Journal of Applied Mathematics and Physics (ZAMP)
Vol. 40, March 1989

0044-2275/89/020285-05 \$ 1.50 + 0.20/0

A constraint on wave, radial and shock velocities in non-linear electromagnetic materials

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1. Introduction

Several classes of evolution phenomena are appropriately described by hyperbolic systems. When these systems are non linear, one may have solutions that are not continuous at some times. The replacement of the pertinent differential equations by suitable conditions on the discontinuity of the field is possible when the former are written in the conservative form

$$\partial_\alpha f^\alpha(u) = f(u), \qquad f^0 = u, \tag{1}$$

where $f^\alpha(\alpha = 0, 1, 2, 3)$ and f are N dimensional column vectors.

Here the meaning of hyperbolicity is that the eigenvalues $\lambda(u, \mathbf{n})$ of the matrix $A_n = A^i n_i = n_i\, \partial f^i/\partial u$, which represent the velocities of the wave surfaces in the direction of the normal unit vector $\mathbf{n}$, are real and that the matrix has N linearly independent eigenvectors. One should note that if $N = \pm 2, \pm 3, \pm 4$ (mod.8) [1], it is not possible for the eigenvalues to be all simple for all directions $\mathbf{n}$, so that the construction of a basis of eigenvectors may not be immediate.

In Physics there is usually a supplementary conservation law that concerns a convex energy (or entropy) density h^0

$$\partial_\alpha h^\alpha(u) = g(u) \tag{2}$$

which may be obtained by multiplying (1) by the principal field

$$u'(u) = \partial h^0/\partial u$$

and which allows one to write the system in the symmetrical form of Friedrichs-Godunov. In this manner hyperbolicity is guaranteed.

According to the theory of relativity, the speed of light may be exceeded neither by the wave velocities nor by the radial velocities

$$\boldsymbol{\Lambda} := \lambda \mathbf{n} + \frac{\partial \lambda}{\partial \mathbf{n}} - \left(\mathbf{n} \cdot \frac{\partial \lambda}{\partial \mathbf{n}}\right)\mathbf{n}$$

which characterize the transport of wave perturbations, and which usually are larger

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than λ, since

$$\mathbf{\Lambda} \cdot \mathbf{n} = \lambda.$$

The same constraint applies to the shock velocity s in the Rankine-Hugoniot equations

$$n_i\{f^i(u_1) - f^i(u_0)\} = s(u_1 - u_0) \quad i = 1, 2, 3$$

which link the values u_1 and u_0 of the fields ahead and behind the shock.

Let us assume now that for each four-vector ξ^α that is time like (i.e. $\xi^\alpha \xi_\alpha > 0$) and for each $\delta u' \neq 0$ one has

$$Q := \xi_\alpha\, \delta u' \cdot \delta f^\alpha > 0. \tag{3}$$

For $\xi_0 = 1$ and $\xi_i = 0$, inequality (3) implies the positive definiteness of the hessian matrix $H = \partial u'/\partial u$; hence the convexity of the function $h^0(u)$ and the hyperbolicity of the system are guaranteed. Moreover, one can prove [2] that (3) is a necessary and sufficient condition in order that the radial and shock velocities do not exceed the speed of light.

Condition (3) on the positive definiteness of Q has been studied previously in the framework of Born-Infeld nonlinear electrodynamics [3].

Here we study electromagnetic materials, in which electric and magnetic permeabilities are field dependent. Applying the theorem quoted above, we obtain that under suitable conditions on the two permeabilities the speed of light is indeed an upper bound for the wave and radial velocities—whose explicit expressions are quite cumbersome[4]—and for the shock velocities.

2. Non linear electricity and magnetism

We consider the equations of the electric and magnetic fields

$$\mathbf{D}_t - \operatorname{curl} \mathbf{H} = \mathbf{0} \qquad \mathbf{B}_t + \operatorname{curl} \mathbf{E} = \mathbf{0} \tag{4}$$

with the constitutive equations

$$\mathbf{D} = \varepsilon(E^2)\mathbf{E} \qquad \mathbf{B} = \mu(H^2)\mathbf{H}$$

which characterize an isotropic material that may exhibit ferromagnetic as well as ferroelectric properties (see for ex. [5]).

