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On wave propagation in elastic tubes conducting rotational flows

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1. Introduction

Wave propagation in compliant ducts filled with a streaming fluid concerns not only the basic fluid dynamics, but also the physiology and the medicine, especially of the cardio-vascular and respiratory systems (McDonald 1974, Shapiro 1977a, Hyatt et al. 1979, Pedley 1980). The most obvious physiological illustration of downstream waves is that of the pulse wave, which propagates from the heart towards the periphery of the vascular tree. Under some pathological circumstances, the blood velocity is high enough to significantly increase the pulse propagation velocity (Anliker et al. 1971). An interesting feature occurs with upstream waves when the flow becomes "supercritical". This happens, for example in a Starling Resistor, when the volume flow ceases to increase in spite of stronger aspiration. The explanation is that in a supercritical flow the mean fluid velocity exceeds, at some place of the compliant tube, the local propagation speed of small perturbations. It follows that signals of stronger aspiration, which are generated in the downstream reservoir, cannot be propagated up to the upstream reservoir. In consequence, the flow does not increase any more. Such a mechanism should for example explain the phenomenon of expiratory flow limitation (Dawson and Elliott 1977). Several other examples with clinical relevancy are listed in the extensive study of Kamm and Shapiro (1979).

In this general context, a basic problem takes on special interest, namely the determination of the relationship between C, $\bar{U}$ and a, where C is the propagation speed of perturbations in a long elastic tube conducting a fluid flow, $\bar{U}$ is the cross-sectional mean velocity of this flow and a is the wave speed when $\bar{U} = 0$. Because of the experimental difficulties, the relationship $C(\bar{U}, a)$ has only been established on theoretical basis. The interest for this problem is illustrated by the following (non-exhaustive) list of publications: Streeter et al. 1963, Rudinger 1966, Olsen and Shapiro 1967, Jones 1969, Skalak 1972, Histand and Anliker 1973, Rumberger and Nerem 1977, Jan et al. 1983, Holenstein et al. 1984, Cancelli and Pedley 1985, Niederer 1985, Rooz et al. 1985, Shimizu 1985. Several other works are cited in the extensive monograph of Pedley (1980). In these publications a relationship $C(\bar{U}, a)$ has been either derived and/or it has been used to interpret experimental results. The underlying theoretical models differ among each other in many respects. However, under the appropriate unifying assumptions, all these models lead to the same relationship $C = \bar{U} \pm a$, which implies that the perturbations propagate at the speed a relative to the fluid.

Besides their own specificity, these theoretical models have in common two basic approximations. The general assumption after which the wavelenght of the perturbations is much larger than the tube radius leads to the classical long wave approximation: (i) the radial acceleration and pressure variation are negligible, so that the radial momentum equation is ignored. The second approximation is (ii) to assume that the velocity profile of the axial flow is rectangular.

As a consequence, little is known about the influence of the flow velocity profile on this wave propagation phenomenon, and the present study should contribute to fill this gap.

2. Governing equations

Considered is the wave propagation phenomenon in the system composed of an arbitrary fast fluid flow in a long elastic tube.

The problem shows axial symmetry and no body forces act on the inviscid and incompressible fluid. Perturbations of small amplitude are superposed on the basic flow, which is steady and parallel to the tube axis. We shall seek for traveling wave solutions of the form

$$\Psi(x, r, t) = \Phi(r) \cdot e^{i\alpha(x - Ct)}, \tag{1}$$

$$p(x, r, t) = \chi(r) \cdot e^{i\alpha(x - Ct)}, \tag{2}$$

where Ψ is the stream function of the perturbation, p is the pressure perturbation and t is the time; the co-cordinate in the direction of the basic flow is x, while r is the radial co-ordinate normal to that direction; α is the wave number and C is the wave propagation speed. Retaining the linearized convective acceleration terms and using the relations $u = 1/r \cdot \partial\Psi/\partial r$ and $v = -1/r \cdot \partial\Psi/\partial x$ we obtain the momentum equations

