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On the regularity of optimal controls

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1. The regularity problem

In any mathematical theory which deals with problems whose solutions are functions, the issue of the *regularity of solutions* ranks together with those of *existence* and *uniqueness* as one of the most basic questions. Usually, existence is proved by working on a very large space F, which contains very pathological functions. (Often, F has to be taken to be so large that the elements of F are not even ordinary functions, and can only be regarded as functions in a "generalized" sense.) It then becomes important to find conditions under which the solutions can be proved to be better than just an arbitrary member of F.

In the particular case of Optimal Control Theory, there are two types of solutions one is interested in; namely,

(i) "open loop" controls, and
(ii) "closed loop" (i.e. *feedback*) controls.

In each case, it is natural to ask whether the solutions necessarily have to possess some nontrivial regularity properties. We could also be less demanding, and only ask that, whenever a solution exists, it should follow that a solution with extra regularity exists as well. The latter is, in our view, the better question to ask, since there are degenerate problems where *every* admissible control is a solution, and therefore the solutions can be as pathological as any arbitrary control. (For instance, suppose x, y evolve according to $\dot{x} = 1$, $\dot{y} = u$, and the control u satisfies $|u| \leqq 1$. If we want to steer $(0, 0)$ to $(1, 0)$ in minimum time, then any control $u(\cdot)\colon [0, 1] \to [-1, 1]$ such that $\int_0^1 u(t)\,dt = 0$ will be a solution. Such controls can be very pathological, e.g. with a set of positive measure of discontinuities. However, one particular solution is $u(t) \equiv 0$, which is obviously very regular.)

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So we formulate the general regularity question as follows:

Given a class $\mathscr{A}$ of *"admissible controls" and a collection* $\mathscr{P}$ *of problems, such that each* $P \in \mathscr{P}$ *gives rise to a set* $S(P) \subseteqq \mathscr{A}$ *of solutions of* P, *find "interesting" classes* $\mathscr{R} \subseteqq \mathscr{A}$ *such that, whenever* $P \in \mathscr{P}$ *and* $S(P) \neq \emptyset$, *then* $S(P) \cap \mathscr{R} \neq \emptyset$.

(That is, find regularity conditions such that, whenever a problem in $\mathscr{P}$ has a solution, then it has one that satisfies the regularity conditions.) An $\mathscr{R}$ with the above property will be called *sufficient for solving* $\mathscr{P}$ *in* $\mathscr{A}$.

A classical example is provided by linear, time-optimal control. For a system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in K \subseteqq \mathbb{R}^m, \tag{1}$$

where K is a polyhedron and A, B are matrices of the appropriate sizes, one can easily prove a *bang-bang theorem with bounds on the number of switchings*, which says that there exists an integer $N > 0$ such that, whenever a point x_1 can be steered to a point x_2 in time T by means of some measurable K-valued control, then x_1 can be steered to x_2 in time T by a control which is bang-bang (i.e. with values in the set of vertices of K) and has at most NT switchings. Here we may let $\mathscr{A}_K$ denote the class of all measurable K-valued controls defined on $[0, T]$ for some T, and we may let $\mathscr{P}_{A,B,K}$ be the class of all $\mathscr{P}_{A,B,k,x,y,t}$, where $P_{A,B,K,x,y,t}$ is the problem of finding, for given A, B, K, x, y, t, a control that steers x to y in time t. Or we may take $\tilde{\mathscr{P}}_{A,B,K}$ to be the class of all $\tilde{P}_{A,B,K,x,y}$, where $\tilde{P}_{A,B,k,x,y}$ is the problem of steering x to y in minimum time. (So $\mathscr{P}_{A,B,K}$ is a class of "reachability" problems, and $\tilde{\mathscr{P}}_{A,B,K}$ consists of "optimal reachability" problems.) We let $\mathscr{R}_{N,K}$ denote the class of those $u(\cdot) \in \mathscr{A}_K$ such that, if $u(\cdot)$ is defined on $[0, T]$, then $u(\cdot)$ is bang-bang with at most NT switchings. Then the theorem says that for every A, B, K there is an N such that $\mathscr{R}_{N,K}$ is sufficient for solving $\mathscr{P}_{A,B,K}$ in $\mathscr{A}_K$. Since every solution of $\tilde{P}_{A,B,K,x,y}$ is a solution of $P_{A,B,K,x,y,t}$ for some t, we see that $\mathscr{R}_{N,K}$ is also sufficient for solving $\mathscr{P}_{A,B,K}$ in $\mathscr{A}_K$.