It is easy to show that we can write system (4) in the conservative form (1) provided we set

$$u = f^0 \equiv \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \qquad f^i \equiv \begin{pmatrix} \mathbf{H} \wedge \mathbf{e}_i \\ \mathbf{e}_i \wedge \mathbf{E} \end{pmatrix} \qquad f \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

where $(\mathbf{e}_i)$ is an orthonormal basis of R^3.

Multiplying (4) by the principal field

$$u' \equiv (\mathbf{E}, \mathbf{H}) \tag{6}$$

one can show that the system (4) admits the well known additional conservation law (2) with

$$h^0 = \tfrac{1}{2}\{F(E^2) + G(H^2)\} \qquad h^i = (\mathbf{E} \wedge \mathbf{H}) \cdot \mathbf{e}_i,$$

$$F(E^2) = \int (\varepsilon + 2\varepsilon' E^2)\, dE^2 \qquad \varepsilon' = \partial\varepsilon/\partial E^2,$$

$$G(H^2) = \int (\mu + 2\mu' H^2)\, dH^2 \qquad \mu' = \partial\mu/\partial H^2.$$

Taking into account (5) and (6), we can write condition (3) in the form

$$Q = \xi_0\{\varepsilon\, \delta E^2 + 2\varepsilon'(\mathbf{E} \cdot \delta\mathbf{E})^2 + \mu\, \delta H^2 + 2\mu'(\mathbf{H} \cdot \delta\mathbf{H})^2\} + 2(\boldsymbol{\xi} \wedge \delta\mathbf{E}) \cdot \delta\mathbf{H} > 0 \quad \forall\xi_0 > |\boldsymbol{\xi}|. \tag{7}$$

3. Electromagnetic materials (constant $\mu > 0$)

Equation (7) written in the form

$$\xi_0 Q' = \mu\left(\xi_0\, \delta\mathbf{H} + \frac{\boldsymbol{\xi}}{\mu} \wedge \delta\mathbf{E}\right)^2 + \frac{1}{\mu}(\boldsymbol{\xi} \cdot \delta\mathbf{E})^2 + \varepsilon\left(\xi_0^2 - \frac{\xi^2}{\varepsilon\mu}\right)\delta E^2 + 2\varepsilon'\xi_0^2(\mathbf{E} \cdot \delta\mathbf{E})^2 \tag{8}$$

(where Q' is Q for μ constant) reduces to $\varepsilon\left(\xi_0^2 - \frac{\xi^2}{\varepsilon\mu}\right)\delta E^2$ if we set

$$\delta\mathbf{H} = -\frac{\boldsymbol{\xi}}{\xi_0\mu} \wedge \delta\mathbf{E}, \tag{9}$$

$\delta\mathbf{E} \perp \boldsymbol{\xi}, \mathbf{E}$. Therefore one must have

$$\varepsilon > 0 \tag{10}$$

(taking $\boldsymbol{\xi} = \mathbf{0}$) and also

$$\varepsilon \geq \frac{1}{\mu}. \tag{11}$$

Thus it is apparent that conditions (10) (11), which are necessary for the positive definiteness of Q, are also sufficient when $\varepsilon' \geq 0$. If on the contrary $\varepsilon' < 0$, we can observe that

$$\xi_0 Q' \geq \mu\left(\xi_0\, \delta\mathbf{H} + \frac{\boldsymbol{\xi}}{\mu} \wedge \delta\mathbf{E}\right)^2 + \frac{1}{\mu}(\boldsymbol{\xi} \cdot \delta\mathbf{E})^2 + \left((\varepsilon + 2\varepsilon' E^2)\xi_0^2 - \frac{\xi^2}{\mu}\right)\delta E^2.$$

Then for $\boldsymbol{\xi} = \mathbf{0}$ and $\delta\mathbf{H} = \mathbf{0}$ the requirement $\varepsilon + 2\varepsilon' E^2 > 0$ follows. Subsequently we have

$$\varepsilon + 2\varepsilon' E^2 \geq \frac{1}{\mu} > 0 \tag{12}$$

which is a sufficient condition. It is also a necessary condition as can be seen by retaining (9) and choosing $\delta\mathbf{E} \| \mathbf{E}$, $\boldsymbol{\xi} \perp \delta\mathbf{E}$. Thus inequality (12) replaces (11).

