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Boundary effects on granular shear flows of smooth disks

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Introduction

In developing recent theories for rapidly flowing granular materials workers have exploited the analogy between the fluctuating nature of rapid granular motion and the random molecular motion within a dense gas. The grains are assumed to interact vigorously with their neighbors through energy dissipating binary collisions, while the effects of enduring contacts and static friction between grains are ignored. Based upon a statistical description of the particles' velocities, it is possible to define mean fields such as density, velocity, and granular temperature, and to derive balance equations corresponding to each. Constitutive quantities may then be identified through their appearances in these equations, and constitutive relations may be obtained through appropriate statistical averaging. The appearance in these constitutive theories of a rate of dissipation due to inelastic collisions is the most striking departure from the kinetic theory for dense gases.

Complete constitutive theories derived in this manner for assemblies of identical, nearly elastic spheres include those obtained by Jenkins and Savage [1983], Lun et al. [1984], and Jenkins and Richman [1985a]. Jenkins and Richman [1985b] have extended their work to assemblies of rough disks while Jenkins and Mancini [1987] have obtained theories for both binary mixtures of nearly elastic spheres and disks. Apart from physical differences, these theories vary in complexity according to the assumptions made regarding the distribution functions governing the flow statistics.

Several other theories in which a somewhat simpler averaging technique was employed have also been advanced. Among these are the theories obtained for homogeneous shear flow by Shen and Ackermann [1984] for rough disks, and by Raymond and Shen [1986] for rough spheres. Recently Shen and Ackerman [1986] have considered more general flows of rough disks and have obtained a constitutive relation for the flux of fluctuation energy.

Although all these theories account in some detail for the interactions between the grains within the flow, very little attention has been paid to the interactions between the grains and the boundaries that contain the flow. When these theories are applied to homogeneous shearing flow, it is imagined that there exist boundaries that can sustain such a simple flow. When the stresses predicted in this hypothetical situation are then compared with experimentally determined stresses, it is assumed that the shearing surfaces of the experimental apparatus are precisely those imagined. When the apparent shear rate associated with the shear cell is interpreted as the actual shear rate experienced by the contained granular material, slip at the boundary is neglected.

In their numerical simulations of the shearing of rough inelastic disks and smooth inelastic spheres, Walton and Braun [1986a, 1986b] ignored all boundary effects by periodically imaging particles above and below the primary calculational cell. However, Campbell and Brennen [1982, 1985] performed numerical simulations of the shearing of rough inelastic disks between parallel walls and found the flows to be critically affected by the nature of the walls. In one case, no tangential slip was permitted between the particles' peripheries and the walls after each collision. Only in a second case in which particles experienced no change in spin but assumed the tangential velocity of the wall after collision did homogeneous shear flows result. Significant boundary effects may also be inferred from experimental results. The upper and lower surfaces in the shearing device of Savage and Sayed [1984] were roughened by attaching to each a layer of sandpaper. Hanes and Inman [1985] used virtually the same device, except with less dissipative shearing surfaces roughened by cementing to each a grain layer of the material being tested. They measured stresses that were as much as three times higher than those of Savage and Sayed at equivalent solid fractions and shear rates. Savage and Mckeown [1983] specifically examined the effect of wall roughness on the stresses developed in sheared granular materials. They carried out experiments and found that the resulting stresses induced when the internal surfaces of their device were roughened were typically higher than those induced when the surfaces were smooth.

Without conditions that explicitly ensure the separate balances of momentum and energy fluxes at the boundaries it is necessary, when analyzing a granular flow, to specify the slip velocity and either the granular temperature or the flux of fluctuation energy at the containing surfaces. It was in this manner, for example, that Haff [1983] was able to employ his phenomenological theory to analyze several simple granular flows. If the appropriate conditions were known, then it would be possible to determine the slip velocity, granular temperature, and energy flux at the boundary in terms of the geometry and dissipative character of the boundary. Hui et al. [1984] have proposed phenomenological boundary conditions on the shear stress and energy flux and have used them with Haff's flow theory to analyze shear flow between parallel plates. However, these conditions contain several unknown parameters related to the geometry of the boundary. Johnson and Jackson [1987] have improved upon these conditions by accounting for the mechanism by which fluctuation energy may be supplied to the flow.

In the spirit of the kinetic theories for granular flows, Jenkins and Richman [1986] have obtained boundary conditions on the stresses and energy flux at bumpy walls that interact with flows of smooth, nearly elastic particles. The wall's roughness was completely characterized in terms of its geometry and it was possible for the wall to supply fluctuation energy to the flow. However, the statistical averaging carried out in calculating the rates of momentum and energy transferred at the wall was based upon a simple Maxwellian. In the analysis of shear flow, therefore, they used a constitutive theory based upon the same distribution. This theory contained no contributions from particle transport to the energy flux or the shear stress. Moreover, the sole transport contribution to the normal stress was neglected.

In this paper we improve upon the work of Jenkins and Richman [1986] in several ways. We repair a minor defect in their averaging procedure by expanding the distribution function about a point near the wall that guarantees positive slip velocities. More importantly, in calculating the rates of momentum and energy transferred at the boundary, we employ a distribution function that contains corrections to the Maxwellian. It is then possible to use a more elaborate constitutive theory based upon this distribution function in our analysis of the shear flow between bumpy walls. This theory includes the effects of particle transport on the shear stress, normal stress, and energy flux. We view these as significant improvements because, when the solid fraction is 0.5 for example, the transport normal stress and energy flux are approximately one-third as great as their collisional counterparts, while the transport shear stress is about one-fourth as great. Furthermore, the lowest order terms of the pair distribution function do not contribute to either the collisional shear stress or collisional energy flux. Consequently, the additions to both the collisional shear stress and energy flux introduced by small corrections to the pair distribution based upon the improved Maxwellian are of the same order as the contributions from the pair distribution function based upon the Maxwellian alone. This is true even though the corrections to the Maxwellian are small compared to the Maxwellian itself.

In presenting the results of our shear flow analysis, we include the detailed profiles of mean velocity, granular temperature, and solid fraction, and we indicate how these profiles are influenced by variations in boundary geometry. In addition, we place special emphasis on the manner in which the shear and normal stresses vary with boundary characteristics and solid fraction, primarily because the stresses are the quantities most easily measured by the experimentalist. We also demonstrate the sensitivity of the slip velocity to boundary geometry. We do so because in many cases the variation in slip velocity is at least in part responsible for the corresponding variations in the stresses.

Balance Equations and Constitutive Theory

Here we present a theory for an idealized granular material consisting of identical, smooth, nearly elastic, circular disks that are constained to move in their common plane. This is a specialization of a theory developed by Jenkins and Richman [1985b] for slightly rough disks.