$$\frac{1}{r}(U - C)\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r}\Phi = -\frac{1}{\varrho}\chi \quad \text{in } x\text{-direction}, \tag{3}$$

$$\frac{\alpha^2}{r}(U - C)\Phi = -\frac{1}{\varrho}\frac{\mathrm{d}\chi}{\mathrm{d}r} \quad \text{in } r\text{-direction}, \tag{4}$$

where $U(r)$ is the velocity of the basic flow and ϱ is the fluid density. The elimination of χ between (3) and (4) leads to the Rayleigh-like equation

$$(U - C)\left(\Phi'' - \alpha^2 \frac{R_0^2}{4y}\Phi\right) - U''\Phi = 0, \tag{5}$$

where the prime denotes differentiation with respect to the variable $y \equiv r^2/R_0^2$ (R_0 is the undisturbed tube radius). This variable substitution allows to enhance the similarity between (5) and the Rayleigh equation in the hydrodynamic stability theory.

The elastic tube is an infinitely long, thin walled cylinder of radius R. Its cross-sectional area A only depends on the local transmural pressure and the surrounding pressure is constant. The equation for the motion of the tube reduce then to the the simple "tube law"

$$A = A\,[p_w(x, t)]\,, \tag{6}$$

where $p_w \equiv p\,(x, R, t)$ is the pressure perturbation at the tube wall.

The linearized boundary condition at the impermeable, moving wall requires $v = \partial R/\partial t + U_s\,\partial R/\partial x$, where $U_s \equiv U\,(R_0)$ is the velocity of the basic flow at the wall (slip velocity). Replacing v by $-1/r \cdot \partial\Psi/\partial x$ in the previous boundary condition and using (1), (2), (6), we obtain

$$-2\pi\,\Phi = \frac{\mathrm{d}A}{\mathrm{d}p_w}\,(U_s - C)\,\chi \quad \text{for} \quad r = R_0\,, \tag{7}$$

where $\mathrm{d}A/\mathrm{d}p_w$ is constant since the theory is linearized. The axial symmetry requires $v = 0$ on the axis. This condition, together with (1) and $v = -1/r \cdot \partial\Psi/\partial x$, leads to

$$\frac{\Phi}{r} \to 0 \quad \text{as} \quad r \to 0\,. \tag{8}$$

3. A particular solution

We now solve the differential Eq. (5) when the basic flow velocity profile is given by the polynomial

$$U = 2(\bar{U} - U_s)\,(1 - y) + U_s\,, \tag{9}$$

where $y \equiv r^2/R_0^2$ and $\bar{U}$ is the cross-sectional mean velocity:

$$\bar{U} \equiv \frac{2}{R_0^2}\int_0^{R_0} U\,r\,\mathrm{d}r = \int_0^1 U\,\mathrm{d}y\,. \tag{10}$$

Both $\bar{U}$ and U_s being arbitrary, $U\,(r)$ represents any quadratic profile, in particular the rectangle ($\bar{U} = U_s$) or the no-slip parabola ($U_s = 0$).

Since U'' is now zero, (5) reduces to $\Phi'' - \alpha^2\,(R_0^2/4\,y)\,\Phi = 0$, the solution of which is a linear combination of the modified Bessel functions $I_1\,(\alpha\,R_0\,y^{1/2})$ and $K_1\,(\alpha\,R_0\,y^{1/2})$. Considering that K_1 has a pole at the origin, the solution satisfying the boundary condition (8) is

$$\Phi = k\,\alpha\,r\,I_1\,(\alpha\,r)\,, \tag{11}$$

where k is the integration constant. The introduction of (11) in (3) gives the amplitude of the pressure perturbation:

$$-\frac{R_0^2}{2\varrho}\chi = \frac{k}{2}\alpha^2 R_0^2\left[2(\bar{U}-U_s)\left(1-\frac{r^2}{R_0^2}\right)+U_s-C\right]\cdot I_0(\alpha r)$$
$$+\,k\,2(\bar{U}-U_s)\,\alpha r\cdot I_1(\alpha r). \tag{12}$$

