Journal of Applied Mathematics and Physics (ZAMP)
Vol. 40, July 1989

0044-2275/89/040501-09 $ 1.50 + 0.20

Magnetohydrodynamic shock wave decay

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1. Introduction

The motion of a non-uniform shock wave is an important and difficult problem, the exact solution of which would require a solution of the non-isentropic equations of motion. Generally, approximate analytical techniques which yield the principal characteristics of the solution must be utilized.

The fact that the difference between a shock transition and a simple wave transition with the same strength and initial state involves only terms of third and higher order in the strength formed the basis for an approximate theory, useful for describing the motion of relatively weak shocks, developed by Friedrichs [1, 2], who replaced the actual shock conditions by the transition through a corresponding simple compression wave. That theory gave an approximate description of the decay of a non-uniform shock wave.

Germain and Gundersen [3–5] developed a theory of non-isentropic perturbations of uniform and simple wave flows which has been used to give an improved treatment of the decay of a shock wave, applicable to shocks of arbitrary strength, though only the initial stages of the decay were considered. The Friedrichs theory was contained as a special case. See Burnside and Mackie [6].

The theory governing the motion of a compressible fluid whose electrical conductivity may be assumed to be infinite and the theory of conventional gas dynamics are quite similar. It is possible to develop a theory of shock waves and simple waves and approximate analytical techniques in a manner quite parallel to that employed in conventional gas dynamics. Specifically, it is possible to develop a theory of perturbations of initially uniform magnetohydrodynamic shock waves and non-isentropic perturbations of simple wave flows [7–10]. These solutions have been utilized to discuss the dacay of a magnetohydrodynamic shock wave [11–12] in a form which included the magnetic extension of the Friedrichs theory [13] and the solution for conventional gas dynamics as limiting cases. An interesting conclusion from these alternative discussions is that the Friedrichs theory is useful for much stronger shocks than might be expected.

Later, another discussion of the decay of a shock wave was given by Ardavan-Rhad [14], whose analysis was based on a particular solution of the non-isentropic equations of motion, obtained through the use of a modified hodograph transformation. An approximate representation of the shock path, valid for relatively weak shocks, was obtained.

The present paper shows that this later method may be extended to magnetohydrodynamic flows in a form which includes the solution for conventional gas dynamics as a special limiting case. Since the basic problem is the same as that considered previously [11] in which a detailed discussion of the entire flow field was given, only the approximate shock path will be discussed. The exact solution of the non-isentropic equations may prove useful in other problems.

2. The basic equations

For the problem considered, the equations which govern the one-dimensional unsteady flow of an ideal, inviscid, perfectly conducting compressible fluid, subjected to a transverse magnetic field, may be written as

$$P = \exp[(s - s_0)/c_v]\rho^{\gamma} \tag{1}$$

$$u_t + uu_x + 2\omega^2 c_x/(\gamma - 1)c - c^2 s_x/\gamma(\gamma - 1)c_v = 0 \tag{2}$$

$$c_t + uc_x + (\gamma - 1)cu_x/2 = 0 \tag{3}$$

$$s_t + us_x = 0 \tag{4}$$

where $P, \rho, u, c, B, \mu, b^2 = B^2/\mu\rho,\ s, s_0, \omega = (b^2 + c^2)^{1/2}$ and γ are, respectively, the pressure, density, particle velocity, local speed of sound, magnetic induction, permeability, square of the Alfvén speed, specific entropy, specific entropy at some reference state, true speed of sound and ratio of specific heats at constant pressure c_p and at constant volume c_v. All dependent variables are assumed to be functions of one space variable, x, and the time, t. Partial derivatives are denoted by subscripts.

When the flow is isentropic, this system of equations may be integrated to give

$$u/2 + w/(\gamma - 1) = \alpha, \quad -u/2 + w/(\gamma - 1) = \beta \tag{5}$$

where

$$w = \int (\omega/c)\, dc$$

and (α, β) denote generalized Riemann invariants. The characteristics of the system of equations (2)–(4) are given by

$$dx/dt = u, \quad u + \omega, \quad u - \omega$$

i.e., the particle paths, the α-characteristics and the β-characteristics.

Introduction of the change of variables

$$w = \exp[(\gamma - 1)\eta/2] \tag{6}$$

$$\xi = \int (\omega w/c^2)\, d\eta - (s - s_0)/\gamma(\gamma - 1)c_v \tag{7}$$

transforms equations (2)–(4) into

$$u_t + uu_x + c^2\xi_x = 0 \tag{8}$$

$$\eta_t + u\eta_x + \omega w^{-1}u_x = 0 \tag{9}$$

$$\xi_t + u\xi_x + \omega^2 u_x/c^2 = 0 \tag{10}$$

Note that for isentropic flow, $\xi = \eta$.