We adopt a statistical description of the disks' velocities and positions by introducing a single particle distribution function f. This function is defined such that $f(\boldsymbol{c}, \boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{r}$ gives the number of particles with velocity $\boldsymbol{c}$ within the range $\mathrm{d}\boldsymbol{c}$ whose centers are located at $\boldsymbol{r}$ within the area element $\mathrm{d}\boldsymbol{r}$ at time t. The number of the flow disks $n(\boldsymbol{r}, t)$ per unit area is then

$$n = \int f(\boldsymbol{c}, \boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{c}, \tag{1}$$

where integration is taken over all velocities. If each disk is of mass m and diameter σ, then the mean density ϱ of the flow is mn and the solid fraction ν is $n\pi\sigma^2/4$.

Any particle property $\phi(\boldsymbol{c})$ has a mean value denoted by $\langle\phi\rangle$ and defined by the weighted average

$$\langle\phi\rangle \equiv (1/n) \int \phi(\boldsymbol{c}) f(\boldsymbol{c}, \boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{c}. \tag{2}$$

The mean velocity $\boldsymbol{u}$ is simply $\langle \boldsymbol{c} \rangle$, and the fluctuation velocity $\boldsymbol{C}$ is the difference $\boldsymbol{c} - \boldsymbol{u}$. The granular temperature T is a mean measure of the kinetic energy associated with these fluctuations and is given by $\langle C^2 \rangle/2$.

The balance equations for the mean fields ϱ, $\boldsymbol{u}$, and T are:

$$\dot{\varrho} + \varrho \nabla \cdot \boldsymbol{u} = 0, \tag{3}$$

where an overdot indicates the time derivative with respect to the mean motion;

$$\varrho \dot{\boldsymbol{u}} = -\nabla \cdot \boldsymbol{P} + n\boldsymbol{F} \tag{4}$$

where $\boldsymbol{P}$ is the pressure tensor and $\boldsymbol{F}$ is the body force per disk; and

$$\varrho \dot{T} = -\nabla \cdot \boldsymbol{Q} - \mathrm{tr}(\boldsymbol{P} \cdot \boldsymbol{D}) - \gamma, \tag{5}$$

where $\boldsymbol{Q}$ is the flux of fluctuation energy, γ is the rate of dissipation per unit area, and $\boldsymbol{D} \equiv (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$.

The dissipation γ is due entirely to inelastic collisions, and is given in terms of the mean fields by

$$\gamma = 4(1 - e)\, \kappa T/\sigma^2, \tag{6}$$

where e is the coefficient of restitution between flow disks, and

$$\kappa \equiv 2\varrho\sigma G T^{1/2}/\pi^{1/2}, \tag{7}$$

in which $G(\nu) \equiv \nu(16 - 7\nu)/16(1 - \nu)^2$. The pressure tensor $\boldsymbol{P}$ and the energy flux $\boldsymbol{Q}$ measure, respectively, the flux of momentum and fluctuation energy within the flow. As such they are composed of contributions from particle collisions as well as from the transport of particles between collisions. Here we employ a constitutive theory that accounts for both of these effects. In this theory, the pressure tensor is given by

$$\boldsymbol{P} = (2\varrho GFT - \kappa \operatorname{tr} \boldsymbol{D})\boldsymbol{I} - \kappa J \hat{\boldsymbol{D}}, \tag{8}$$

where $F(\nu) \equiv 1 + 1/2G$, $J(\nu) \equiv 1 + \pi(1 + 1/G)^2/8$, and $\hat{\boldsymbol{D}}$ is deviatoric part of $\boldsymbol{D}$. The energy flux is

$$\boldsymbol{Q} = -\kappa M \nabla T, \tag{9}$$

where $M(\nu) = 1 + \pi(3/2 + 1/G)^2/4$. This constitutive theory applies only to circumstances in which $(1 - e)$ is of the same order as $\varepsilon \equiv \sigma/L$, where L is a characteristic length over which the mean fields vary. However, we emphasize that, unlike those theories that neglect the effects of particle transport, this theory is not restricted to dense flows. Finally, we point out that the constitutive theory admits all values of solid fraction that are less than one. This is because the expression for the frequency of collisions employed in deriving the theory was based upon the radial distribution function of Verlet and Levesque [1982], which diverges only when $\nu = 1$, but which gives reasonable results for all values of ν up to the value at which enduring contacts dominate particle interactions.

Boundary Conditions

Here we derive conditions that must be satisfied at an impenetrable boundary of a two-dimensional granular flow. We consider a unit length of this boundary that has a unit inward normal $\boldsymbol{N}$. The collisional rate of supply of momentum to the flow by this segment is $\boldsymbol{M}$, and the rate of energy absorbed from the flow due to inelastic collisions with this segment is D. We focus attention on a rectangle of unit length within the flow that has this segment of the boundary as one of its sides. As its width goes to zero, the balance of momentum within the rectangle requires that

$$\boldsymbol{M} = \boldsymbol{P} \cdot \boldsymbol{N}. \tag{10}$$

The balance of energy requires that

$$\boldsymbol{M} \cdot \boldsymbol{v} - D = \boldsymbol{Q} \cdot \boldsymbol{N}, \tag{11}$$

where the slip velocity $\boldsymbol{v}$ is the velocity $\boldsymbol{U}$ of the boundary minus the flow velocity $\boldsymbol{u}$ at the boundary. The slip-work term $\boldsymbol{M} \cdot \boldsymbol{v}$ is due to equal tractions on opposite sides of the rectangle acting through velocities that differ by an amount $\boldsymbol{v}$. This is the mechanism by which the boundaries may supply fluctuation energy to

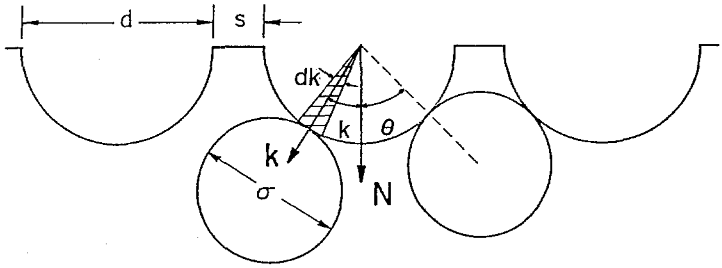

Figure 1
Surface geometry.

the flow, although it is the relative magnitudes of the slip-work and the dissipation rate D that ultimately determine the direction of the energy flux normal to the boundary. When these two exactly balance, the boundary neither supplies nor absorbs energy from the flow.

