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# An effective boundary element method for inhomogeneous partial differential equations

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## 1. Introduction

In recent years, the boundary integral method (BIM) has become a popular technique for solving boundary-value problems that involve a linear partial differential equation (PDE) [1]. This approach replaces the original problem with one of solving an integral equation that is defined on the boundary. Since the new problem is of lower dimension, its numerical solution can be far less demanding than the equivalent finite element or difference schemes. In particular, for the case of an homogeneous PDE, it will only be necessary to consider a boundary discretization. Unfortunately, this major advantage disappears when the PDE is inhomogeneous since the resulting integral equation will include an integral over the interior and so necessitate a discretization of the same complexity as the equivalent finite element procedure. Furthermore, if the solution is required for a large amount of post-processing, the extra integral can prove a costly feature.

One way of removing the interior integral is to consider a particular solution to the homogeneous PDE. The remainder of the solution will then satisfy a homogeneous PDE and hence lead to an integral equation with only boundary contributions. The idea behind combining a particular solution to the solution of a homogeneous problem was discussed in Buzbee et al. [2], who named the method the method of the capacitance matrix method. Although the procedure sounds simple, the shortage of closed form particular solutions makes a general implementation difficult. The problem, however, can be easily overcome if one is prepared to accept an approximate particular solution. This can be derived in several ways, but arguably the simplest approach is to represent the inhomogeneity in terms of simpler functions for which the solution is known. In reference [3], this procedure was adopted for a specialised PDE and it is the purpose of the present work to show how the method can be extended to other equations.

The secret to success lies with the choice of approximation basis. Although it is possible to use trigonometric [4] or Chebyshev polynomials, such representations can require a great number of terms. This is especially true when the inhomogeneity has a localised anomaly such as a peak. This is partly offset by the time saving afforded by the use of the fast Fourier Transform. However, such features can often lead to a deterioration in the approximation, even at points distant from the anomaly. A further disadvantage is the difficulty of deriving representations in the case of irregular boundaries. All of these problems would suggest a piecewise approximation of the type that has been effective in the finite element method (FEM); for examples, the works by Nardini and Brebbia [5], Banerjee et al. [6], Henry and Banerjee [7], Saigal et al. [8], and Wilson et al. [9]. The corresponding basis functions, however, would yield complicated particular solutions with two distinct regions of analyticity. What is needed is a basis that yields simple particular solutions with only one region of analyticity. Furthermore, each basis element should give negligible contribution outside a finite neighbourhood and so result in an almost piecewise approximation. The Gaussian distribution has the required properties and was found to be effective in the works of Coleman [3], and Zheng et al. [10]. Gaussians, however, come from a more general class known as radial basis functions [11] which, in recent years, have become popular for multidimensional interpolation. Since many of this class have the desired behaviour, together with other useful properties, the present work concentrates on them.

## 2. Poisson equation

Consider the Poisson equation

$$\nabla^2\Phi = f(\boldsymbol{r}) \tag{1}$$

over a region $R$ with boundary $\partial R$. Given suitable boundary values, a solution can be derived from the integral formulation

$$c(\boldsymbol{r}_0)\Phi(\boldsymbol{r}_0) + \int_{\partial R}\left(G(\boldsymbol{r}, \boldsymbol{r}_0)\frac{\partial\Phi}{\partial n}(\boldsymbol{r}) - \Phi(\boldsymbol{r})\frac{\partial G}{\partial n}(\boldsymbol{r}, \boldsymbol{r}_0)\right) dS(\boldsymbol{r}) + \int_R f(\boldsymbol{r})G(\boldsymbol{r}, \boldsymbol{r}_0)\, dV(\boldsymbol{r}) = 0 \tag{2}$$

where $G$ satisfies

$$\nabla^2 G = \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

and

$$c = \begin{cases} \frac{1}{2} & \boldsymbol{r}_0 \in \partial R \text{ and } \partial R \text{ smooth at } \boldsymbol{r}_0 \\ 1 & \boldsymbol{r}_0 \in R - \partial R \end{cases}$$

