Journal of Applied Mathematics and Physics (ZAMP) 0044-2275/89/010069-09 $ 1.50 + 0.20/0
Vol. 40, January 1989

The plane turbulent plume in a magnetic field

By L. J. Crane, School of Mathematics, Trinity College, Dublin 2, Ireland

1. Introduction

The motion of stellar atmospheres is set up and controlled by the interaction of temperature and magnetic fields. This paper is an attempt to understand some aspects of this interaction by constructing a simple two dimensional model in which an upwardly moving turbulent plume of electrically conducting gas is retarded by a uniform horizontal magnetic field, the horizontal direction being chosen for maximum effect on the upward flow.

As far as the author is aware no treatment of such a plume exists in the literature; however Gray [1] has considered the laminar plume in a magnetic field which so varies with height as to produce a similarity solution.

It is shown that at a comparatively small height above the source the flow is that of a simple plume which is unaffected by the magnetic field (in the above statement "comparative" means compared to a unit of height which can be constructed out of the basic thermal and magnetic constants of the flow); as the height increases the magnetic field gradually retards the upward flow till eventually buoyancy and magnetic forces are in almost perfect equilibrium; when this occurs the flow is shown to be that of a two dimensional turbulent jet.

2. Equations of motion

A system of rectangular coordinates (x, y, z) is taken with z-axis lying along the horizontal source and x-axis vertically above the source; (u, v, w) are the corresponding unsteady velocity components; the magnetic field is of constant strength B_0 and is in the y-direction.

In fully developed free turbulent flow viscous stresses may be neglected; thus the unsteady vertical momentum equation is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g\beta\theta - Ku, \tag{1}$$

where g is the acceleration due to gravity, β is the coefficient of volumetric expansion, θ is the excess of temperature over the surroundings and

$$K = \frac{\sigma B_0^2}{\rho}, \tag{2}$$

where σ is the electrical conductivity of the fluid and ρ is the density. The basic assumptions underlying the right side of (1) are:

(i) that density differences may be neglected except in the buoyancy term $g\beta\theta$; this is the Boussinesq approximation,
(ii) that the magnetic field B_0 is of sufficient strength to mask the magnetic field due to the electric current produced by the motion of the conducting fluid.

The equation of heat transfer is

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = 0; \tag{3}$$

in this equation molecular heat transfer has been neglected.

Finally the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

where, once again, density differences have been neglected.

3. An exact solution

Equations (1), (3), (4) are satisfied by

$$g\beta\theta = Ku$$

where u satisfies the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0. \tag{5}$$

So far u, v, w, θ have been instantaneous variables. Take now mean values, which are denoted by a bar, then

$$g\beta\bar{\theta} = K\bar{u}; \tag{6}$$

taking the mean value of (5) and noting that the mean flow is independent of the z-coordinates gives:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial x}\overline{(u')^2} - \frac{\partial}{\partial y}\overline{(u'v')}, \tag{7}$$

where primes denote fluctuations from the mean; finally the above equation reduces to

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=-\frac{\partial}{\partial y}\overline{(u'v')}, \tag{8}$$

when the boundary layer assumption, namely

$$\frac{\partial}{\partial x}\ll\frac{\partial}{\partial y}, \tag{9}$$

is made.

In addition to (8) the flow must satisfy the mean continuity equation namely:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0 \tag{10}$$

and the boundary conditions which are:

$$\frac{\partial \bar{u}}{\partial y}=\frac{\partial \bar{\theta}}{\partial y}=0 \quad \text{on} \quad y=0 \tag{11}$$

and

$$\bar{u}, \bar{\theta}\to 0 \tag{12}$$

at the edges of the plume.

So far no reference has been made to the flow of heat from the line source; this may be introduced by considering the mean value of (3), which reduces to

$$\bar{u}\frac{\partial \bar{\theta}}{\partial x}+\bar{v}\frac{\partial \bar{\theta}}{\partial y}=-\frac{\partial}{\partial y}\overline{(v'\theta')}, \tag{13}$$

when the boundary layer assumption (9) is made.