2. Feedback controls

We now give two examples that involve feedback controls. We define a feedback control as follows. Suppose Σ is a control system

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in U \tag{2}$$

for which a class $\mathscr{U}$ of "admissible controls" is specified. A *feedback control* on a subset Ω of M is a mapping $u(\cdot)\colon \Omega \to U$ with the property that for every $x \in \Omega$ there exists an $\varepsilon > 0$ such that, on the interval $0 \leqq t < \varepsilon$, there is a unique solution $t \to x(t) \in \Omega$ of the initial value problem

$$\dot{x}(t) = f(x(t), \quad u(x(t))), \quad x(0) = x. \tag{3}$$

It then follows that, if $u(\cdot)$ is a feedback control on Ω, then for every $x \in \Omega$ there exists a *maximal trajectory* for $u(\cdot)$ starting at x, i.e. a solution $t \to x(t) \in \Omega$ of (3), defined for $0 \leqq t < T(x)$, such that every solution of (3) is the restriction of this one to a subinterval of $[0, T(x))$. A feedback $u(\cdot)$ *steers* Ω *to a point* $\bar{x} \in M$ if $\lim_{t \to T(x)-} x(t) = \bar{x}$ for every $x \in \Omega$.

In that case:
(a) if $T(x) = +\infty$ for all $x \in \Omega$, we say that $u(\cdot)$ *asymptotically steers* Ω to $\bar{x}$; (b) if, for every $x \in \Omega$, $T(x)$ is the optimal time for steering x to $\bar{x}$, we say that $u(\cdot)$ is a *time-optimal feedback on* Ω *with target* $\{\bar{x}\}$.

Our first example of a regularity result for feedback controls involves the problem of stabilizing a system. We first quote the classical result for linear systems. Let $\mathscr{A}_{n,m}$ be the class of all maps $u(\cdot)$ from $\mathbb{R}^n - \{0\}$ to $\mathbb{R}^m$. Let A, B be an $n \times n$ and an $n \times m$ matrix, respectively. Let $P^S_{A,B}$ be the problem of finding a stabilizing feedback for (A, B), i.e. a $u(\cdot) \in \mathscr{A}_{n,m}$ which is a feedback that asymptotically steers $\mathbb{R}^n - \{0\}$ to 0, for the linear system $\dot{x} = Ax + Bu$.

If we let $L_{n,m}$ denote the class of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, then we know that $P^S_{A,B}$ has a solution if and only if (A, B) is stabilizable, in which case $P^S_{A,B}$ has a solution in $L_{n,m}$. So $L_{n,m}$ is sufficient for solving $\mathscr{P}^S_{n,m}$ in $\mathscr{A}_{n,m}$, where $\mathscr{P}^S_{n,m} = \{P^S_{A,B}: A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\}$.

This shows that, for linear systems, the stabilization problem has the best possible regularity properties: if a solution exists at all, then there is one which is actually a linear map. For nonlinear systems the situation is much worse. An example was given in Sussmann [10] which shows that, for a completely controllable real analytic sytem, a continuous stabilizing feedback may fail to exist. The example of [10] is of a global nature. (Essentially, the reason for the nonexistence of a continuous feedback is that there is an "obstacle" near the target, which every trajectory from a point x to the target $\bar{x}$ necessarily has to avoid.) More recently, Brockett [4] obtained a necessary condition for the existence of a continuous stabilizing feedback which implies, for instance, that no such feedback exists for the system

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = uy - vx. \tag{4}$$

Brockett's results have the remarkable feature of being purely local in nature.