Expressions for $\boldsymbol{M}$ and D in terms of the mean fields depend upon the geometry and dissipative character of the boundary. Here we consider the bumpy surface shown in Figure 1. Halves of identical, smooth, nearly elastic, circular disks of diameter d are equally spaced a distance s apart. If we define $\Delta \equiv s/d$ and $r \equiv \sigma/d$, then the fraction of the periphery of each wall disk that is accessible to flow disks is $2\theta/\pi$, where $\sin\theta = (1 + \Delta)/(1 + r)$. We restrict our attention to spacings between wall disks that prevent the flow disks from colliding with the flat wall; therefore $0 < \Delta < -1 + (1 + 2r)^{1/2}$. In addition, we consider values of r of order unity only. The boundary becomes effectively rougher as either Δ is increased or r is decreased. Consequently $\sin\theta$ appears to be a natural measure of wall roughness. Also shown in Figure 1 is the unit vector $\boldsymbol{k}$ directed from the center of a wall disk to the center of a flow disk at impact and the element of angle $\mathrm{d}k$ centered about $\boldsymbol{k}$. At impact the distance $\bar{\sigma}$ between centers is $(\sigma + d)/2$. The coefficient of restitution between wall disks and flow disks is e_w, and we assume that $(1 - e_w)$ is of order ε.

We focus on a collision between a wall disk centered at $\boldsymbol{p}$ and a flow disk with velocity $\boldsymbol{c}$ in $\mathrm{d}\boldsymbol{c}$ such that the point of contact lies within $\mathrm{d}k$ centered at $\boldsymbol{k}$. The frequency of such collisions per unit length of flat wall is

$$(1/2)\,\chi \csc\theta f(\boldsymbol{c}, \boldsymbol{p} + \bar{\sigma}\boldsymbol{k})\,(\boldsymbol{g}\cdot\boldsymbol{k})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}, \tag{12}$$

in which $\boldsymbol{g}$ is the relative velocity $\boldsymbol{U} - \boldsymbol{c}$ and $\boldsymbol{g}\cdot\boldsymbol{k}$ must be positive for collisions to occur. The dimensionless factor $\chi(v)$ accounts for the influence of excluded area and the shielding of flow disks from wall disks by other flow disks on the collision frequency.

The rates $\boldsymbol{M}$ and D are statistical averages of the changes per collision in the appropriate flow disk property weighted by the frequency of each wall-flow disk collision. The change in momentum per collision experienced by a flow disk is $m(1 + e_w)\,(\boldsymbol{g}\cdot\boldsymbol{k})\,\boldsymbol{k}$. The expression for $\boldsymbol{M}$ is therefore

$$\boldsymbol{M} = (1/2)\,\chi m(1 + e_w)\,\csc\theta \int \boldsymbol{k} f(\boldsymbol{c}, \boldsymbol{p} + \bar{\sigma}\boldsymbol{k})\,(\boldsymbol{g}\cdot\boldsymbol{k})^2\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}, \tag{13}$$

where integration is carried out over all velocities $\boldsymbol{c}$ for which $\boldsymbol{g} \cdot \boldsymbol{k} > 0$ and over all angles k for which $-\theta < k < \theta$. The loss in energy per collision as seen from the moving wall is $m(1 - e_w^2)\,(\boldsymbol{g} \cdot \boldsymbol{k})^2/2$, so that the expression for D is

$$D = (1/4)\,\chi m(1 - e_w^2)\,csc\theta \int f(\boldsymbol{c}, \boldsymbol{p} + \bar{\sigma}\boldsymbol{k})\,(\boldsymbol{g} \cdot \boldsymbol{k})^3\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}, \tag{14}$$

where the integration is again carried out over $(\boldsymbol{g} \cdot \boldsymbol{k}) > 0$ and $-\theta < k < \theta$.

In the integral expressions for $\boldsymbol{M}$ and D, we use the single particle distribution function obtained by Jenkins and Richman [1985 b] in their derivation of the consitutive relations (6)–(9). They employed the method of moments, in which improvements to the Maxwellian distribution were obtained by requiring that the balance equations for certain second and third moments of velocity be satisfied to lowest order. Their scheme of approximation is self-consistent if we assume that the dimensionless velocity gradient $\sigma \nabla \boldsymbol{u}/T^{1/2}$ is of order $\varepsilon^{1/2}$ while the dimensionless gradients $\sigma \nabla \nu$ and $\sigma \nabla T/T$ are of order ε. The expression for the single particle distribution function that contains the lowest order correction to the Maxwellian is then

$$f(\boldsymbol{c}, \boldsymbol{r}) = (n/2\pi T)\,[1 - (2^{1/2}\sigma B/\pi^{1/2} T^{3/2})\,\boldsymbol{C} \cdot \hat{\boldsymbol{D}} \cdot \boldsymbol{C}]\,\exp(-C^2/2T), \tag{15}$$

in which $B(\nu) \equiv \pi(1 + 1/G)/8\sqrt{2}$ and where all mean fields are evaluated at $\boldsymbol{r}$. Because the corrections to the Maxwellian that involve the gradients of solid fraction and granular temperature have been neglected, expression (15) contains an error of order ε.

Here we expand the distribution function $f(\boldsymbol{c}, \boldsymbol{p} + \bar{\sigma}\boldsymbol{k})$ about the point $\boldsymbol{r} \equiv \boldsymbol{p} + \bar{\sigma}\boldsymbol{N}$. In this manner, the boundary is located at a distance $\bar{\sigma}\boldsymbol{N}$ from the flat wall. For the integrations required by Eqs. (13) and (14), it is convenient to write the result in terms of the relative velocity $\boldsymbol{g} = \boldsymbol{U} - \boldsymbol{c}$ and the slip velocity $\boldsymbol{v} = \boldsymbol{U} - \boldsymbol{u}(\boldsymbol{r})$. If we assume that $\boldsymbol{v}/T^{1/2}$ is of order $\varepsilon^{1/2}$, then to within an error of order ε,

$$\begin{aligned} f(\boldsymbol{c}, \boldsymbol{p} + \bar{\sigma}\boldsymbol{k}) = (n/2\pi T)\,\{1 &+ (\boldsymbol{g} \cdot \boldsymbol{v}/T) - (\bar{\sigma}/T)\,[(\boldsymbol{k} \cdot \nabla - \boldsymbol{N} \cdot \nabla)\,\boldsymbol{u}] \cdot \boldsymbol{g} \\ &- (2^{1/2}\sigma B/\pi^{1/2} T^{3/2})\,\boldsymbol{g} \cdot \hat{\boldsymbol{D}} \cdot \boldsymbol{g}\}\,\exp(-g^2/2T), \end{aligned} \tag{16}$$