When (13) is integrated over a horizontal section of the plume, and the boundary conditions (11), (12) applied, it follows that

$$\int_{-\infty}^{\infty}\bar{u}\bar{\theta}\,dy$$

is constant; this constant may be expressed in terms of the heat flux at the source; thus

$$g\beta\int_{-\infty}^{\infty}\bar{u}\bar{\theta}\,dy=b, \tag{14}$$

where the specific buoyancy flux b is defined by

$$b=\frac{Q_0 g\beta}{C_p\rho}, \tag{15}$$

where Q_0 is the heat flux per unit length of source and C_p is the specific heat at constant pressure of the fluid.

On substitution of $\bar{\theta}$ from (6) relation (14) becomes

$$\rho \int_{-\infty}^{\infty} \bar{u}^2 \, dy = \frac{b\rho}{K} \equiv m_0; \tag{16}$$

that is, the vertical momentum flux m_0 is conserved in that flow given by the exact solution (6).

Clearly the flow which satisfies (8), (10), (16) is a plane turbulent jet whose momentum flux is m_0. This jet solution is an asymptotic flow to which the plume approximates far above the source.

4. An approximate solution

In this section an approximate Pohlhausen method will be used to determine the height at which the exact solution of section (3) is valid.

Two internal relations must be satisfied, these are: the heat flux equation (14) and the momentum integral equation, namely:

$$\frac{d}{dx} \int_0^{\infty} \bar{u}^2 \, dy = g\beta \int_0^{\infty} \bar{\theta} \, dy - K \int_0^{\infty} \bar{u} \, dy; \tag{17}$$

this equation is derived from the mean value of equation (1), namely:

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = - \frac{\partial}{\partial y} \overline{(u'v')} + g\beta\bar{\theta} - K\bar{u}, \tag{18}$$

by integration over a horizontal section subject to conditions (11), (12).

The profiles of velocity and temperature are now taken as:

$$\left.\begin{aligned} \bar{u} &= U_0(x) \ \operatorname{sech}^2 \eta \\ \bar{\theta} &= \theta_0(x) \operatorname{sech}^2 \eta \\ \eta &= \frac{y}{\Delta(x)} \end{aligned}\right\} \tag{19}$$

Substituting (19) into (14), (17) gives

$$g\beta U_0 \theta_0 \Delta = \tfrac{3}{4} b \tag{20}$$

and

$$\frac{d}{dx} (U_0^2 \Delta) = \tfrac{3}{2} g\beta\theta_0 \Delta - \tfrac{3}{2} K U_0 \Delta. \tag{21}$$

One further equation is needed to close the solution; this is found by taking the value of equation (18) on the x-axis, i.e.

$$U_0 \frac{dU_0}{dx} = -\left[\frac{\partial}{\partial y}\overline{(u'v')}\right]_{\text{axis}} + g\beta\theta_0 - KU_0. \tag{22}$$

when a coefficient of eddy viscosity ϵ_T is introduced, such that

$$-\frac{\partial}{\partial y}[\overline{u'v'}] = \epsilon_T \frac{\partial^2 \bar{u}}{\partial y^2}, \tag{23}$$

equation (22) becomes:

$$U_0 \frac{dU_0}{dx} = -\frac{2\epsilon_T U_0}{\Delta^2} + g\beta\theta_0 - KU_0 \tag{24}$$

Equation (22) may be further simplified by using Reichardt's [2] hypothesis for free turbulent flows, namely: that ϵ_T is proportional to the product of the axial velocity and the width of the flow; thus:

$$\epsilon_T = \alpha U_0 \Delta, \tag{25}$$

where α is a numerical constant which is fixed by reference to experiment; finally (24) becomes:

$$U_0 \frac{dU_0}{dx} = -\frac{2\alpha U_0^2}{\Delta} + g\beta\theta_0 - KU_0. \tag{26}$$