Equations (8)–(10) comprise a system of three equations for three dependent variables which are functions of two independent variables. The procedure to be utilized is to interchange the dependent variables (ξ, η) and the independent variables (x, t), carrying a function of the particle velocity as a to be determined function of the new independent variables. This leads to the following system of equations.

$$u_\xi x_\eta - u_\eta x_\xi + u\psi - c^2 t_\eta = 0 \tag{11}$$

$$x_\xi - ut_\xi - \omega w^{-1}\psi = 0 \tag{12}$$

$$x_\eta - ut_\eta + \omega^2\psi/c^2 = 0 \tag{13}$$

where

$$\psi = t_\xi u_\eta - u_\xi t_\eta \tag{14}$$

Replacement of the values of x_ξ from equation (12) and x_η from equation (13) in equation (11) leads to the result

$$c^2 t_\eta + [\omega^2 u_\xi/c^2 + \omega w^{-1}u_\eta]\psi = 0 \tag{15}$$

Assuming the second-order partial derivatives are continuous, cross-differentiation of equations (12)–(13) leads to the linear differential equation for ψ

$$\omega w^{-1}\psi_\eta + \omega^2\psi_\xi/c^2 = -\psi[1 + (\omega w^{-1})_\eta] \tag{16}$$

which has the general solution

$$\psi = (w/\omega)\exp\left[-\int (w/\omega)\, d\eta\right] g\left[\xi - \int (w\omega/c^2)\, d\eta\right] \tag{17}$$

in terms of an arbitrary differentiable function g. Consequently, the problem is reduced to solving equations (14)–(15) with ψ given by equation (17). Equations (12)–(13) then give $x = x(u, t)$.

Assuming the second-order partial derivatives are continuous, solving equations (14)–(15) for t_ξ and t_η and cross-differentiating gives the single second-order partial differential equation for $u = u(\xi, \eta)$

$$\frac{\partial}{\partial \xi}\{[\omega^2 c^{-2} u_\xi + \omega w^{-1} u_\eta] w \omega^{-1} c^{\{-2+2/(1-\gamma)\}} g\}$$
$$= \frac{\partial}{\partial \eta}\{w\omega^{-1} c^{2/(1-\gamma)}[u_\xi(\omega^2 c^{-2} u_\xi + \omega w^{-1} u_\eta) c^{-2} g - g] u_\eta^{-1}\} \tag{18}$$

Since the solution for u must reduce to the value given by the generalized Riemann invariant ($\beta = \beta_0$, a constant) when the flow is isentropic, a trial solution may be assumed to be

$$u = 2wh(\xi - \eta)/(\gamma - 1) - 2\beta_0 \tag{19}$$

where h is an arbitrary differentiable function.

Substitution of equation (19) into equation (18) leads to the solution

$$g = (\gamma + 1)[h - 2\omega w h'/(\gamma - 1)c^2] f(\sigma)/2 \tag{20}$$

where the prime denotes differentiation with respect to the argument,

$$f(\sigma) = \exp\left[-\int wR\omega^{-1} d \log(h^2 - 1)/2(\gamma - 1)\right]$$
$$\times \exp\left[\int \{Rc^2/2\omega^2(h^2 - 1)\}\, d\sigma\right] \tag{21}$$

$$\sigma = -(s - s_0)/\gamma(\gamma - 1)c_v$$

and

$$R = [3\omega^2 + (\gamma - 2)c^2]/\omega^2$$

Since the equations defining t may be written as

$$-t_\eta = wc^{\{-2+2/(1-\gamma)\}} gh$$

$$t_\xi = c^{2/(1-\gamma)}[1 + 2hh_\eta/(\gamma - 1)]g[h + 2h_\eta/(\gamma - 1)]^{-1}$$

the solution for t is found to be

$$t = (\gamma + 1)\omega f c^{(\gamma+1)/(1-\gamma)}/Rc \tag{22}$$

Equations (12)–(13) then give the solution for x as

$$x = [u + \omega h(\sigma)]t + \{-\omega + w\, d\omega/dw\} \int t\, dh \tag{23}$$

Equations (19)–(23) give the solution for the non-isentropic flow in terms of h. In the limit of vanishing magnetic field, this solution reduces exactly to the solution obtained for conventional non-conducting flow.

