\ &\quad = -\,q\,\varrho^{q-1}\cos(q\,\theta^{(i)}_{q,0})\,\tilde{g}_2(\varrho, \theta^{(i)}_{q,0}, \tilde{\lambda}^{(i)}(\varrho))\,\frac{\partial h^{(i)}}{\partial \varrho}(\varrho, \tilde{\lambda}^{(i)}(\varrho)). \end{aligned} \tag{4.36}$$

Assuming (H4) it is then straightforward to calculate the first nonzero term in the Taylor expansion of this expression; comparing with (4.27) and (4.29) then gives (4.24).

We recall from the introduction (remark 4) that we say that a bifurcating subharmonic solution is stable if its two characteristic multipliers near 1 are on the unit circle, and unstable if these characteristic multipliers are real. Using the notation of lemma 8 this means that we have stability if $\alpha^{(i)}(\varrho)^2 < 0$, and instability if $\alpha^{(i)}(\varrho)^2 > 0$. From (4.24) we then obtain the following conclusion.

Theorem 2. Assume (H1)–(H4). Then all bifurcating subharmonic solutions given by theorem 1 are unstable if $q = 3$ or if $q = 4$ and $|A_1(0)| < |g_2(0,0)|$. If $q \geqq 5$ or if $q = 4$ and $|A_1(0)| > |g_2(0,0)|$ then the subharmonic solutions along the q branches with $i = 1$ are stable if $q_2(0,0)A_1(0) < 0$ and unstable if $g_2(0,0)A_1(0) > 0$; for the q branches with $i = 2$ the opposite holds.

Remark 6. It follows from theorem 1 and 2 that under the hypotheses (H1)–(H4) the equation (1.1) has for fixed and sufficiently small $\lambda \neq 0$ the following nontrivial subharmonic solutions with period $2\pi q$ near $x = 0$:

(i) no such solutions if $q \geqq 5$ and $\lambda A_1(0)A_0'(0) > 0$, or if $q = 4$, $|A_1(0)| > |g_2(0,0)|$ and $\lambda A_1(0)A_1'(0) > 0$;
(ii) q such solutions if $q = 3$ or if $q = 4$ and $|A_1(0)| < |q_2(0,0)|$; these q subharmonic solutions are obtained one from the other by phase shifts over multipliers of 2π, and they are all unstable;
(iii) $2q$ such solutions if $q \geqq 5$ and $\lambda A_1(0)A_0'(0) < 0$, or if $q = 4$, $|A_1(0)| > |g_2(0,0)|$ and $\lambda A_1(0)A_0'(0) < 0$; q of these subharmonic solutions are stable, the other q are unstable; the stable ones are related to each other by phase shifts over multipliers of 2π, and the same holds for the unstable ones.

For $\lambda = 0$ the equation (1.1) has no nontrivial $2\pi q$-periodic solutions near $x = 0$.

Remark 7. The coefficients $A_1(0)$ and $g_2(0,0)$, which determine the direction of the bifurcation and the stability properties of the bifurcating solutions, can be calculated from higher order coefficients in the Taylor expansion of $f(x, 0)$. In particular, the expression for $A_1(0)$ involves terms up to third order, while that for $g_2(0,0)$ involves terms up to order $(q - 1)$. The hypothesis (H4) (as well as (H3)) is generically satisfied for one-parameter problems satisfying (H1) and (H2).

5. Perturbations

One of the conclusion of section 4 was that if $g_2(0,0) \neq 0$ there is bifurcation of precisely $2q$ branches of subharmonic solutions; these bifurcating subharmonics have a particular symmetry (see (4.17) and (4.18)), and there are no other $2\pi q$-periodic solutions of (1.1) near $x = 0$ and for small λ. To see what happens if $g_2(0,0) = 0$ we consider in this section two-parameter equations of the form

$$\dot{x} = f(t, x, \lambda, \varepsilon). \tag{5.1$_\varepsilon$}$$

In this equation we consider $\lambda \in \mathbb{R}$ as the bifurcation parameter, and $\varepsilon \in \mathbb{R}$ as a (small) perturbation parameter; this means that we look upon $(5.1)_\varepsilon$ for $\varepsilon \neq 0$ as a perturbation of $(5.1)_0$, which is supposed to satisfy the hypotheses (H1)–(H3), but with $g_2(0,0) = 0$.