Further substitution of (11) in the boundary condition at the wall (7) leads to

$$-k\,2\pi\alpha R_0 I_1(\alpha R_0)=\frac{\mathrm{d}A}{\mathrm{d}p_w}(U_s-C)\,\chi(R_0). \tag{13}$$

The eigenvalue relation is obtained by eliminating χ between (12) and (13). This leads to a quadratic equation for the unknown C, the roots of which are

$$C = U_s+(\bar{U}-U_s)\,N\pm[a^2N+(\bar{U}-U_s)^2N^2]^{1/2}. \tag{14}$$

The abbreviations N and a are defined as

$$N\equiv\frac{2}{\alpha R_0}\frac{I_1(\alpha R_0)}{I_0(\alpha R_0)}, \tag{15}$$

$$a^2\equiv\frac{A_0}{\varrho}\Big/\frac{\mathrm{d}A}{\mathrm{d}p_w}, \tag{16}$$

where a is the Moens-Korteweg wave speed and $A_0=\pi R_0^2$. The behaviour of $N(\alpha R_0)$ for small and large values of αR_0 can be obtained from the corresponding expansions of I_0 and I_1. It follows then from (15)

$$N\cong 1-(\alpha R_0)^2/8+O(\alpha^4R_0^4)\quad\text{as}\quad|\alpha R_0|\to 0, \tag{17}$$

$$N\cong\frac{2}{\alpha R_0}+O(\alpha^{-2}R_0^{-2})\quad\text{as}\quad|\alpha R_0|\to\infty. \tag{18}$$

In the following we shall only consider periodic solutions in x, which implies real (positive) wave numbers α. As a consequence, also $N(\alpha R_0)$ is real. This function reaches its maximum, namely one, at the origin and decreases monotonically as αR_0 increases, the asymptotic value being zero. This behaviour of N accounts for the dispersion phenomenon, the longest waves traveling the fastest.

Since N is non-negative, C_+ and C_-, the two wave speeds defined by (14), are real and the physical solutions are readily computed from (11) and (12):

$$u=k\alpha^2 I_0(\alpha r)\cos[\alpha(x-Ct)], \tag{19}$$

$$v=k\alpha^2 I_1(\alpha r)\sin[\alpha(x-Ct)], \tag{20}$$

$$-\frac{p}{\varrho}=k\alpha^2\left\{\left[2(\bar{U}-U_s)\left(1-\frac{r^2}{R_0^2}\right)+U_s-C\right]\cdot I_0(\alpha r)\right.$$
$$\left.+\frac{4}{\alpha^2R_0^2}(\bar{U}-U_s)\,\alpha r\cdot I_1(\alpha r)\right\}\cos[\alpha(x-Ct)]. \tag{21}$$

By definition, downstream waves propagate with the stream at the speed $C_+ > 0$, while upstream waves propagate in the opposite direction at the speed $C_- < 0$. In consequence, upstream wave propagation vanishes as soon as $C_- > 0$. The value of $\bar{U}$ for which $C_- = 0$ is the critical velocity $\bar{U}^*$ and the latter can be computed from (14):

$$\bar{U}^* = U_s\left(1 - \frac{1}{2N}\right) + \frac{a^2}{2U_s}. \tag{22}$$

For a rectangular profile, $U_s = \bar{U} = \bar{U}^*$ and (22) leads to

$$\bar{U}^* = a N^{1/2}. \tag{23}$$

By contrast, if the profile satisfies the no-slip condition $U_s = 0$, then $\bar{U}^* = \infty$. In this case upstream wave propagation occurs for all values of $\bar{U}/a$.