in which all mean fields are evaluated at $\boldsymbol{r}$. When this approximation is used in Eqs. (14) and (13), the expressions for D and $\boldsymbol{M}$ in terms of the mean fields follow directly from integration. We find that D is given by

$$D = (2/\pi)^{1/2} \varrho \chi\,(1 - e_w)\,T^{3/2}\,\theta\,csc\theta, \tag{17}$$

up to an error of order $\varepsilon^{3/2}$, and that the Cartesian components of $\boldsymbol{M}$ are given by

$$\begin{aligned} M_\alpha = \varrho \chi T\{N_\alpha &+ (2/\pi T)^{1/2}\,v_\alpha(\theta\,csc\theta - \cos\theta) \\ &+ (2/\pi T)^{1/2}\,\bar{\sigma} u_{\gamma,\beta}[(1 + \sigma B/\bar{\sigma})\,I_{\alpha\beta\gamma} + N_\beta I_{\alpha\gamma}]\}, \end{aligned} \tag{18}$$

up to an error of order ε. The tensors $I_{\alpha\gamma}$ and $I_{\alpha\beta\gamma}$ depend on the boundary geometry and are defined as

$$I_{\alpha\gamma} \equiv (\theta \csc\theta + \cos\theta)\, N_\alpha N_\gamma + (\theta \csc\theta - \cos\theta)\, \tau_\alpha \tau_\gamma, \tag{19}$$

where $\boldsymbol{\tau}$ is the unit vector normal to $\boldsymbol{N}$, and

$$I_{\alpha\beta\gamma} \equiv [(2/3)\sin^2\theta - 2]\, N_\alpha N_\beta N_\gamma - (2/3)\sin^2\theta (N_\alpha \tau_\beta \tau_\gamma + N_\beta \tau_\alpha \tau_\gamma + N_\gamma \tau_\alpha \tau_\beta). \tag{20}$$

Although the expression for D obtained here is identical to that obtained by Jenkins and Richman [1986], the expression for $\boldsymbol{M}$ differs in two ways. We have based all statistical averaging upon a more elaborate velocity distribution function that was obtained through the method of moments. We have done so in order to derive boundary conditions that are consistent with the constitutive theory given in Eqs. (6)–(9). The simple Maxwellian distribution used by Jenkins and Richman may be recovered by setting $B = 0$ in Eq. (15). In addition, by expanding the distribution function in Eqs. (13) and (14) about the center $\boldsymbol{p}$ of a wall disk, Jenkins and Richman artificially extended the possible locations of the center of a flow disk to the flat part of the wall. This led generally to an underestimation of the slip velocities and, in particular, allowed the possibility of negative slip velocities. Here we have repaired this defect by expanding the distribution function about the location $\boldsymbol{p} + \bar{\sigma}\boldsymbol{N}$ of the center of a flow disk when it collides at an angle $k = 0$. This modification introduced the term involving $\boldsymbol{N}$ on the right hand side of Eq. (16) and, after integration, the last term on the right hand side of Eq. (18).

Shear Flow

We consider steady rectilinear flow driven by the relative motion of two identical parallel walls to which half-disks have been attached. In an $x - y$ Cartesian coordinate system the boundaries are located at $x = \pm L/2$. The upper boundary moves in the x-direction with a constant speed U while the lower boundary moves with the same speed in the opposite direction. The velocity u in the x-direction, the granular temperature T, and the solid fraction ν are functions of y only.

In this case, the balance of mass (3) is satisfied identically. In the absence of gravity, the x- and y-momentum balances (4) may be integrated to show that both P_{xy} and P_{yy} are constants; we call these constants $-S$ and N, respectively. From Eq. (8) we have

$$S = \kappa J u'/2, \tag{21}$$

where a prime denotes differentiation with respect to y, and

$$N = 2\varrho G F T. \tag{22}$$

By combining the definition (7) of κ with Eq. (22) we find that

$$\kappa = N\sigma/(\pi T)^{1/2} F, \tag{23}$$

and by using this in Eq. (21) we obtain

$$u' = 2(\pi T)^{1/2} FS/N\sigma J. \tag{24}$$

Differentiation of (22) with respect to y yields

$$v' = -(T'/T)\,[d\ln(vGF)/dv]^{-1}, \tag{25}$$

which is a relation between the spatial derivative of the solid fraction and the spatial derivative of the granular temperature.

The energy Eq. (5) simplifies to

$$Q_y' - Su' + \gamma = 0. \tag{26}$$

For Q_y and γ we use the constitutive relations (9) and (6), respectively; both of these involve κ. Then u' and κ are eliminated by employing Eqs. (24) and (23). The resulting equation may be written in terms of the derivatives of T only by using Eq. (25) to eliminate v'. When cast in terms of $w \equiv T^{1/2}$, the energy equation becomes

$$(w''/w) = (\lambda/L)^2 - (w'/w)^2 H, \tag{27}$$

where

$$\lambda^2 \equiv (L/\sigma)^2[2(1-e) - \pi F^2 S^2/N^2 J]/M, \tag{28}$$

and

$$H \equiv 2[d\ln(F/M)/dv]\,[d\ln(vGF)/dv]^{-1}. \tag{29}$$

When the transport contributions to the normal stress are neglected, F is equal to one. When the transport contribution to the energy flux and its effect on the collision frequency are neglected, M is also equal to one. Under these circumstances, H vanishes and (27) is a linear equation for $w(y)$ that is uncoupled from $v(y)$. The variation of H with v is shown in Figure 2. As v decreases, the transport contributions are responsible for increasingly larger fractions of each constitutive quantity; consequently, H increases.

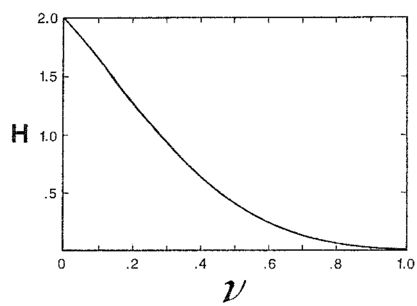

Figure 2
$H(v)$ versus v.

The y-component of Eq. (10) is the normal stress condition at the upper wall. With expression (18) for M_y and the fact that both I_{yyx} and I_{yx} vanish, we have

$$\varrho \chi T = N, \tag{30}$$

in which all quantities are evaluated at $y = L/2$. In this, we employ (22) to obtain

$$\chi = 1 + 2G, \tag{31}$$

in which each function of v is evaluated at the solid fraction v_w at the wall. Equation (31) determines the function $\chi(v)$ in such a way that the normal stress boundary condition does not restrict the solid fraction at the wall. Consequently, v_w may be treated as a parameter upon which the solutions depend. This treatment of the normal stress boundary condition is motivated by shear cell experiments such as those conducted by Hanes and Inman [1985], in which steady state stresses were obtained over a range of solid fraction. By contrast, Jenkins and Richman [1986] assumed that χ was a known function of v and used the normal stress condition to fix the solid fraction at the wall.