More precisely we will assume that f is smooth and satisfies (H1) for all (λ, ε) and (H4) at $(\lambda, \varepsilon) = (0, 0)$. Then the equation (1.5) will also depend on ε, as will its characteristic multiplier $\mu^*(\lambda, \varepsilon) = \exp(i\psi(\lambda, \varepsilon))$. We have $\psi(0, 0) = 2\pi p/q$, and we replace (H3) by

(H3)' $\dfrac{\partial \psi}{\partial \lambda}(0, 0) \neq 0.$

Using a redefinition of the parameters we may then without loss of generality assume that

$$\psi(0, \varepsilon) = 2\pi p/q, \qquad \forall \varepsilon. \tag{5.2}$$

(We will only work with small values of ε.) We can then carry through the analysis of the foregoing sections, with this difference that now everything depends also on ε. It follows from (5.2) that $D_x M(0, 0, \varepsilon)$ has a two-dimensional nullspace, spanned by vectors of the form $\zeta_1 + w_1(\varepsilon)$ and $\zeta_2 + w_2(\varepsilon)$, with $w_i(\varepsilon) \in V$. It then follows easily that $w_i(\varepsilon) = D_u v^*(0, 0, \varepsilon) \cdot \zeta_i$ $(i = 1, 2)$; the formula for $g_1(0, \lambda, \varepsilon)$, similar to (3.29), then shows that

$$A_0(0, \varepsilon) = 0, \qquad \forall \varepsilon. \tag{5.3}$$

We still find that for each fixed ε there is at $\lambda = 0$ bifurcation of at least $2q$ branches of $2\pi q$-periodic solutions of $(5.1)_\varepsilon$; of course the precise form of the branches, described by $\lambda = \tilde{\lambda}^{(i)}(\varrho, \varepsilon)$, will depend on ε. We now make the following assumption for the function $g_2(z, \lambda, \varepsilon)$:

(H5) $g_2(0, 0, 0) = 0$ and $\dfrac{\partial g_2}{\partial \varepsilon}(0, 0, 0) \neq 0.$

Restricting to the case $q \geqq 5$ we see from (H5) and the results of section 4 that the stability properties along the $2q$ branches will change as ε crosses zero; therefore we may expect some further bifurcation to take place.

Until now we have only considered solutions of the bifurcation equation (4.1) for which $\sin q\theta = 0$. However, when (H5) holds then we can find other solutions of (4.1) by solving the system

$$\begin{aligned} \tilde{g}_1(\varrho, \theta, \lambda, \varepsilon) &= 0 \\ \tilde{g}_2(\varrho, \theta, \lambda, \varepsilon) &= 0. \end{aligned} \tag{5.4}$$

From (5.3), (H3)' and (H5) we have

$$\tilde{g}_1(0, \theta, 0, 0) = \tilde{g}_2(0, \theta, 0, 0) = 0$$

and

$$\frac{\partial(\tilde{g}_1, \tilde{g}_2)}{\partial(\lambda, \varepsilon)}(0, \theta, 0, 0) = \frac{1}{2\pi}\frac{\partial \psi}{\partial \lambda}(0, 0)\frac{\partial g_2}{\partial \varepsilon}(0, 0, 0) \neq 0.$$

Hence we can solve (5.4) by the implicit function theorem, and obtain $(\lambda, \varepsilon) = (\lambda^*(\varrho, \theta), \varepsilon^*(\varrho, \theta))$, with $\lambda^*(0, \theta) = \varepsilon^*(0, \theta) = 0$,

$$\lambda^*\left(\varrho, \theta + \frac{2\pi}{q}\right) = \lambda^*(\varrho, \theta) = \lambda^*(\varrho, -\theta), \tag{5.5}$$

and similar properties for $\varepsilon^*(\varrho, \theta)$. Using a result similar to lemma 7 it follows that $\lambda^*(\varrho, \theta)$ and $\varepsilon^*(\varrho, \theta)$ will have the form

$$\begin{aligned} \lambda^*(\varrho, \theta) &= \sum_{l=1}^{r} c_l \varrho^{2l} + 0(\varrho^q), \\ \varepsilon^*(\varrho, \theta) &= \sum_{l=1}^{r} d_l \varrho^{2l} + 0(\varrho^q). \end{aligned} \tag{5.6}$$