When (2) is solved by numerical means, the process will constitute a BIM. For the homogeneous case $f \equiv 0$, this can result in considerable saving over the equivalent FEM since only the boundary need be discretized. Obviously, this advantage is lost when $f \not\equiv 0$. There is, however, an alternate approach that retains some of the advantage of the homogeneous BIM. This consists of finding a particular solution $\Phi_p(\nabla^2 \Phi_p = f)$ and then solving the integral equation

$$c(\boldsymbol{r}_0)\Phi_c(\boldsymbol{r}_0) + \int_{\partial R} \left( G(\boldsymbol{r}, \boldsymbol{r}_0) \frac{\partial \Phi_c}{\partial n}(\boldsymbol{r}) - \Phi_c(\boldsymbol{r}) \frac{\partial G}{\partial n}(\boldsymbol{r}, \boldsymbol{r}_0) \right) dS(\boldsymbol{r}) = 0 \tag{3}$$

for the remainder of the solution ($\Phi = \Phi_c + \Phi_p$). In general, however, a closed form particular solution is impractical and so recourse must be made to an approximate solution. Nevertheless, such a solution can still be far less demanding than the corresponding integral over $R$.

In the present work, the particular solution is obtained by representing the inhomogeneity in terms of basis functions $\psi_i$ ($i = 1$ to $N$) for which the particular solutions are known, that is

$$f(\boldsymbol{r}) \simeq \sum_{i=1}^{N} \alpha_i \psi_i(\boldsymbol{r}) \tag{4}$$

The functions $\psi_i$ are chosen to have the form

$$\psi_i(\boldsymbol{r}) = \psi\left( \frac{|\boldsymbol{r} - \boldsymbol{r}_i|}{\beta_i} \right) \tag{5}$$

where $\beta_i$ is some constant and $\psi$ is a function of a single variable. Such functions are known as radial basis functions [11]. If $\phi(r)$ is a radial solution corresponding to the inhomogeneity $\psi(r)$, then

$$\phi_i(\boldsymbol{r}) = \beta_i^2 \phi\left( \frac{|\boldsymbol{r} - \boldsymbol{r}_i|}{\beta_i} \right) \tag{6}$$

will be a solution corresponding to inhomogeneity $\psi_i(r)$. Since $\phi(r)$ satisfies an ordinary differential equation, there is rarely any difficulty in finding a solution. If a closed form analytic solution is not available, standard techniques will provide a numerical solution.

Consider the 2D Laplace equation with inhomogeneity $\psi(r) = \exp(-r^2)$, then

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = \exp(-r^2) \tag{7}$$

from which

$$\frac{\partial \phi}{\partial r} = \frac{1}{2r}(1 - \exp(-r^2))$$

after imposing the condition $\partial\phi/\partial r = 0$ at $r = 0$. Choosing $\phi = 0$ at $r = 0$, this can be further integrated to yield

$$\phi(r) = \tfrac{1}{2}\left(\ln(r) + \tfrac{1}{2}E_1(r^2) + \frac{\gamma}{2}\right) \tag{8}$$

where $E_1$ is the exponential integral and $\gamma$ is Euler's constant.

A further example is given by $\psi(r) = (1 + r^2)^n$, for which

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = (1 + r^2)^n \tag{9}$$

On integrating,

$$\frac{\partial\phi}{\partial r} = \frac{(r^2+1)^{1+n} - 1}{2r(1+n)} \quad n \neq -1 \tag{10}$$

after imposing the condition $\partial\phi/\partial r = 0$ at $r = 0$. Integrating again,

$$\phi(r) = \begin{cases} \tfrac{1}{3}\left(\dfrac{(r^2+1)^{3/2}}{3} + (r^2+1)^{1/2} - \ln(1 + (r^2+1)^{1/2})\right) & \text{for } n = \tfrac{1}{2} \\ (r^2+1)^{1/2} - \ln(1 + (r^2+1)^{1/2}) & \text{for } n = -\tfrac{1}{2} \\ \ln(1 + (r^2+1)^{1/2}) & \text{for } n = -\tfrac{3}{2} \end{cases}$$

after imposing the condition $\phi = 0$ at $r = 0$. The above choice of $n$ yield some popular radial basis functions. It should be noted, however, that the case $n = \tfrac{1}{2}$ does not have the required decay properties.