4. Variable electric and magnetic permeability

Setting in (7) first $\delta\mathbf{H} = \mathbf{0}$, $\delta\mathbf{E} \| \mathbf{E}$ and then $\delta\mathbf{E} = \mathbf{0}$, $\delta\mathbf{H} \| \mathbf{H}$ we obtain $\varepsilon + 2\varepsilon' E^2 > 0$, $\mu + 2\mu' H^2 > 0$ respectively.

Let us consider separately two cases

Case I. $\mu' \geq 0$

Since

$$\xi_0 Q = \xi_0 Q' + 2\mu' \xi_0^2 (\mathbf{H} \cdot \delta \mathbf{H})^2, \tag{13}$$

inequalities (10), (11) or (12) remain sufficient conditions for the positive definiteness of Q. In order to see if they are also necessary conditions, we consider identity (13) with $\xi_0 Q'$ written in the form (8). Then if

i) $\varepsilon' \geq 0$

setting $\boldsymbol{\xi} \| \mathbf{H}$, $\delta \mathbf{E} \| \boldsymbol{\xi} \wedge \mathbf{E}$ and retaining (9), we obtain

$$\xi_0 Q = \varepsilon \left(\xi_0^2 - \frac{\xi^2}{\varepsilon \mu} \right) \delta E^2$$

which is positive by (10) and (11); thus conditions (10) and (11) are necessary. On the other hand, if

ii) $\varepsilon' < 0$

setting $\delta \mathbf{E} \| \mathbf{E}$, $\boldsymbol{\xi} \| (\mathbf{E} \wedge \mathbf{H}) \wedge \mathbf{E}$ and retaining once again (9), we obtain

$$\xi_0 Q = \left((\varepsilon + 2\varepsilon' E^2) \xi_0^2 - \frac{\xi^2}{\mu} \right) \delta E^2$$

which is positive if conditions (10) and (12) hold.

Case II. $\mu' < 0$

i) If $\varepsilon' \geq 0$, one proceeds as in I(ii) since ε and μ play symmetrical roles. In this manner one obtains

$$\varepsilon, \mu > 0 \quad \mu + 2\mu' H^2 \geq \frac{1}{\varepsilon}.$$

ii) If $\varepsilon' < 0$, using (7) we can write

$$\begin{aligned}
\xi_0 Q &\geq \xi_0^2 \left((\varepsilon + 2\varepsilon' E^2)\, \delta E^2 + (\mu + 2\mu' H^2)\, \delta H^2 \right) + 2\xi_0 (\boldsymbol{\xi} \wedge \delta \mathbf{E}) \cdot \delta \mathbf{H} \\
&= (\mu + 2\mu' H^2) \left(\xi_0\, \delta \mathbf{H} + \frac{\boldsymbol{\xi} \wedge \delta \mathbf{E}}{\mu + 2\mu' H^2} \right)^2 + \frac{(\boldsymbol{\xi} \cdot \delta \mathbf{E})^2}{\mu + 2\mu' H^2} \\
&\quad + \left(\xi_0^2 (\varepsilon + 2\varepsilon' E^2) - \frac{\xi^2}{\mu + 2\mu' H^2} \right) \delta E^2 \\
&\geq \left(\xi_0^2 (\varepsilon + 2\varepsilon' E^2) - \frac{\xi^2}{\mu + 2\mu' H^2} \right) \delta E^2
\end{aligned}$$

and this extreme value can be attained when $\mathbf{E} \perp \mathbf{H}$, $\delta \mathbf{E} \| \mathbf{E}$, $\delta \mathbf{H} \| \mathbf{H}$ and $\boldsymbol{\xi} \| \mathbf{E} \wedge \mathbf{H}$. Since $\xi_0^2 > \xi^2$, the necessary and sufficient condition

$$\varepsilon + 2\varepsilon' E^2 \geq \frac{1}{\mu + 2\mu' H^2} > 0.$$

ensues.

Acknowledgements

This research has been supported in part by G.N.F.M. of C.N.R. and by M.P.I. research funds.

One of the authors (A.G.) thanks I.N.d.A.M. for a research fellowship.

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Abstract

In a non linear electromagnetic isotropic medium the wave, radial and shock velocities do not exceed light speed iff some simple conditions are satisfied.

Sunto

Si danno condizioni necessarie e sufficienti par la limitatezza delle velocità d'onda, radiale e d'urto nei materiali elettromagnetici isotropi non lineari.

(Received; September 8, 1988)