4. Comparison with other works: long wave approximation

The theoretical literature, in which the relationship between the wave speed and the flow velocity has been explicited, neglects the momentum equation in radial direction. This implies, however, the long wave assumption $\alpha R_0 \ll 1$, in which case Eqs. (11), (14) and (19) to (21) become

$$\Phi = \frac{k}{2}\alpha^2 r^2 + O(\varepsilon^4), \tag{24}$$

$$u = k\alpha^2\left(1 + \frac{\alpha^2 r^2}{4}\right)\cos[\alpha(x - Ct)] + O(\varepsilon^4), \tag{25}$$

$$v = \frac{k}{2}\alpha^3 r \sin[\alpha(x - Ct)] + O(\varepsilon^3), \tag{26}$$

$$-\frac{p}{\varrho} = u\{\bar{U} - U_s \mp (a^2 + (\bar{U} - U_s)^2)^{1/2}\} + O(\varepsilon^4), \tag{27}$$

$$C = \bar{U} \pm (a^2 + (\bar{U} - U_s)^2)^{1/2} + O(\alpha^2 R_0^2), \tag{28}$$

where $\varepsilon \equiv \alpha r$. Two velocity profiles are of particular interest, namely the rectangle and the parabola. It follows then from (28):

$$C = \bar{U} \pm a \quad \text{for} \quad U_s = \bar{U}, \tag{29}$$

$$C = \bar{U} \pm (a^2 + \bar{U}^2)^{1/2} \quad \text{for} \quad U_s = 0. \tag{30}$$

4.1. Linearized theories

Apparently, only two publications deal with linearisation around a state which is not the rest and which, simultaneously, satisfies the no-slip condition at the tube wall. Morgan and Ferrante (1955), while considering a Poiseuille flow, used a linearized perturbation method and, for $\bar{U} \ll a$, they approximately solved the problem. If the fluid viscosity, the wall mass and the Poisson's ratio are neglected in their Eqs. 6 and 7 for the motion of the tube, these reduce to our simple tube law (6). Their Eq. 79, in our notation, simplifies then to

$$C = \bar{U} \pm a\left(1 + \frac{\bar{U}^2}{2a^2}\right). \tag{31}$$

Now, the brackets of (31) contain nothing more than the first two terms of the power serie in $\bar{U}/a$ into which the square root of (30) can be expanded. In the validity range of Morgan and Ferrante's work, namely $\bar{U} \ll a$, both approaches are naturally equivalent. As to the report of Womersley (1957), its section X is briefly presented in the discussion.

4.2. One-dimensional, non-linear theories

In the simplest form of the one-dimensional, non-linear approach, the mass conservation equation and the fluid momentum equation are given by

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(A\bar{U}) = 0, \tag{32}$$

$$\frac{\partial \bar{U}}{\partial t} + \bar{U}\frac{\partial \bar{U}}{\partial x} = -\frac{1}{\varrho}\frac{\partial P}{\partial x}. \tag{33}$$

The cross-sectional mean velocity $\bar{U}$ is defined as

$$\bar{U}(x, t) = \frac{2}{R^2}\int_0^R U(x, r, t)\, r\, \mathrm{d}r, \tag{34}$$

where $U(x, r, t)$ is the unknown axial velocity component. The tube law $A = A(P)$, (32) and (33) define a non-linear system which has been integrated by the method of the characteristics, for the first time, by Lambert (1958). This method has been extended by Streeter et al. (1963) and later by many authors already cited in the introduction. The improvements in Lambert's model include seepage, tapering, and viscoelatic tube properties as well as fluid frictional terms. Common to all these models, however, remains the rectangular velocity profile $\bar{U}(x, t)$. Although the characteristic equations differ from one model to the other, the equation of the slope of the characteristics is the same for all, namely

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv C(x, t) = \bar{U}(x, t) \pm a(x, t). \tag{35}$$

Now, since the non-linear theory should be able to account for the linearized one, (35) should be a generalization of (28). As seen by a simple inspection this happens only for the rectangular profile (29). A comparison of (30) and (35) further shows that the non-linear theory can account for the linearized one with a parabolic profile only if $\bar{U}/a \ll 1$. Since the validity of the linearized result (30) is not limited to $\bar{U}/a \ll 1$, this restriction necessarily applies to the non-linear result (35).