For a given inhomogeneity, the next stage is to decide on values for the $\boldsymbol{r}_i$ and $\beta_i$. Although it is possible to place $\boldsymbol{r}_1$ to $\boldsymbol{r}_N$ on a regular grid, this can be wasteful. Since the method is not dependent on such a grid, it is perhaps better to concentrate the $\boldsymbol{r}_i$ in regions where the function has its most rapid variation. Obviously, the corrsponding $\beta_i$ must be carefully chosen since too small $\beta_i$ will result in an approximation of isolated peaks and too large $\beta_i$ will make the approximation procedure ill-conditioned. In practice, it was found that $\beta_i$ shoud be of the same order as the distance of the closest neighbours to $\boldsymbol{r}_i$ and an effective value was found to be the average distance of immediate neighbours. Collocation at the points $\boldsymbol{r}_1$ to $\boldsymbol{r}_N$ will then provide a system of $N$ equations that determine the $N$ unknowns $f_1$ to $f_N$. Although the solution of the resulting system can be computationally expensive, the use of localised basis functions makes the system so well conditioned that iterative procedures such as the conjugate gradient method are rapidly convergent. Furthermore, especially in the case $\psi(r) = \exp(-r^2)$, the system matrix will be effectively sparse. Once the $\alpha_i$ are known, the particular integral will be given by

$$\Phi_p \simeq \sum_{i=1}^{N} \alpha_i\phi_i(\boldsymbol{r}) \tag{11}$$

The boundary conditions on $\Phi_c$ can now be determined and $\Phi_c$ itself derived from boundary integral equation (3) with kernel

$$G(\boldsymbol{r}, \boldsymbol{r}_0) = \frac{1}{2\pi} \ln|\boldsymbol{r} - \boldsymbol{r}_0| \tag{12}$$

The solution is now complete.

The above considerations can be readily extended to the 3 dimensional case. For the exponential inhomogeneity $\psi(r) = \exp(-r^2)$,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \exp(-r^2) \tag{13}$$

from which

$$\frac{\partial^2 \hat{\phi}}{\partial r^2} = r \exp(-r^2)$$

where $\hat{\phi} = r\phi$. On integrating,

$$\frac{\partial \hat{\phi}}{\partial r} = -\tfrac{1}{2} \exp(-r^2)$$

after imposing $\partial\hat{\phi}/\partial r = -\frac{1}{2}$ at $r = 0$. Furthermore

$$\phi(r) = -\frac{\sqrt{\pi}}{4r} \operatorname{erf}(r)$$

after imposing $\phi = -\frac{1}{2}$ at $r = 0$. Corresponding to

$$\psi(r) = (1 + r^2)^n$$

it can be shown that

$$\phi(r) = \begin{cases} \frac{1}{2}\ln(r^2+1) - 2 + \dfrac{2}{r}\tan^{-1}(r) & \text{for } n = -1 \\ \frac{1}{2}\left(\sqrt{r^2+1} + \dfrac{\ln(r + \sqrt{r^2+1})}{r}\right) & \text{for } n = -\frac{1}{2} \\ \dfrac{(r^2+1)^{3/2}}{12} + \dfrac{r^2+1}{8} + \dfrac{\ln(r+\sqrt{r^2+1})}{8r} & \text{for } n = \frac{1}{2} \end{cases}$$

It should be noted that the kernel for (3) will be given by

$$G(\boldsymbol{r}, \boldsymbol{r}_0) = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_0|} \tag{14}$$

in this case.