When θ_0, x are eliminated from (20), (21), and (26) it is found, and after some simplification, that

$$\frac{d\Delta}{dn} + \frac{1}{3}\frac{\Delta}{n} = \frac{4}{\sqrt{3}}\frac{\alpha}{K}\sqrt{\frac{m_0}{\rho}}\frac{\sqrt{\frac{n}{\Delta}}}{(1-n)} \tag{27}$$

where nm_0 is the vertical momentum flux which is given by

$$nm_0 = \tfrac{4}{3}\rho U_0^2 \Delta, \tag{28}$$

when the profiles (19) are used.

Since m_0 is the asymptotic value of the momentum flux, see (15), it follows that the dimensionless quantity n ranges from zero at the source to unity at large vertical heights.

The solution of (27) for which

$$\Delta = 0 \quad \text{when} \quad n = 0,$$

is

$$U_0 \Delta^2 = \frac{3\alpha}{K}\frac{m_0}{\rho} f(n) \tag{29}$$

where

$$f(n) = -n - \log(1-n) \tag{30}$$

From (28), (29) it is readily shown that

$$U_0^3 = \frac{3}{16\alpha} b \frac{n^2}{f} \tag{31}$$

and

$$\Delta^3 = 12\alpha^2 \frac{b}{K^3} \frac{f^2}{n}, \tag{32}$$

while θ_0 may be found from (19).

The height x may be introduced by integrating equation (21), thus:

$$\frac{K}{b^{1/3}} (18\alpha)^{1/3} x = \int_0^n \frac{n^{2/3}\, dn}{(1-n) f^{1/3}}. \tag{33}$$

Two cases are of special interest; first close to the source, n is small and (31), (32), (20) give:

$$\left.\begin{aligned} U_0 &= \left(\frac{3}{8\alpha}\right)^{1/3} b^{1/3}, \\ \Delta &= 3\alpha x, \\ g\beta\theta_0 &= \left(\frac{1}{24\alpha^2}\right)^{1/3} \frac{b^{2/3}}{x}; \end{aligned}\right\} \tag{34}$$

these correspond to the plane non-magnetic plume.

Second, when n is close to unity:

$$\left.\begin{aligned} U_0 &= \left[\frac{3}{16\alpha} \frac{m_0}{\rho x}\right]^{1/2}, \\ \Delta &= 4\alpha x, \\ g\beta\theta_0 &= \tfrac{1}{4} b \left[\frac{3\rho}{m_0 \alpha x}\right]^{1/2}; \end{aligned}\right\} \tag{35}$$

equations which correspond to the jet solution of section (3); indeed the jet profile

$$u = U_0 \operatorname{sech}^2 \frac{y}{\Delta} \tag{36}$$

with U_0, Δ given by (35) is the exact solution of equations (10), (18) when the eddy viscosity (23), (25) is used; for details see Schlichting [3]. Equation (36) is in excellent agreement with the experimental data of Förthmann [4] when the constant α is taken to be .033.

When this value of α is used the approximate solution for small n, namely (34), becomes:

$$\left.\begin{aligned} U_0 &= 2.2b^{1/2}, \\ \Delta &= .10x, \\ g\beta\theta_0 &= 3.4b^{2/3}x^{-1}. \end{aligned}\right\} \tag{37}$$

Now Chen and Rodi [5] have collected and summarised the published experimental data on plane plumes. The experimental values which they recommend for the numerical coefficients of U_0 and $g\beta\theta_0$ in (37) are 1.9 and 2.4 respectively; they also give the half width of the velocity and temperature profiles as $.12x$ and $.13x$ respectively; values which may be compared with the theoretical half width calculated from (37) and (19), which is $.09x$.