Using the uniqueness of the solutions as given by the implicit function theorem one also has

$$\lambda^*(\varrho, \theta_{q,j}^{(i)}) = \tilde{\lambda}^{(i)}(\varrho, \varepsilon^*(\varrho, \theta_{q,j}^{(i)})), \qquad i = 1, 2; \; j = 0, 1, \ldots, q-1. \tag{5.7}$$

Assuming for example that $d_1 > 0$ this gives the following bifurcation picture. For fixed $\varepsilon \leqq 0$ we have precisely $2q$ branches bifurcating at $\lambda = 0$ from $x = 0$. For $\varepsilon > 0$ we still have these $2q$ branches, but now there is also a closed curve of $2\pi q$-periodic solutions of $(5.1)_\varepsilon$, given by

$$\{(x^*(\varrho, \theta), \lambda^*(\varrho, \theta)) \mid \varepsilon^*(\varrho, \theta) = \varepsilon\}, \tag{5.8$_\varepsilon$}$$

with $x^*(\varrho, \theta) = \chi(\varrho e^{i\theta}) + v^*(\chi(\varrho e^{i\theta}), \lambda^*(\varrho, \theta), \varepsilon^*(\varrho, \theta))$. These additional solutions have no particular symmetry, and because of (5.7) the curve of solutions $(5.8)_\varepsilon$ connects the $2q$ branches bifurcating from $x = 0$. Hence we have secondary bifurcation, and at $\varepsilon = 0$ we see the onset of this secondary bifurcation. Figure 1 shows the projection of the solution set on the two-dimensional space $U = N(D_x M(0, 0, 0))$ for the cases $\varepsilon \leqq 0$ and $\varepsilon > 0$ respectively, while Fig. 2 shows the corresponding curves in a (λ, ϱ)-diagram.

We conclude this paper with a brief discussion of what happens when the perturbation parameter ε in (5.1) breaks the time-reversibility; that means that we suppose that $(5.1)_\varepsilon$ is for $\varepsilon \neq 0$ no longer time-reversible. We assume that

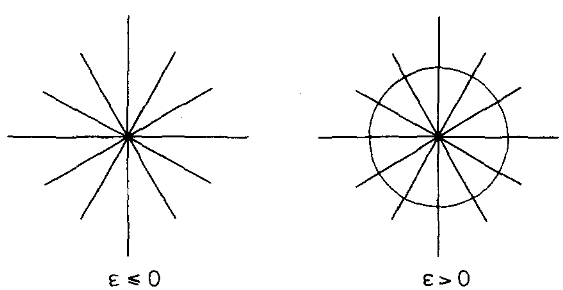

Figure 1
Projection of solution set of $(5.1)_\varepsilon$ onto U.

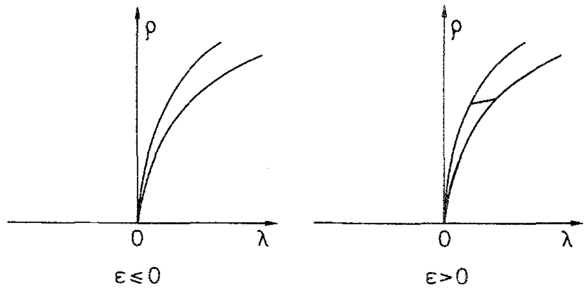

Figure 2
Bifurcation diagram for $(5.1)_\varepsilon$.

(H1) (i) and (ii) is satisfied for all (λ, ε), while (H1) (iii) holds only for $\varepsilon = 0$. We also assume that (H2) holds at $(\lambda, \varepsilon) = (0, 0)$. The characteristic multiplier $\mu^*(\lambda, \varepsilon)$ will no longer be on the unit circle for $\varepsilon \neq 0$, and hence $\psi(\lambda, \varepsilon)$ may take complex values for $\varepsilon \neq 0$. In fact we will assume that

(H3)″ $\dfrac{\partial \psi}{\partial \lambda}(0, 0) \neq 0$ and $\operatorname{Im} \dfrac{\partial \psi}{\partial \varepsilon}(0, 0) \neq 0$.