## 2.1. Example

As a particular example, consider the Poisson equation

$$\nabla^2\Phi = \sin(\pi x)\sin(\pi y) \tag{15}$$

subject to $\Phi = 0$ on the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. This has the exact solution

$$\Phi = -\frac{1}{2\pi^2}\sin(\pi x)\sin(\pi y) \tag{16}$$

Simulations were performed for rectangular grids of uniformly spaced interpolation points. Furthermore, for the boundary integral aspect of the solution, a discretisation of 128 equal length constant elements was chosen. After several experiments, this was judged to ascribe the majority of error to the particular integral aspect. The relative error for the basis functions $\exp(-r^2)$, $(1+r^2)^{1/2}$, $(1+r^2)^{-1/2}$, and $(1+r^2)^{-3/2}$ is shown in Tables 1 to 4 respectively. As can be seen, the best performance is given by the basis function $(1+r^2)^{1/2}$. This, however, does not have the required decay properties. Of the basis functions that do, the Gaussian is clearly the best. The simulations were performed using the preferred value of $\beta_i$. For $\psi(r) = \exp(-r^2)$, and a mesh spacing of .125, doubling and halving this value gave relative errors at point $(\frac{1}{2}, \frac{1}{2})$ of .00013 and .27714 respectively. In the case of $\psi(r) = (1+r^2)^{-1/2}$, these errors were .00018 and .02381 at the same point. Clearly, doubling $\beta_i$ gives considerable improvement and halving is a disaster. It should, however, be noted that too large $\beta_i$ will lead to ill-conditioning problems in the interpolation process.

Table 1
Relative error for basis function $\exp(-r^2)$

| Grid spacing | error at (.5, .5) | error at (.25, .25) | error at (.25, .5) |
|---|---|---|---|
| .25 | .00313 | .01635 | .00968 |
| .125 | .00088 | .00115 | .00102 |
| .0625 | .00026 | .00026 | .00026 |

Table 2
Relative error for basis function $(1+r^2)^{1/2}$

| Grid spacing | error at (.5, .5) | error at (.25, .25) | error at (.25, .5) |
|---|---|---|---|
| .25 | .00193 | .00979 | .00606 |
| .125 | .00023 | .00032 | .00027 |
| .0625 | .00003 | .00006 | .00004 |

Table 3
Relative error for basis function $(1+r^2)^{-1/2}$

| Grid spacing | error at (.5, .5) | error at (.25, .25) | error at (.25, .5) |
|---|---|---|---|
| .25 | .00872 | .02022 | .01459 |
| .125 | .00184 | .00217 | .00201 |
| .0625 | .00045 | .00045 | .00045 |

Table 4
Relative error for basis function $(1+r^2)^{-3/2}$

| Grid spacing | error at (.5, .5) | error at (.25, .25) | error at (.25, .5) |
|---|---|---|---|
| .25 | .03381 | .04520 | .03956 |
| .125 | .01510 | .01559 | .01533 |
| .0625 | .01048 | .01046 | .01047 |

## 3. Extension to other equations

It is obvious that the procedures of the last section can be applied to other than the Poisson equation. The major difficulty, however, is the derivation of a particular solution corresponding to a given radial basis function. In the previous section, this problem was reduced to that of solving an ODE and so, at worst, resulted in a numerical solution consisting of an array of radial values. Indeed, the numerical solution can have advantage over the analytic form if the latter involves the evaluation of complex transcendental functions. What is required, however, is for the equation of interest to have a radial solution corresponding to a radial inhomogeneity. This is the case for several equations as illustrated in the following sections.