The x-component of Eq. (10) is the shear stress boundary condition. In expression (18) for M_x we use $I_{xyx} = (2/3)\sin^2\theta$, $I_{xx} = (\theta \csc\theta - \cos\theta)$, and $\varrho\chi T = N$. After u' has been eliminated through Eq. (24), the resulting boundary condition may be written in terms of the slip velocity, $\upsilon = U - u(L/2)$, as

$$\upsilon/w = (\pi/2)^{1/2}(S/N)\, Z, \tag{32}$$

where Z depends upon the geometry of the boundary and is given by

$$Z \equiv \frac{1 - (4\sqrt{2}\,F\bar{\sigma}/3J\sigma)\,(1 + \sigma B/\bar{\sigma})\sin^2\theta}{(\theta\csc\theta - \cos\theta)} + \frac{2\sqrt{2}\,F\bar{\sigma}}{J\sigma}. \tag{33}$$

Here and in what follows, all functions of v that appear in Z are evaluated at v_w. We note that Z, and therefore the slip velocity υ, is positive for all solid fractions and boundary geometries under consideration.

The energy flux boundary condition is given by Eq. (11), in which we recognize that $\varrho\chi T = N$, $M_x = S$ and $Q_y = -2\kappa w w'$. We first write the result in terms of S/N by using Eq. (23) to eliminate κ and Eq. (32) to eliminate υ/w. Then, from Eq. (28), we write S/N in terms of λ. The resulting boundary condition is

$$\begin{aligned} \sigma w'/w = {} & (F/2^{1/2}M)\,[J(1-e)\,Z/F^2 - (1-e_w)\,\theta\csc\theta] \\ & - (\sigma^2 J Z \lambda^2/2^{3/2} L^2 F), \end{aligned} \tag{34}$$

in which all functions of v are evaluated at v_w.

Equations (22), (24), and (27) determine $v(y)$, $u(y)$, and $w(y)$ to within three constants of integration. These constants, as well as S and N are determined by conditions (30), (32), and (34) at $y = L/2$ and by the requirements that both u and w' vanish at $y = 0$. In order to simplify the solution procedure, we define the average solid fraction $\bar{v}$ by

$$\bar{v} \equiv (2/L) \int_0^{L/2} v(y)\, dy, \tag{35}$$

and replace v in Eqs. (24) and (27) by $\bar{v}$. In these equations, then, we interpret the functions F, J, M, and H as those evaluated at $\bar{v}$. In terms of the variable $q \equiv w'/w$, the energy equation becomes

$$q' = (\lambda/L)^2 - (1 + H)\, q^2, \tag{36}$$

which, subjected to the condition $q(0) = 0$, yields

$$q = (\lambda/L\sqrt{1 + H})\, \tanh(\lambda\sqrt{1 + H}\, y/L). \tag{37}$$

Solving for $w(y)$, we find

$$w = A\, \cosh^{\frac{1}{1+H}}(\lambda\sqrt{1 + H}\, y/L), \tag{38}$$

where A is an as yet undetermined constant. Integration of Eq. (24) yields the velocity profile

$$u = (2\pi^{1/2} F S/N\sigma J)\, A \int_0^y \cosh^{\frac{1}{1+H}}(\lambda\sqrt{1 + H}\,\xi/L)\, d\xi, \tag{39}$$

in which we have ensured that $u(0) = 0$.

The three remaining unknown constants are S, N, and A, or equivalently, λ, N, and A. By employing both Eqs. (38) and (39) evaluated at $y = L/2$ in boundary condition (32) we obtain the relation between λ and A:

$$U/A = (\pi/2)^{1/2}(S/N)\,[Z \cosh^{\frac{1}{1+H}}(\lambda\sqrt{1 + H}/2) + (2^{3/2} F/\sigma J) \int_0^{L/2} \cosh^{\frac{1}{1+H}}(\lambda\sqrt{1 + H}\, y/L)\, dy], \tag{40}$$

where S/N is given in terms of λ by Eq. (28). Here, all functions of v are evaluated at $\bar{v}$.

Finally, we combine Eq. (37) at $y = L/2$ with the energy flux boundary condition (34) to obtain the transcendental equation that determines λ:

$$(\lambda\sqrt{1 + H}/2)\, \tanh(\lambda\sqrt{1 + H}/2) + (\sigma J Z/2^{1/2} L F)\, (\lambda\sqrt{1 + H}/2)^2 = (F L/2^{3/2} M \sigma)(1 + H)\, [J(1 - e)\, Z/F^2 - (1 - e_w)\, \theta \csc\theta]. \tag{41}$$

In this, all functions of v except $H = H(\bar{v})$ are evaluated at v_w. When the quantity in the square brackets is positive, λ is real and $w(y)$ and $u(y)$ are expressible in terms of hyperbolic functions. The temperature increases from the center line to walls, which in this instance supply fluctuation energy to the flow. According to Eq. (22), the solid fraction therefore decreases from the centerline to the walls.

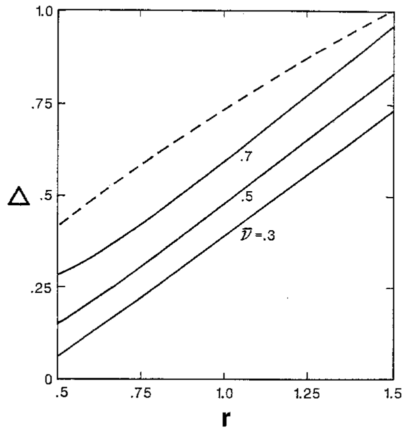

Figure 3
Curves that correspond to the critical case $\lambda = 0$ for $\bar{\nu} = 0.3$, 0.5, and 0.7 and $(1 - e)/(1 - e_w) = 0.5$. The dashed curve gives the maximum value of $\Delta = -1 + (1 + 2r)^{1/2}$.