4.3. *Quasi-one-dimensional, non-linear theory*

Also in the quasi-one-dimensional, non-linear method proposed by Barnard et al. (1966) the radial fluid momentum equation is neglected and the equations are integrated by the method of the characteristics. Here, however, the axial flow profile is not prespecified, so that the momentum equation

$$\frac{\partial \bar{U}}{\partial t} + \frac{\bar{U}}{A}(1-B)\frac{\partial A}{\partial t} + B\bar{U}\frac{\partial \bar{U}}{\partial x} = -\frac{1}{\varrho}\frac{\partial P}{\partial x} + \frac{2\nu}{R}\frac{\partial U}{\partial r}\bigg|_{r=R} \tag{36}$$

contains an unknown function $B(x,t)$ defined by

$$B \equiv \frac{2}{R^2\bar{U}^2}\int_0^R U^2 r\,\mathrm{d}r\,. \tag{37}$$

This unknown function reduces to an unknown constant after introduction of the variable separation

$$U(x,r,t) = \bar{U}(x,t)\cdot g(r/R)\,. \tag{38}$$

Barnard et al. (1966) obtain then for the slope of the characteristics

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv C(x,t) = B\bar{U} \pm (a^2 + B(B-1)\bar{U}^2)^{1/2}. \tag{39}$$

The value of B depends on the assumed velocity profile. In the case more particularly considered by Barnard et al., namely this of a parabola, $g = 2(1 - r^2/R^2)$ and $B = 4/3$. However, even then, (39) does not represent a generalization of the linearized result (30).

The reason lies in the separation of the variables (38). Substituting (38) in the differential form of the mass conservation equation we obtain after a single integration

$$V = \frac{\bar{U}}{2A}\frac{\partial A}{\partial x} r\,g(r/R) - \frac{1}{A}\frac{\partial}{\partial x}(A\bar{U})\frac{1}{r}\int_0^r z\,g(z/R)\,\mathrm{d}z\,. \tag{40}$$

Introducing now (38) and (40) in the differential form of the (inviscid) momentum equation in x-direction we find

$$-\frac{1}{\varrho}\frac{\partial P}{\partial x} = \left[\frac{\partial \bar{U}}{\partial t} + g(s)\,\bar{U}\frac{\partial \bar{U}}{\partial x}\right] g(s) - \frac{\bar{U}}{A}\frac{\partial A}{\partial t}\left[\frac{s}{2} - \frac{1}{s}\int_0^s n\,g(n)\,\mathrm{d}n\right]\frac{\mathrm{d}g}{\mathrm{d}s}, \tag{41}$$

where $s \equiv r/R$. The definitions (34) of $\bar{U}$ and (38) of g imply that the integral in (41) equals 1/2 for $s = 1$. In consequence

$$-\frac{1}{\varrho}\frac{\partial P}{\partial x}\bigg|_{r=R} = \left[\frac{\partial \bar{U}}{\partial t} + g(1)\,\bar{U}\,\frac{\partial \bar{U}}{\partial x}\right] g(1)\,. \tag{42}$$

For a rectangular profile $g(r/R) \equiv 1$ and (42) reduces to the classical form of the inviscid, one-dimensional, non-linear momentum Eq. (33). However, for any profile satisfying the no-slip condition at the wall $g(1) = 0$ and (42) becomes $\partial P/\partial x = 0$ for $r = R$. This unrealistic pressure distribution being a direct consequence of (38), the latter is inadequate.

The separation of the variables (38) is convenient in order to satisfy the no-slip condition, but it represents nevertheless a too particular constraint since it allows only affine velocity profiles. A similar situation exists in boundary layers, where affine velocity profiles also require particular pressure distributions (Schlichting 1965).