To summarise: the Pohlhausen solution is in excellent agreement with the experimental data for the plane jet when α is chosen as .033; this corresponds to the limit as n tends to unity. At the other extreme, as n tends to zero and the flow reduces to a non-magnetic plume, the Pohlhausen method:

(i) overestimates axial velocity by 15%,
(ii) overestimates axial temperature by 30%, and
(iii) underestimates the width by 30%.

Table 1 gives numerical values of the approximate Pohlhausen solution in dimensionless form.

Table 1
Non-dimensional parameters of the flow as calculated by the approximate Pohlhausen method.

n	$xK\left(\frac{\alpha}{b}\right)^{1/3}$	$U_0\left(\frac{\alpha}{b}\right)^{1/3}$	$g\beta\theta_0\left(\frac{\alpha}{b}\right)^{1/3}$	$K\frac{\Delta K}{(\alpha^2 b)^{1/3}}$
0	0	.721	∞	0
0.1	.050	.704	7.01	0.152
0.2	.105	.698	4.84	0.318
0.3	.165	.669	2.22	0.505
0.4	.232	.646	1.62	0.718
0.5	.309	.623	1.25	0.964
0.6	.400	.619	0.96	1.261
0.7	.512	.566	0.81	1.633
0.8	.625	.526	0.67	2.142
0.9	.895	.474	0.53	2.972

This table gives, in dimensionless form; vertical distance, axial velocity, axial temperature and width of the plume corresponding to different values of n (the ratio of momentum flux to its asymptotic value).

5. Discussion

Two and only two physical parameters govern the turbulent magnetic plume; these are the specific buoyancy flux b, defined by equation (15) and the magnetic parameter K defined by equation (2), the dimensions of these parameters being $(\text{velocity})^3$ and $(\text{time})^{-1}$, respectively. Out of these parameters an intrinsic unit of vertical height, namely

$$b^{1/3}/K$$

may be formed. This unit of height characterises the different regions of the flow; when

$$x \ll b^{1/3}/K$$

the magnetic field is unimportant and the flow is that of a simple plume when

$$x \approx b^{1/3}/K$$

a jet type flow occurs. Indeed it may be deduced from (33) that at a height

$$x = 2.7\,\frac{b^{1/3}}{K} \tag{38}$$

the flow is sensibly that of a two dimensional jet whose momentum flux is within 10% of its ultimate value.

The principal effect of the magnetic field is to retard the upward velocity of the plume; the rate of spread of the plume is virtually unchanged in its transition from simple plume to jet flow; consequently to ensure a constant vertical flux of heat the magnetic plume must be hotter than the simple plume; indeed while the axial temperature in the simple plume decays as the inverse of the vertical height, the magnetic plume temperature decays much more slowly in fact as the inverse square root of the vertical height.

The author suggests that it should be possible to observe the jet in the laboratory by using mercury in a horizontal magnetic field of 1000 gauss, the heating being provided by a horizontal source (Q_0) of strength 10 W/cm length. This would result in a time scale (K^{-1}) of about 1.4 seconds and a velocity scale ($b^{1/3}$) of about 2 cm/s.

Moreover it can be deduced from (38) that such a turbulent jet should be well established at a height of 7 cm above the source.

References

[1] D. D. Gray, App. Sci. Res., *33*, 437 (1977).
[2] H. Reichardt, VDI—Forschungsheft, *414* (1942), quoted in [3].
[3] H. Schlichting, *Boundary layer theory*, McGraw Hill, New York (1979).

[4] E. Förthmann, Ing-Arch., *5*, 42 (1934).
[5] C. J. Chen and W. Rodi, *Vertical turbulent buoyant jets*, Pergamon Press, Oxford (1980).

Summary

This paper traces the development of a two dimensional turbulent plume in a magnetic field from its origin, where it is virtually unaffected by the magnetic field up to its ultimate jet-like form.

(Received: January 6, 1987; revised: October 7, 1988)