On the other hand, it was proved in Sussmann [10] that a *piecewise analytic* stabilizing feedback always exists, for a controllable analytic system. Sontag [8] has proved the existence of piecewise linear feedbacks.

Four our second example, we let U be a polyhedron in $\mathbb{R}^m$, and we let $\mathscr{A}_n(U)$ be the class of all U-valued maps on $\mathbb{R}^n$. We let $P^{TO}_{A,B,U}$ be the problem of finding a *time-optimal feedback* for the system $\dot{x} = Ax + Bu$, $u \in U$, with target 0. It can be proved that, for $\mathscr{P}^{TO}_n(U) = \{P^{TO}_{A,B,U}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\}$, there is a sufficient class $RS^{TO}_n(U)$ that consists of the U-valued maps on $\mathbb{R}^n$ that are *bang-bang C^ω regular syntheses.* (A map is bang-bang if it takes values in the set of vertices of U. A C^ω *regular synthesis* is a map $u(\cdot)$ which is "piecewise real analytic" in a sense which will not be described in detail here. Roughly, it means that there is a locally finite partition $\mathcal{S}$ of $\mathbb{R}^n$ into embedded real-analytic connected submanifolds, such that $u(\cdot)$ is real-analytic on each $S \in \mathcal{S}$. For a complete definition, cf. [11].) This result is essentially due to P. Brunovsky, who only considered systems that satisfy an extra normality condition (cf. [5]). However, Brunovsky's proof can be generalized so as to do away with the normality requirement. Moreover, Brunovsky's method yields other regularity results as well. The main idea is as follows. Suppose we prove a regularity result for optimal trajectories, which gives a family $\mathcal{F}$ of trajectories, γ^α, parametrized by a finite-dimensional parameter α in some set A such that, whenever a point x can be optimally steered to x', then x can be optimally steered to x' by means of one of the γ_α. Suppose A is a subanalytic subset of some C^ω manifold N. Moreover, assume that the γ_α depend analytically on α, in the sense that γ_α is defined on $[a(\alpha), b(\alpha)]$, and the map $u\colon \alpha \to (a(\alpha), b(\alpha), \gamma(a(\alpha)), \gamma(b(\alpha)), J(\gamma_\alpha))$ is proper and real-analytic. (Here $J(\gamma_\alpha)$ is the cost of γ_α.) Let a target point $\bar{x}$ be fixed. It then follows from a simple application of the theory of subanalytic sets that the graph of the Bellman function V is a subanalytic set. Under some extra technical conditions, one can actually prove the existence of a regular synthesis. The crucial point is the analyticity of u and the finite-dimensionality of α. Typically, this is established by first showing that the optimal trajectories can be taken to be finite concatenations of pieces that are trajectories of vector fields in some family X that depends analytically on a finite-dimensional parameter. The finite-dimensionality of α then follows if the number of pieces that occur in the concatenations is uniformly bounded. That is, one has to prove results on *piecewise regularity with bounds on the number of switchings* (PRBNS). Therefore the study of PRBNS results is important not only because of its intrinsic interest, but also as a tool for proving regularity theorems for Bellman functions and optimal feedback controls.

3. Time-optimal control for a single-input system

We study the problem of time-optimal control for a system

$$\Sigma\colon \dot{x} = f(x) + u\,g(x), \quad |u| \leqq 1, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}$$

where f and g are smooth (or analytic) vector fields. (Admissible controls are measurable functions with values in $[-1, 1]$.) The *aim* is to *characterize the local structure of time-optimal trajectories near a reference point p through the Lie bracket configuration at p.* In particular we are interested in conditions which guarantee that time-optimal trajectories are finite concatenations of bang and singular arcs near p with a bound on the number of pieces.

It turns out that even for this most simple class of nonlinear processes the problem is anything but easy. Therefore the approach taken was to study the low-dimensional systems first and try to develop a set of techniques and tools which might be applicable to higher dimensional systems as well.