The abstract equation (1.8) is then replaced by the equation

$$M(x, \lambda, \varepsilon) = 0 \tag{5.9}$$

with

$$M(\gamma x, \lambda, \varepsilon) = \gamma M(x, \lambda, \varepsilon), \qquad \forall (x, \lambda, \varepsilon) \in X_q \times \mathbb{R}^2 \tag{5.10}$$

and

$$M(\sigma x, \lambda, 0) = -\sigma M(x, \lambda, 0), \quad \forall (x, \lambda) \in X_q \times \mathbb{R}. \tag{5.11}$$

This means that M is D_q-equivariant for $\varepsilon = 0$, but only C_q-equivariant for $\varepsilon \neq 0$ (C_q is the group generated by the rotation in the plane over an angle $2/q$).

We can again apply the equivariant Liapunov-Schmidt reduction of section 3 to solve (5.9) near the origin; the result is the bifurcation equation

$$G(z, \lambda, \varepsilon) = 0, \tag{5.12}$$

where G inherits the equivariance properties (5.10) and (5.11) of M; so we have

$$G(\mu_0 z, \lambda, \varepsilon) = \mu_0 G(z, \lambda, \varepsilon), \qquad \forall (z, \lambda, \varepsilon) \in \mathbb{C} \times \mathbb{R}^2, \tag{5.13}$$

and

$$G(\bar{z}, \lambda, 0) = -\overline{G(z, \lambda, 0)}, \qquad \forall (z, \lambda) \in \mathbb{C} \times \mathbb{R}. \tag{5.14}$$

As we already pointed out in remark 5 this implies that $G(z, \lambda, \varepsilon)$ has the form

$$G(z, \lambda, \varepsilon) = i g_1(z, \lambda, \varepsilon) z + i g_2(z, \lambda, \varepsilon) \bar{z}^{q-1}, \tag{5.15}$$

with $g_i\colon \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$ smooth, and such that

$$g_i(\delta_q z, \lambda, \varepsilon) = g_i(z, \lambda, \varepsilon) = g_i(\bar{z}, \lambda, \varepsilon), \quad i = 1, 2. \tag{5.16}$$

It follows from (5.14) and lemma 6 that

$$\operatorname{Im} g_i(z, \lambda, 0) = 0, \qquad \forall (z, \lambda) \in \mathbb{C} \times \mathbb{R}. \tag{5.17}$$

As in section 4 we put $z = \varrho \exp(i\theta)$ in (5.12); for nontrivial solutions this equation is then equivalent to

$$\tilde{g}_1(\varrho, \theta, \lambda, \varepsilon) + \varrho^{q-2} \tilde{g}_2(\varrho, \theta, \lambda, \varepsilon) e^{-iq\theta} = 0. \tag{5.18}$$

We have $\tilde{g}_i(0, \theta, 0, 0) = 0$, and a calculation as in the proof of lemma 7 shows that

$$D_\varepsilon g_1(0, 0, 0) = \frac{1}{2\pi} \frac{\partial \psi}{\partial \varepsilon}(0, 0); \tag{5.19}$$

it follows that

$$\frac{\partial(\operatorname{Re}\tilde{g}_1, \operatorname{Im}\tilde{g}_1)}{\partial(\lambda, \varepsilon)}(0, \theta, 0, 0) = \frac{1}{(2\pi)^2} \frac{\partial \psi}{\partial \lambda}(0, 0) \cdot \operatorname{Im} \frac{\partial \psi}{\partial \varepsilon}(0, 0),$$

which is different from zero by (H3)″. Hence we can split (5.18) in its real and imaginary part, and solve by the implicit function theorem for λ and ε; we obtain $(\lambda, \varepsilon) = (\bar{\lambda}(\varrho, \theta), \bar{\varepsilon}(\varrho, \theta))$, with smooth functions $\bar{\lambda}(\varrho, \theta)$ and $\bar{\varepsilon}(\lambda, \varrho)$ satisfying $\bar{\lambda}(0, \theta) = \bar{\varepsilon}(0, \theta) = 0$ and

$$\bar{\lambda}(\varrho, \theta + 2\pi/q) = \bar{\lambda}(\varrho, \theta), \qquad \bar{\varepsilon}(\varrho, \theta + 2\pi/q) = \bar{\varepsilon}(\varrho, \theta). \tag{5.20}$$