When the quantity in the square brackets is negative, λ is imaginary and the solutions are trigonometric. Here the temperature decreases while the solid fraction increases from the centerline to the walls, which now absorb fluctuation energy from the flow. The critical case occurs when the quantity in the square brackets is zero. This corresponds to the condition $w'(L/2) = 0$ in which the walls neither supply nor absorb fluctuation energy. In this case, $\lambda = 0$ so that the temperature, and therefore the solid fraction, are constant across the gap while the velocity varies linearly. In Figure 3 we have plotted the curves in $r - \Delta$ space that give rise to this critical case. We have shown these curves for three values of $\bar{\nu} = 0.3$, 0.5, and 0.7, while we have fixed the ratio $(1 - e)/(1 - e_w) = 0.5$. The area below each curve corresponds to boundaries that supply fluctuation energy, and the area above corresponds to those that absorb energy. The curves also depend upon the ratio $(1 - e)/(1 - e_w)$. For a fixed value of e, an increase in $(1 - e)/(1 - e_w)$ corresponds to a decrease in the inelasticity of wall-flow disk collisions. This, in turn, results in fewer circumstances under which the boundaries can absorb energy and a corresponding upward shift of these curves. We have found, for example, that when $(1 - e)/(1 - e_w) = 1$ all boundaries supply fluctuation energy to the flow provided that $\bar{\nu}$ is greater than 0.33.

The general solution procedure is as follows. If we pick a value of $\bar{\nu}$, and guess at a value of ν_w, then Eq. (41) determines λ. The constant A is then determined by Eq. (40) and $w(y)$ is fixed through Eq. (38). The normal stress N is then set by Eq. (22) evaluated at $y = L/2$. In turn, Eq. (22) may be inverted at all other values of y to obtain $\nu(y)$. As a check on the initial guess for ν_w, we calculate $\bar{\nu}$ according to Eq. (35) and compare this to the value of $\bar{\nu}$ that was chosen originally. After a suitable number of iterations on ν_w to ensure good agreement of $\bar{\nu}$, the velocity profile $u(y)$ may be obtained from Eq. (39). Finally, we solve Eq. (28) for the shear stress S.

Using this procedure, we have examined the effects of various boundaries and average solid fractions on the profiles within the flow. Here and in what

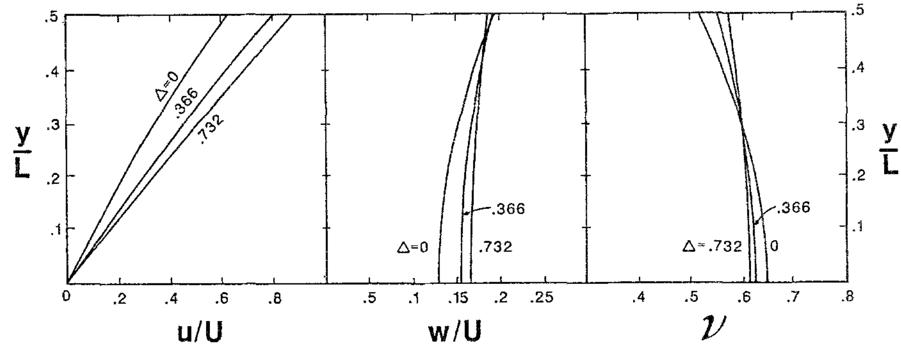

Figure 4
Profiles of u/U, w/U, and ν for $\Delta = 0$, 0.366, and 0.732. Here $r = 1$, $e = 0.8$, $e_w = 0.9$, $\bar{\nu} = 0.6$, and $\sigma/L = 1/11$.

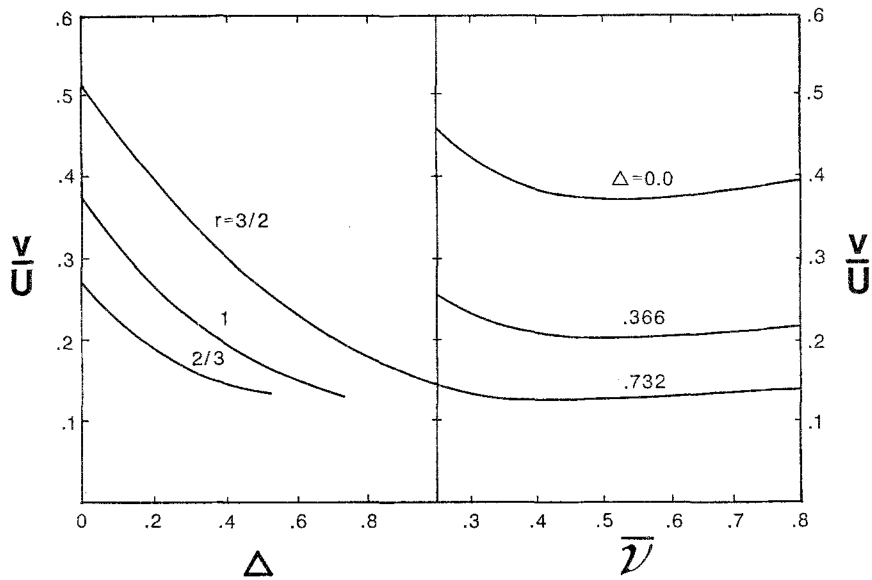

Figure 5
Dependence of v/U: on Δ for $r = 2/3$, 1, and 3/2 when $\bar{\nu} = 0.6$; and on $\bar{\nu}$ for $\Delta = 0$, 0.366, and 0.732 when $r = 1$. In both $e = 0.8$, $e_w = 0.9$, and $\sigma/L = 1/11$.

follows, we set the parameters $r = 1$, $\Delta = 0$, $\bar{\nu} = 0.6$, $e_w = 0.9$, $e = 0.8$, and $\sigma/L = 1/11$, whenever they are not otherwise specified. The variations of u/U, w/U, and ν with y/L are shown in Figure 4 for $\Delta = 0$, 0.366, and 0.732. The dependence of the dimensionless slip velocity v/U on Δ for $r = 2/3$, 1, 3/2 is shown in Figure 5. As r decreases or Δ increases the boundaries become rougher, v/U decreases, and the boundaries supply less fluctuation energy to the flow. Also shown in Figure 5 is the dependence of v/U on $\bar{\nu}$ for $\Delta = 0$, 0.366, and 0.732.