4.4. *Experimental work*

Already Müller (1950), before Morgan and Ferrante (1955), had emphasized the importance of the no-slip condition and questioned the validity of $C = \bar{U} \pm a$. To our our knowledge, Müller is the only one who has experimentally established a relationship between C, $\bar{U}$ and a. Müller, however, only considers the case $\bar{U}/a \cong 0.1$, so that his results are of little help in the present controversy. Indeed, the wave speed (35) predicted by the classical theory and that (30) predicted by the present theory for a parabola markedly differ only if $\bar{U}$ is not much smaller than a.

5. Discussion

The purpose of the present work is the evaluation of the wave propagation speed of small amplitude perturbations in an elastic tube conducting a fluid flow of arbitrary speed. Besides the linearisation, the main underlying simplifications are that (i) the cross-section of the tube only depends on the local transmural pressure, (ii) the fluid is inviscid and (iii) the basic flow velocity profile is described by a quadratic function of the radial co-ordinate and has an arbitrary slip velocity on the tube wall. These simplifications are justified first by the simplicity of the resulting analytical solutions, so that the influence of the velocity profile on the wave propagation is easy to evaluate. Secondly, all these simplifications can be readily incorporated in other partly more general models. Then, the latter become particular cases of the present model, so allowing valid comparisons.

According to our central result (14), the wave propagation speed depends not only on the mean velocity of the basic flow, but also on its slip-velocity, i.e. on the velocity profile. Morgan and Ferrante (1955), in an linearized analysis limited to a low speed Poiseuille-flow ($\bar{U} \ll a$), already came to an equivalent conclusion. Surprisingly, the latter seems to have passed unnoticed and their work, when cited, is usually classified among those neglecting the convective acceleration.

The same questionable classification concerns Womersley's work since he has solved, in the section X of his otherwise famous report (Womersley 1957), the same boundary value problem as Morgan and Ferrante (1955), but for arbitrary values of $\bar{U}/a$. As to the eigenvalue problem, Womersley had the time to solve it only in the same particular case as Morgan and Ferrante ($\bar{U} \ll a$). Womersley's untimely death (McDonald 1974), in 1958, stopped his advance towards a more general solution (Womersley 1957, p. 97 and 107).

On the basis of the one-dimensional, non-linear theory, in which a rectangular flow profile is assumed, Shapiro (1977b) has emphasized the analogy which should exist between the present wave propagation problem and those in compressible flows and in free-surface channel flows. In consequence, upstream wave propagation should vanish as soon as $\bar{U} > a$. The present linearized approach leads to the same conclusion, but only if the profile is rectangular (9). By contrast, a parabolic profile allows countercurrent propagation regardless of $\bar{U}/a$ (22, 30). Consequently, the velocity profile of the basic flow markedly affects this wave propagation phenomenon as soon as the axial convective acceleration becomes significant.

By definition, the one-dimensional theories, linearized or not, rely upon an axial flow with a rectangular velocity profile. This pivotal simplification needs however a firm validation, particulary then when fluid dynamical non-linearities seem to become important. In classical hydraulics, typically a "hard ware" field, these non-linearities probably are unimportant since the pipes are so rigid that $\bar{U}/a$ remains small. This might explain why Lambert (1958) and Streeter et al. (1963) made no attempt at all to validate the one-dimensional flow assumption when they introduced the method of the characteristics in the present "soft ware" field. Although this gap has apparently not been filled, the method is nowadays well established. Actually, the method of the characteristics is known to be readily feasible only for one-dimensional flows. As an illustration, the quasi-one-dimensional, non-linear method of Barnard et al. (1966) which relies upon a parabolic flow is not convincing (see previous section).