It also follows from (5.17) and the analysis of section 4 that

$$\bar{\varepsilon}(\varrho, \theta_{q,j}^{(i)}) = 0, \qquad \forall \varrho, \qquad i = 1, 2; \qquad j = 0, 1, \ldots, q-1. \tag{5.21}$$

The function $g_1(z, \lambda, \varepsilon)$ has again the form (3.24), but now with functions $A_l(\lambda, \varepsilon)$ which depend also on ε, and which may be complex for $\varepsilon \neq 0$. It follows that for $q \geqq 5$ we have

$$\bar{\lambda}(\varrho, \theta) = \sum_{l=1}^{r-1} b_l p^2 + 0(\varrho^{q-2}) \qquad \text{as} \qquad \varrho \to 0. \tag{5.22}$$

with certain coefficients $b_l \in \mathbb{R}$; $\bar{\varepsilon}(\varrho, \theta)$ has a similar form. Now the equation (5.1) has nontrivial $2\pi q$-periodic solutions in a neighborhood of $x = 0$ for those parameter values (λ, ε) which belong to the range of the mapping $(\varrho, \theta) \to (\bar{\lambda}(\varrho, \theta), \bar{\varepsilon}(\varrho, \theta))$. By (5.20), (5.21) and (5.22) this range forms a cusp-like region along the λ-axis, with width of the order $d^{(q-2)/2}$ at a distance d from the origin (see Fig. 3, (i)). Taking as a new (complex) parameter the characteristic multiplier $\mu = \mu^*(\lambda, \varepsilon)$ (this is allowed by (H3)″) this region transforms into the so called "Arnold tongue" for the system (5.1) [1] (see Fig. 3, (ii)). Here the tongue has the peculiarity of being tangent to the unit circle, instead of "licking" the unit circle, as is the case when the unperturbed system is not time-reversible (see Fig. 4; a brief discussion of this case can be found in [5]).

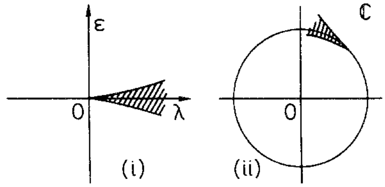

Figure 3
The Arnold tongues for (5.1).

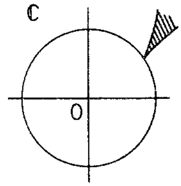

Figure 4
A "generic" Arnold tongue.

We conclude with the remark that (H3)″ implies that (H2) will be satisfied (for different values of p and q) at an infinity of parameter values $(\lambda, 0)$ near the origin; we will also have an infinity of Arnold tongues along the unit circle, each tongue corresponding to subharmonics of different periods $2\pi q$ (see also Remark 3). Hence a multitude of subharmonic solutions will coexist for certain small values of (λ, ε). It is beyond the scope of this paper to try a description of the resulting complicated dynamic behaviour of the system (5.1).

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Abstract

We study bifurcation of $2\pi q$-periodic solutions in one-parameter families of 2π-periodic time-reversible systems. We obtain generically satisfied conditions which imply the bifurcation of $2q$ branches of such subharmonic solutions. When $q \geqq 5$ the solutions along q of these branches are unstable, while the solutions along the other q branches are stable in a weak sense. Special results hold for $q = 3$ and $q = 4$. We also describe a situation in which there is secundary bifurcation and give a brief discussion of what happens under a perturbation which breaks the time-reversibility.

Zusammenfassung

Wir untersuchen die Verzweigung von $2\pi q$-periodischen Lösungen in einer einparametrigen Familie von 2π-periodischen reversiblen Differentialgleichungssystemen. Wir erhalten Bedingungen, welche die Verzweigung von $2q$ solcher subharmonischer Lösungen garantieren und die generisch erfüllt sind. Für $q \geqq 5$ sind q Lösungen instabil und q Lösungen (linear) stabil. Spezielle Resultate gelten für $q = 3$ und $q = 4$. Wir untersuchen auch einen Fall in dem sekundäre Verzweigung eintritt und diskutieren kurz die Wirkung einer Störung, die die Reversibilität zerstört.

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