We have examined the sensitivity of the dimensionless normal stress $N^* \equiv \pi N/4m(2U/L)^2$ and dimensionless shear stress $S^* \equiv \pi S/4m(2U/L)^2$ to changes

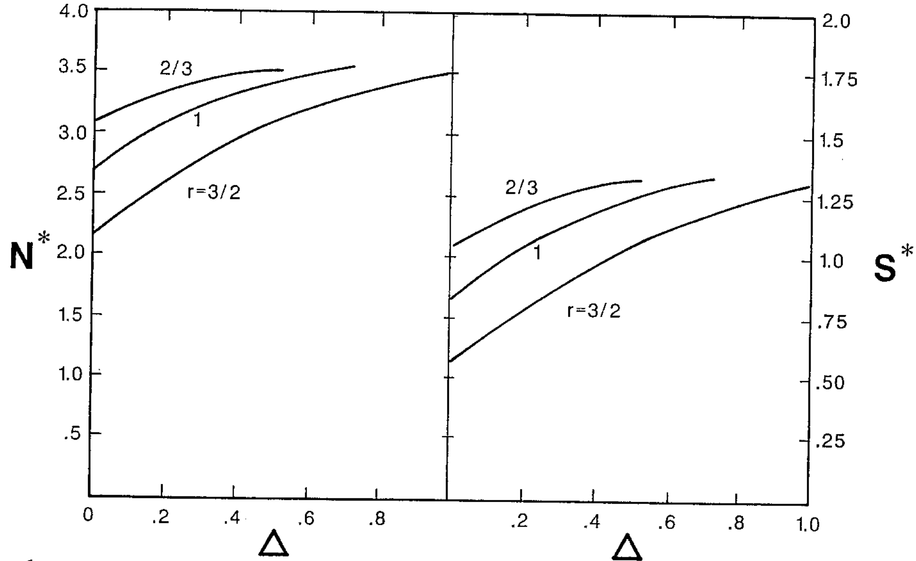

Figure 6
Variations of N^* and S^* with Δ for $r = 2/3$, 1, and 3/2. Here $e = 0.8$, $e_w = 0.9$, $\bar{\nu} = 0.6$, and $\sigma/L = 1/11$.

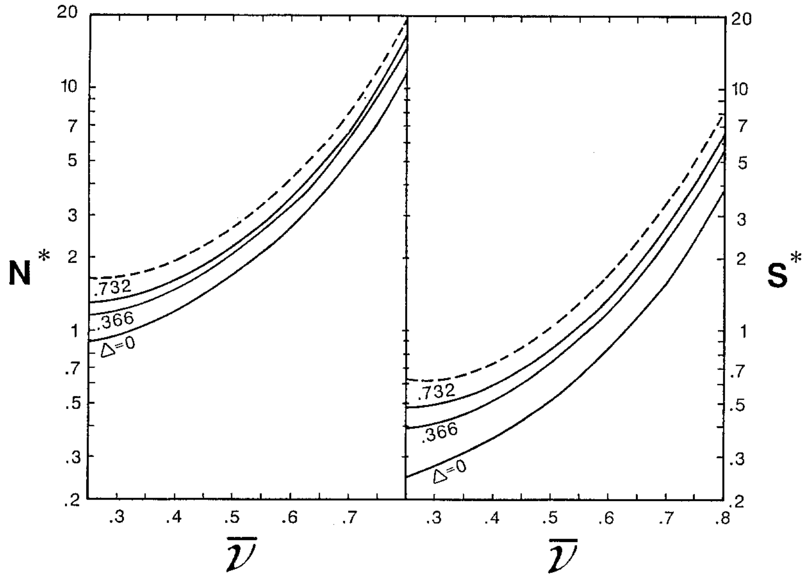

Figure 7
Variations of N^* and S^* with $\bar{\nu}$ for $\Delta = 0$, 0.366, and 0.732. Here $r = 1$, $e = 0.8$, $e_w = 0.9$, and $\sigma/L = 1/11$. The dashed curves correspond to homogeneous shearing with zero slip.

in boundary properties. The variations of N^* and S^* with Δ for $r = 2/3$, 1, and 3/2 are shown in Figure 6. The stresses generally increase as the boundaries becomes rougher. Finally, we plot N^* and S^* versus $\bar{\nu}$ for $\Delta = 0$, 0.366, and 0.732 in Figure 7. Also shown are dashed curves corresponding to a homogeneous shear flow in which there is no slip at the boundary.

Discussion

Based upon a velocity distribution function that contains corrections to the Maxwellian, we have obtained boundary conditions for two-dimensional flows of nearly elastic disks that interact with bumpy walls. Using a kinetic constitutive theory based upon the same distribution function, we have analyzed an idealized granular shear flow driven by boundaries at which these conditions apply. Here, in contrast to the simpler treatment of Jenkins and Richman [1986], it was impossible to eliminate the solid fraction ν from the energy equation. Consequently, both the temperature and velocity profiles depend on ν. We have incorporated this feature of the solutions in a simple manner, by replacing ν in the energy equation with the average solid fraction $\bar{\nu}$ across the gap.

The curves shown in Figure 3, which separate those boundaries that supply fluctuation energy to the flow from those that absorb energy, also depend on $\bar{\nu}$. To understand this dependence we note that these curves correspond to instances of homogeneous shearing; where the rate of energy dissipated due to boundary-flow interactions, which is proportional to $N \cdot w$, exactly balances the slip work $S \cdot v$. From Figure 7 and the second of Figure 5 we find that both S/N and v are relatively insensitive to changes in $\bar{\nu}$. However, as $\bar{\nu}$ increases, w must decrease in order to maintain an energy balance within the flow. The corresponding increase in the slip work is therefore greater than the increase in the dissipation rate at the boundary. Consequently, the boundaries supply fluctuation energy to the flow more readily at higher solid fractions.

The first panel of Figure 5 demonstrates that the slip velocity is very sensitive to boundary geometry. Decreases in v/U indicate increases in the mean shear rates within the flow, which in turn result in higher temperatures. These effects give rise to increases in both normal and shear stresses, as shown in Figure 6. When $r = 3/2$, for example, as Δ varies between zero to one, v/U varies between 0.52 and 0.14. The corresponding dimensionless normal stress N^* nearly doubles from 2.15 to 3.51, while the dimensionless shear stress S^* more than doubles from 0.58 to 1.31.

The variations of the stresses with average solid fraction $\bar{\nu}$ are shown in Figure 7. Because the slip velocity is relatively insensitive to changes in $\bar{\nu}$ between 0.25 to 0.8, these curves all have shapes similar to their dashed curve counterparts, which are based upon a constant shear rate $2U/L$ regardless of $\bar{\nu}$. The dashed curves predict values for the stresses that are too high because they are based upon an overestimation of both the mean shear rate and the resulting temperature necessary to achieve an energy balance within the flow. When $\Delta = 0$, for example, the normal stress is typically only about sixty percent of the value predicted by the dashed curve, and the shear stress is only about half of its dashed curve value.

The induced stresses also depend on the dissipative character of the boundary through the coefficient of restitution e_w. Because of the restriction on the boundary conditions to nearly elastic wall-flow disk collisions, we have examined this effect by varying e_w between 0.8 and 1.0 only. In this range we have found that as the wall-flow collisions become more elastic the normal stress typically increases by ten to twenty percent while the shear stress increases by five to ten percent.