In [12, 13] smooth systems in the plane were considered and a classification of the local structure of time-optimal trajectories for a generic system was given. For analytic systems a complete trajectory analysis was carried out in [14] and, except for certain "degenerate cases", it followed that every point $p \in \mathbb{R}^2$ has a neighborhood U such that time-optimal trajectories lying in U are finite concatenations of bang and singular arcs with a bound on the number of switchings. This was the key step in the proof of the following

Theorem (Sussmann, [15]): Let Σ be an analytic system in the plane (of the form (5)). Let p be a point where the following "nonexplosion condition" holds:

for every $T > 0$ there exists a compact subset

$K = K(T)$ of $\mathbb{R}^2$ such that, if $\gamma\colon [a, b] \to \mathbb{R}^2$ is a trajectory of Σ with $\gamma(b) = p$ and $b - a \leqq T$, then γ is entirely contained in K.

Then the problem of reaching p in minimum time admits a regular synthesis. □

One of the major problems in carrying out the trajectory analysis is to eliminate the optimality of bang-bang trajectories with too many switchings. What has to be considered too many switchings is not quite evident, but it is clear that it should somehow depend on the dimension n of the space and the degree of degeneracy that the Lie bracket configuration at p has (cf. [12]). As measure for the latter the "codimension" seems an appropriate tool to use. In a nontechnical sense the codimension is roughly the number of independent Lie relations which hold at p. For a nondegenerate situation, i.e. when there are no relations whatsoever imposed at p (codimension 0), the number of switchings should not exceed $n - 1$, since then we have n parameters describing time-optimal bang-bang trajectories and heuristically one can expect that they fill up a whole relative neighborhood of p in the time T reachable set. In general however this is not an easy result, but requires a sophisticated analysis going beyond a simple application of the standard necessary conditions for optimality.

For systems in the plane it was shown in [12] that in a nondegenerate situation bang-bang trajectories with 2 switchings are not time-optimal. The argument uses Stokes Theorem and therefore is limited to dimension 2. The first such result in dimension 3 is due to Bressan [3]. Combining a local approximation procedure in which f and g are approximated by vector fields which generate a nilpotent Lie algebra [1] with a newly developed finite optimality condition [2], he proved:

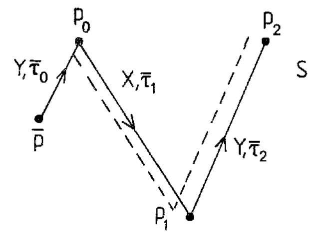
Theorem (Bressan, [3]): Let Σ be a (smooth) system (of the form 5)) in $\mathbb{R}^3$ with $f(p_0) = 0$. Suppose the vectors g, $[f, g]$ and $[f, [f, g]]$ are independent at p_0

and the coefficient k_3 defined by

$$[g, [f, g]](p_0) = k_1 g(p_0) + k_2 [f, g](p_0) + k_3 [f, [f, g]](p_0)$$

satisfies $|k_3| \neq 1$. Then there exists a neighborhood V of p_0 such that every time-optimal trajectory steering p_0 to a point $p \in U$ is of the type BBB or BSB (bang-bang with at most 2 switchings or a concatenation of a bang arc, a singular arc and another bang arc), where some of the pieces may be absent. □

We now briefly describe the ideas behind two techniques which have proven useful in the study of time-optimality for bang-bang trajectories in dimension 3. The first one is based on the analysis of "conjugate points" ([16]). If Γ is a bang-bang extremal in $\mathbb{R}^n$, then n points $q_1, \ldots, q_n$ are called *conjugate* (or a conjugate n-tuple) if Γ has switching-points at $q_1, \ldots, q_n$ in the sense that $\langle \lambda(t_i), g(q_i) \rangle = 0$ where t_i is the corresponding time. Since λ is non-trivial, the vectors which we get when we transport all g's along Γ to one and the same time must be dependent. This gives an equality constraint as necessary condition for optimality of Γ, which we call a "conjugate point relation". We briefly outline now (without giving proofs) how this concept can be used in the study of trajectories. (For a systematic treatment we refer to [16]). We assume $n = 3$ and let $X = f - g$, $Y = f + g$; suppose we have an extremal Γ of the type YXY with times $\bar{\tau}_0$, $\bar{\tau}_1$ and $\bar{\tau}_2$ along Y, X and Y respectively. In a nondegenerate situation, for any such $YX^{\bullet}$-concatenation, i.e. for any pair $(\bar{\tau}_0, \bar{\tau}_1)$, there exists a unique multiplier λ such that the necessary conditions of the Maximum Principle hold.