3. Formulation of the problem

The usual piston model will be utilized. Thus, let the piston, originally at rest, be pushed impulsively with constant speed into fluid originally at rest. At some later time, the piston is abruptly stopped and kept motionless thereafter. At the fluid-piston interface, continuity of the magnetic field is assumed. The originally uniform shock wave thus generated will intersect the centered simple wave generated when the piston is stopped. Within the region of interaction, the entropy change across the shock is no longer constant and the shock is non-uniform. The forward-facing simple wave is centered at some point, say (x_0, t_0), and characterized by $\beta = \beta_0$, a constant. On each characteristic, $dx/dt = u + \omega$, the flow parameters are constant, and these characteristics are straight lines in the (x, t)-plane. The wave may be represented by

$$x - x_0 = (u + \omega)(t - t_0) \tag{24}$$

$$\beta = \beta_0 \tag{25}$$

From the definition of the characteristic parameters of the simple wave flow, equation (5), it follows that

$$u = \alpha - \beta_0 \tag{26}$$

$$w = (\gamma - 1)(\alpha + \beta_0)/2 \tag{27}$$

While it is possible to solve explicitly for the flow parameters in the simple wave flow in the non-conducting case, it is not possible, in general, to do so in the magnetic case (the monatomic fluid is an exception).

Letting U denote the shock speed, the equation of the shock path is given by integrating the differential equation

$$dx/dt = U \tag{28}$$

In the conventional case, it is common to use the pressure ratio across the shock as a basic parameter. The other flow quantities then may be expressed in terms of this parameter through the Rankine-Hugoniot shock conditions. In the conducting case, two parameters are needed; namely, one giving a measure of the shock strength and one giving a measure of the magnetic field. Further, it is much more convenient to use the density ratio across the shock as a measure of the strength. Thus, with subscripts one and two denoting, respectively, the flow in front of and behind the shock, it is convenient to express all flow quantities in terms of the shock strength parameter

$$\pi = -1 + \rho_2/\rho_1 \tag{29}$$

and $m_1 = u_1/c_1$. This derivation is contained in the Appendix.

The shock path is determined from the equation

$$(dx/d\pi)/(dt/d\pi) = U \tag{30}$$

which leads to the following integro-differential equation for $h(\pi)$:

$$\left[\frac{2wh^2}{(\gamma-1)c} - \frac{MwRh}{(\gamma-1)\omega} + \frac{w}{c}\frac{d\omega}{dw}\frac{dh}{d\pi}\right] = \frac{\omega T_1 h^3}{c} + \left[\frac{1}{c}\frac{du}{d\pi} + T_2 - MT_4\right]h^2 + [T_3 - T_1]\frac{\omega h}{c} + M[T_4 - T_3] - \frac{1}{c}\frac{d\omega}{d\pi} - T_2 \tag{31}$$

where

$$T_1 = \frac{2\gamma}{(1-\gamma)c}\frac{dc}{d\pi} + \frac{2}{\omega}\frac{d\omega}{d\pi} - \frac{1}{R}\frac{dR}{d\pi}$$

$$T_2 = \frac{d}{d\pi}\left[w\frac{d\omega}{dw} - \omega\right]\left\{\frac{1}{ct}\int t\frac{dh}{d\pi}\,d\pi\right\}$$

$$T_3 = -\frac{c^2 R}{2\omega^2}\frac{d\sigma}{d\pi}$$

$$T_4 = T_1 - \frac{1}{\omega}\frac{d\omega}{d\pi}$$

$$M = \frac{U-u}{c}$$

and flow quantities are to be evaluated behind the shock, i.e., the subscript two has been omitted in order to simplify the notation.

The initial condition for equation (31) is $h(0) = 1$. The relationship between σ and π is given in the Appendix, so that $h(\sigma)$ may be obtained once $h(\pi)$ has been determined.

The problem of finding $h(\pi)$ is overdetermined since from equation (19), subject to the condition $h(0) = 1$,

$$2w(h-1)/(\gamma-1) = u - 2(w - w_1)/(\gamma-1) \tag{32}$$

The right-hand side of equation (32) represents the transition (change in generalized Riemann invariant) through the simple wave, which, of course, is valid only for a shock of vanishing strength. Thus, the shock path obtained through the use of the solution of equation (31) will give a reasonable approximation to the true shock path only for shocks which are relatively weak. Since the Friedrichs theory has been shown to give a reasonable approximation for shocks stronger than might be expected [11], the same will be true for the present solution.