In this work we have considered only systems of smooth particles. The effect of particle roughness on the shear flow would be to lower the granular temperature, and therefore the stresses, necessary to maintain an energy balance within the flow. However, if the energy dissipated due to normal impact is of the same order as the energy dissipated due to particle roughness, then the effects due to improvements to the Maxwellian on the stresses will be of the same order as those due to particle rotation. Consequently, the improvements to the Maxwellian should not be ignored in an analysis of rough particles.

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References

C. S. Campbell and C. E. Brennen, Computer simulation of shear flows of granular material. *Proc. U.S.-Japan Seminar on New Models and Constitutive Relations in the Mechanics of Granular Materials*, (eds. J. T. Jenkins and M. Satake), Ithaca, New York, pp. 313–326 (1982).

C. S. Campbell and C. E. Brennen, Computer simulation of granular shear flows, *J. Fluid Mech.*, *151*, 167–188 (1985).

P. K. Haff, Grain flow as a fluid-mechanical phenomenon, *J. Fluid Mech. 134*, 401–430 (1983).

D. M. Hanes und D. L. Inman, Observations of rapidly flowing granular-fluid materials, *J. Fluid Mech.*, *150*, 357–380 (1985).

K. Hui, P. K. Haff, J. E. Ungar and R. Jackson, Boundary conditions for high-shear grain flows, *J. Fluid Mech.*, *145*, 223–233 (1984).

J. T. Jenkins and S. B. Savage, A theory for the rapid flow of identical smooth, nearly elastic particles, *J. Fluid Mech.*, *130*, 187–202 (1983).

J. T. Jenkins and M. W. Richman, Grad's 13-moment system for a dense gas of inelastic spheres, *Arch. Rat. Mech. Anal.*, *87*, 355–377 (1985 a).

J. T. Jenkins and M. W. Richman, Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks, *Phys. Fluids*, *28*, 3485–3494 (1985 b).

J. T. Jenkins and M. W. Richman, Boundary conditions for plane flows of smooth, nearly elastic, circular disks, *J. Fluid Mech.*, *171*, 53–69 (1986).

J. T. Jenkins and F. Mancini, Balance laws and constitutive relations for plane flows of a dense, binary mixture of smooth, nearly elastic circular disks, *J. Appl. Mech.*, *54*, 27–34 (1987).

P. C. Johnson and R. Jackson, Frictional-collisional constitutive relations for granular material, with application to plane shearing, *J. Fluid Mech.*, *176*, 67–94 (1987).

C. K. K. Lun, S. B. Savage, D. J. Jeffrey and N. Chepurniy, Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flowfield, *J. Fluid. Mech.*, *140*, 223–256 (1984).

R. K. Raymond and H. H. Shen, Effects of frictional and anisotropic collision on the constitutive relations for a simple shear flow of spheres, *Int. J. Engng. Sci.*, *24*, 1015–1029 (1986).

S. B. Savage and M. Sayed, Stresses developed by dry cohesionless granular materials sheared in an annular shear cell, *J. Fluid Mech.*, *142*, 391–430 (1984).
S. B. Savage and S. Mckeown, Shear stresses developed during rapid shear of concentrated suspensions of large spherical particles between concentric cylinders, *J. Fluid Mech.*, *127*, 453–472 (1983).
H. Shen and N. L. Ackermann, Constitutive equations for a simple shear flow of a disk shaped granular mixture, *Int. J. Engng. Sci.*, *7*, 829–843 (1984).
H. Shen and N. L. Ackermann, Energy diffusion in a granular flow of disk shaped solids, *Int. J. Engng. Sci.*, *24*, 551–556 (1986).
L. Verlet and D. Levesque, Integral equations for classical fluids III. The hard discs system, *Mol. Phys.*, *46*, 969–980 (1982).
O. R. Walton and R. L. Braun, Viscosity, granular-temperature and stress calculations for shearing assemblies of inelastic, frictional disks, *J. Rheol.*, *30*, 949–980 (1986a).
O. R. Walton and R. L. Braun, Stress calculations for assemblies of inelastic spheres in uniform shear, *Acta Mech.*, *63*, 73–86 (1986b).

Summary

We obtain boundary conditions for two-dimensional flows of identical, nearly elastic, circular disks that interact with a flat wall to which identical, evenly spaced half-disks have been attached. Expressions for the transfer of momentum and energy from the boundary to the flow are obtained by statistical averaging over all possible wall-flow disk collisions. We improve upon the expressions derived by Jenkins and Richman [1986] by employing in the averaging process a more elaborate velocity distribution function obtained through the method of moments. In addition we expand the distribution function about a point near the flat wall that guarantees positive slip velocities. With these boundary conditions, we analyze a two-dimensional shear flow driven by parallel bumpy boundaries. The constitutive theory employed includes both the effects of particle collisions and particle transport on the transfer of momentum and energy throughout the flow. We demonstrate how the resulting profiles of velocity, granular temperature, and solid fraction are affected by changes in the geometry of the boundary. We also predict how the induced stresses vary with the geometry of the boundary and the average solid fraction within the flow.

Zusammenfassung

Wir erhalten die Randbedingungen für die zweidimensionale Strömung identischer, beinahe elastischer, runder Scheiben, die sich in Wechselwirkung mit einer geraden Wand befinden, an der in gleichmäßigen Abständen Halbscheiben angebracht sind. Es werden Ausdrücke für die Übertragung von Impuls und Energie vom Rand auf den Strom aufgestellt, die durch den statisch errechneten Durchschnitt aller möglichen Scheibenkollisionen Wand-Strom erhalten werden. Wir verbessern die von Jenkins und Richman (1986) entwickelten Ausdrücke dadurch, daß bei der Berechnung der Mittelwerte eine erweiterte Geschwindigkeitsverteilung, die auf der Momentmethode beruht, einbezogen wurde. Außerdem entwickeln wir die Verteilungsfunktion an einem Punkt so nahe an der Wand, daß positive Gleitgeschwindigkeiten garantiert sind. Wir untersuchen eine zweidimensionale Scherströmung mit diesen Randbedingungen, die durch die parallelen unebenen Ränder getrieben wird. Die konstitutive Theorie, die wir anwenden, beinhaltet sowohl den Einfluß der Teilchenkollisionen als auch den des Teilchentransports auf die Übertragung von Impuls und Energie innerhalb der Strömung. Wir zeigen, wie die Profile der Geschwindigkeit, der Granulartemperatur und des Festkörperanteils, die sich ergeben, durch Veränderungen der Randgeometrie beeinflußt werden. Weiterhin können wir voraussagen, wie die erzeugten Spannungen sich mit der Randgeometrie und dem im Strom enthaltenen Festkörperanteil verändern.

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