The present study shows that the velocity profile of the basic flow and, consequently, that its vorticity $\mathrm{d}U/\mathrm{d}r$ should be taken into consideration. Like in rigid tubes the vorticity of a parabolic basic flow is maximum near the wall. However, the tube compliance profoundly affects the radial velocity component v: while $v = 0$ on a rigid wall, it reaches its maximum on the elastic wall (Eq. 20). The product $v \cdot \mathrm{d}U/\mathrm{d}r$, which is therefore maximum near the wall, appears in the axial fluid momentum equation. While this vorticity-dependent term is neglected in the classical one-dimensional theory, it is considered here, even though only in the linearized form $\Phi \cdot \mathrm{d}U/\mathrm{d}r$ (3). These vorticity considerations remain valid, at least qualitatively, for a class of basic flows wider than the simple parabola. Indeed, the faster is a real basic flow, the blunter is its profile, and the stronger becomes the vorticity near the wall. It is noteworthy that the influence of the basic flow vorticity on the corresponding wave propagation phenomenon is well known in the closely related hydrodynamic stability theory (Schlichting 1965).

In conclusion, the present linearized theory shows that the basic flow velocity profile modifies not only the wave propagation speed (14, 28), but also the amplitude of the pressure perturbations (21, 27). This effect, which becomes noticeable as soon as $\bar{U}/a \ll 1$ is not fulfilled, is the more pronounced the larger is $\bar{U}/a$. The flow disturbances are treated as inviscid, although the generation of the basic flow implies viscous effects. This procedure is classical; the effect of viscosity on the disturbances is left for future work.

As to the practical implications, fluid dynamical non-linearities and high-speed flow effects seem to play an important role in some patho-physiological situations and in the interpretation of clinical tests concerning the cardiovascular as well as the respiratory systems (Anliker et al. 1971, Dawson and Elliott 1977, Shapiro 1977a, Hyatt et al. 1979, Pedley 1980, Niederer 1985). A revaluation of these fluid dynamical effects could possibly bring some new insights in the just mentioned fields.

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Abstract

The propagation of perturbations in liquid filled elastic tubes depends on the stream velocity of the basic flow. This phenomenon is currently analyzed with the method of the characteristics which relies upon a basic flow with a rectangular velocity profile. It seems that this one-dimensional flow approximation has not been convincingly validated, which justifies to consider other, more general velocity profiles.

In the present analytical study the velocity profile is a quadratic function of the radial coordinate. Small amplitude perturbations are superposed on this inviscid, basic state in which the mean velocity $\bar{U}$ is arbitrarily large. A normal mode analysis shows that the velocity profile and therefore the vorticity of the basic flow influence the more the phenomenon the larger is $\bar{U}$. For example, a parabolic profile allows countercurrent wave propagation regardless of $\bar{U}$.

This questions the one-dimensional wave propagation theory in compliant tubes and, consequently, the interpretation of several physiological and medical problems mainly in the respiratory and cardio-vascular systems.

Resumé

La propagation de perturbations dans un tube élastique conduisant un écoulement fluide dépend de la vitesse de l'écoulement de base. Ce phénomène est habituellement étudié avec la méthode des caractéristiques, où l'on suppose que le profil de vitesse de l'écoulement est rectangulaire. Comme cette simplification ne semble pas avoir été bien validée, il paraît indiqué d'étudier l'impact d'autres profils.

Dans la présente étude analytique, ce profil de vitesse est une fonction quadratique de la coordonée radiale. A cet écoulement non visqueux, dont la vitesse moyenne $\bar{U}$ est arbitraire, l'on superpose des perturbations de faible amplitude. Une analyse linéarisée montre que le profil de vitesse et donc le rotationel de l'écoulement de base influencent d'autant plus ce phénomène d'ondes que $\bar{U}$ est élévée.

Ceci met en question la théorie uni-dimensionelle de la propagation d'ondes dans des tubes compliants et, par là-même, l'interpretation de divers problèmes physiologiques et médicaux, avant tout des systèmes respiratoires et cardio-vasculaires.

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