This then defines the time $\bar{\tau}_2$ along Y up to the next switching point. Now vary (τ_0, τ_1) near $(\bar{\tau}_0, \bar{\tau}_1)$ to obtain a 2-parameter family of YXY-extremals. The third switching points, defined via

$$p\, e^{\tau_0 Y} e^{\tau_1 X} e^{\tau_2(\tau_0, \tau_1) Y},$$

describe a surface S, which we call the "conjugate surface". (Here $t \to p\, e^{tX}$ denotes the point obtained by following the integral curve of X starting at p for time t. We let the diffeomorphisms act on the right, since in this way the formal calculations involving the Campell-Hausdorff formula come out right.) If S contains a trajectory of the system Σ through p_2, it is possible to construct a family of trajectories which all steer $\bar{p}$ to p_2 and all take exactly the same time as Γ. If for instance X and Y point to opposite sides of S, then S contains a trajectory of Σ (a convex combination of X and Y is tangent to S) and if Γ is optimal, this trajectory is in fact a singular arc. So we get a family of $YXYS$-trajectories which steer $\bar{p}$ to p_2 and take the same time as Γ. If we can exclude the optimality of one of them (for instance if singular arcs or singular junctions can be excluded) this implies that Γ is not time-optimal.

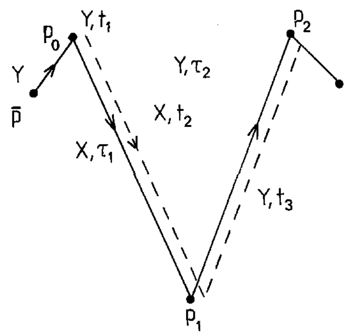
So basically 3 conditions are needed for this to work:

i) the conjugate surface must exist
ii) X and Y should point to opposite sides of the conjugate surface at p_2
iii) there should be some reason that implies that the trajectories with which we compare Γ are not optimal.

It turns out that all this works very smoothly in a nondegenerate situation in $\mathbb{R}^3$ and we can prove (cf. [16]):

Proposition: Let Σ be a (smooth) 3-dimensional system (of the form (5)). Suppose that each of the triples $(f, g, [f, g])$, $(g, [f, g], [f + g, [f, g]])$ and $(g, [f, g], [f - g, [f, g]])$ is a triple of independent vectors at p. Then p has a neighborhood U such that bang-bang trajectories that lie in U and have more than 2 switchings are not time-optimal. □

This result can also be obtained by a more algebraic procedure (cf. [6, 7]). Suppose now Γ is a $YXYX$-trajectory. We would like to exclude the optimality of Γ beyond its third switching point p_2. It is reasonable to compare this $YXYX$-concatenation to a YXY-trajectory (i.e. a trajectory with one less



switching, which does exactly the same, i.e. steers $\bar{p}$ to a point on Γ just after p_2. In a nondegenerate situation such a YXY-trajectory exists: if X, Y and $[X, Y]$ are independent at a reference point p, then the equation

$$p_0\, e^{\tau_1 X}\, e^{\tau_2 Y}\, e^{\tau_1 s X} = p_0\, e^{t_1 Y}\, e^{t_2 X}\, e^{t_3 Y} \tag{6}$$

can be solved by smooth functions $t_i = t_i(s; \tau_1, \tau_2)$, $i = 1, 2, 3$. The difference in the time is given as

$$\Delta := t_1 + t_2 + t_3 - \tau_1 - \tau_2 - \tau_1 s$$

and it turns out that necessary conditions for optimality of Γ are

(I) $\dot{\Delta}(0; \tau_1, \tau_2) = 0$

(II) $\ddot{\Delta}(0; \tau_1, \tau_2) \geqq 0$

where $\cdot$ denotes differentiation with respect to s. Asymptotic expansions for (I) and (II) can be computed using Lie-algebraic formulas and so this can be turned into an applicable criterion. From this the proposition follows easily as well (cf. [6]).