In magnetohydrodynamics, strong shocks can occur in two ways, i.e., for large π or for a very strong applied magnetic field with $\pi > 0$. Thus, the present solution requires that the applied magnetic field must also be relatively weak.

Appendix

The generalized Rankine-Hugoniot magnetic shock conditions relate flow quantities on the two sides of a shock wave in terms of two parameters; namely, one giving a measure of the shock strength and one giving a measure of the applied magnetic field [9]. With the notation $\theta = (\gamma + 1)/(\gamma - 1)$, $\tau = P_2/P_1$, $\rho_2/\rho_1 = 1 + \pi$, $m = b/c$, $n = u/c$, $M = (U - u)/c$, the flow quantities needed for equation (31) will be given as functions of π and m_1.

$$\tau = \frac{\theta\pi + \dfrac{2}{\gamma - 1} + \dfrac{\gamma m_1^2 \pi^3}{2}}{\dfrac{2}{\gamma - 1} - \pi}$$

$$\frac{1}{\tau}\frac{d\tau}{d\pi} = \frac{\dfrac{4\gamma}{(\gamma - 1)^2} + \dfrac{3\gamma m_1^2 \pi^2}{\gamma - 1} - \gamma m_1^2 \pi^3}{\left[\dfrac{2}{\gamma - 1} - \pi\right]\left[\theta\pi + \dfrac{2}{\gamma - 1} + \dfrac{\gamma m_1^2 \pi^3}{2}\right]} \equiv S_1$$

$$\frac{c_2^2}{c_1^2} = \frac{\tau}{1 + \pi}$$

$$\frac{1}{c_2}\frac{dc_2}{d\pi} = \frac{S_1(1 + \pi) - 1}{1 + \pi} \equiv S_2$$

$$\frac{m_2^2}{m_1^2} = \frac{(1 + \pi)^2}{\pi}$$

$$\frac{\omega_2^2}{c_2^2} = 1 + \frac{(1 + \pi)^2 m_1^2}{\pi} \equiv S_3$$

$$\frac{1}{\omega_2}\frac{d\omega_2}{d\pi} = S_2 + \frac{\pi - 1}{2\pi(1 + \pi)} \equiv S_4$$

$$w = \int S_3^{1/2} c_2 S_2 \, d\pi$$

$$M_1^2 = \{2 + m_1^2[2 + (2-\gamma)\pi]\}\,\frac{(1+\pi)}{2-(\gamma-1)\pi}$$

$$M_2 = \frac{M_1}{[(1+\pi)\tau]^{1/2}}$$

$$n_2 = \left[n_1 + \frac{\pi M_1}{1+\pi}\right]\left[\frac{1+\pi}{\tau}\right]^{1/2}$$

$$\frac{1}{c_2}\frac{du_2}{d\pi} = n_2 S_2 + \frac{dn_2}{d\pi}$$

$$R = \frac{3S_3 + (\gamma-3)}{S_3}$$

$$\frac{1}{R}\frac{dR}{d\pi} = \frac{2[3S_3S_4 + (\gamma-2)S_2]}{3S_3+\gamma-2} - 2S_4$$

$$\frac{d\omega}{dw} = \frac{R-2}{\gamma-1}$$

$$\frac{d\sigma}{d\pi} = \frac{1}{(\gamma-1)}\left[\frac{1}{1+\pi} - \frac{S_1}{\gamma}\right]$$

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Summary

A modified hodograph transformation is used to obtain an exact solution of the equations governing the one-dimensional unsteady flow of an ideal, inviscid, perfectly conducting compressible fluid, subjected to a transverse magnetic field. This solution is used to obtain an approximate representation of the path of an initially uniform shock wave which intersects a centered simple wave. In the limit of vanishing magnetic field, the solution reduces exactly to the solution of the corresponding problem for conventional gas dynamics.

Résumé

Une transformation hodographe modifiée est employée pour obtenir une solution exacte des équations relatives aux écoulements unidimensionnels non-stationnaires et non-isentropiques d'un fluide non visqueux idéal, parfaitement conducteur d'électricité et compressible, soumis à l'action d'un champ magnétique transversal. On utilise cette solution pour obtenir une représentation approximative de la trajectoire d'une onde de choc magnétohydrodynamique initialement uniforme, rencontrant une onde simple centrée.

Dans le cas limite d'un champ magnétique nul, la solution se réduit exactement à celle du problème correspondant de la dynamique classique des gaz. C'est là une confirmation de la validité de la théorie.

(Received: December 9, 1988)