With a little bit of extra effort one can then actually prove

Theorem: Let Σ be a (smooth) 3-dimensional system (of the form (5)) and suppose that each of the triples $(f, g, [f, g])$, $(g, [f, g], [f + g, [f, g]])$ and $(g, [f, g], [f - g, [f, g]])$ is a triple of independent vectors at p. Then there exists a neighborhood U of p such that time-optimal trajectories of Σ that lie in U are of type BBB or BSB. □

For analytic systems this implies as a

Corollary: Let Σ be an analytic 3-dimensional system (of the form (5)). Then there exists an analytic subset A of positive codimension such that every $p \notin A$ has a neighborhood U with one of the following two properties:

(a) time-optimal trajectories that lie in U are of the type BBB or BSB
(b) whenever $q_2 \in U$ is reachable from $q_1 \in U$ time-optimally in time T within U, then there exists a bang-bang trajectory with at most 2 switchings which steers q_1 to q_2 in U in time T. □

Case b) has to be included to cover degenerate cases when every trajectory is time-optimal (cf. also [9]). It says that in such a case we can still make a selection of time-optimal trajectories which is nice, i.e. BBB.

The conjugate surface method and the Lie-algebraic approach are clearly related: (I) is exactly the conjugate point relation and (II) relates to the conditions which come up when one wants to check that X and Y point to opposite sides of the conjugate surface. The exact relationship between the two methods has however not been investigated yet.

The conjugate point technique has the advantage that the computations are much simpler than the complicated Lie-algebraic procedures which give the approximations for $\dot{\Delta}$ and $\ddot{\Delta}$. On the other hand, the conjugate point method seems to apply in a straightforward manner only in nondegenerate situations, whereas the asymptotic expansions are the same also for more degenerate situations (such as $g \wedge [f, g] \wedge [f \pm g, [f, g]] = 0$ at p), even though in degenerate cases their application may be delicate. Using them we can show

Theorem (Schättler, [7]): For a (smooth) 3-dimensional system Σ (of the form (5)), which is generic within the class of systems that satisfy $f \wedge g \wedge [f, g] \neq 0$, every point p has a neighborhood U such that bang-bang trajectories that lie in U are not time-optimal if they have more than 7 switchings. □

The generic statement for arbitrary systems in $\mathbb{R}^3$ is still an open problem. Also it is not known at the moment how to fit singular junctions in the picture. If we assume that in addition to $f \wedge g \wedge [f, g] \neq 0$ also $g \wedge [f, g] \wedge [f + g', [f, g]] \neq 0$ or $g \wedge [f, g] \wedge [f - g, [f, g]] \neq 0$ holds, then generically every point has a neighboorhood U such that time-optimal trajectories that lie in U are concatenations of bang and singular arcs with at most 6 pieces ([7]). But even in the case when we still assume that f, g and $[f, g]$ are independent, it is currently not known what type of concatenations between bang and singular arcs can occur generically.

Finally we remark that both techniques are conceptually not restricted to dimension 3. The conjugate surface method is actually quite general and applicable for a broad range of problems (cf. [16]). But both methods seem to have their limitations in the computational complexity which increases rapidly with the dimension. If these problems can be overcome, maybe general results in $\mathbb{R}^n$ can be obtained and this would yield quite a bit of qualitative information about the structure of time-optimal controls for single-input nonlinear systems.

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Abstract

We outline some recent results on the regularity of optimal controls. We formulate the general regularity problem for open-loop and closed-loop controls, and explain how results for the open-loop case have implications for the closed-loop case as well. We then describe a number of results on the regularity of open-loop controls.

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