### *3.1. Biharmonic equation*

Consider the inhomogeneous biharmonic

$$\nabla^4\Phi = f(\mathbf{r}) \tag{17}$$

The solution to the homogeneous form ($f \equiv 0$) is given by $\Phi = \mathbf{r}\cdot\mathbf{r}\Phi_1 + \Phi_2$ where $\Phi_1$ and $\Phi_2$ both satisfy Laplaces equation and hence the integral form (3).

The technique proceeds as for the Poisson equation, the only major difference being the particular solutions that correspond to a given basis function. Let $\psi(r)$ be a radial inhomogeneity, then the corresponding particular solution $\phi(r)$ satisfies

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\right)\right)=\psi(r) \tag{18}$$

in the three dimensional case. Furthermore, for basis functions of the form

$$\psi_i(\boldsymbol{r})=\psi\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_i|}{\beta_i}\right) \tag{19}$$

the corresponding particular solution will be given by

$$\phi_i(\boldsymbol{r})=\beta_i^4\phi\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_i|}{\beta_i}\right) \tag{20}$$

Consider $\hat{\phi}=r\phi$, then

$$\frac{\partial\hat{\phi}^4}{\partial r^4}=r\psi(r)$$

from which

$$\frac{\partial\hat{\phi}^2}{\partial r^2}=\frac{\sqrt{\pi}}{4}\operatorname{erfc}(r)$$

when $\psi(r)=\exp(-r^2)$. Furthermore, it can be shown [11] that

$$\int\int\operatorname{erfc}(r)\,dr=\frac{r^2}{2}\operatorname{erfc}(r)-\frac{r}{2\sqrt{\pi}}(\exp(-r^2)-1)+\tfrac{1}{4}\left(\operatorname{erfc}(r)-1+\frac{2r}{\sqrt{\pi}}\right)$$

and so

$$\phi(r)=-\tfrac{1}{8}\left(\left(r+\frac{1}{2r}\right)\sqrt{\pi}\operatorname{erf}(r)+\exp(-r^2)-2\right) \tag{21}$$

after adding suitable eigensolutions. For two dimensional problems,

$$\phi(r)=\tfrac{1}{8}\left(r^2\left(\ln r-1+\frac{\gamma}{2}\right)+\ln r+\frac{\gamma}{2}+\frac{E_1(r^2)}{2}-\frac{E_2(r^2)}{2}\right) \tag{22}$$

when $\psi(r)=\exp(-r^2)$. Also, for

$$\psi(r)=(1+r^2)^n$$

the particular solutions $\phi(r)$ for three dimensional problems will be

$$\phi(r) = \begin{cases} -\left(\dfrac{r^4}{360} + \dfrac{7r^2}{360} - \dfrac{9}{160}\right)\sqrt{1+r^2} - \left(\dfrac{r}{16} - \dfrac{1}{96r}\right)\ln(r + \sqrt{1+r^2}) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n = \frac{1}{2} \\ \left(\dfrac{13}{48} - \dfrac{r^2}{24}\right)\sqrt{1+r^2} + \left(\dfrac{1}{16r} - \dfrac{r}{4}\right)\ln(r + \sqrt{1+r^2}) & \text{for } n = -\frac{1}{2} \\ -\dfrac{3}{4}\sqrt{1+r^2} + \left(\dfrac{r}{2} - \dfrac{1}{4r}\right)\ln(r + \sqrt{1+r^2}) & \text{for } n = -\frac{3}{2} \end{cases}$$

Results for two-dimensional can also be easily derived, but they are not recorded here for brevity.

### 3.2. *Elasticity equation*

Consider an isotropic linear elastic continuum $R$ with boundary $\partial R$. The governing equations are

$$\sigma_{ij,j} + f_i = 0 \tag{23}$$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \eta(u_{ij} + u_{j,i}) \tag{24}$$

where $\sigma_{ij}$ is the stress tensor, $u_i$ the displacement field, $f_i$ a known inhomogeneity such as body force and $\lambda$ and $\eta$ are the Lamé constants for the medium. Combining equations (23) and (24) yields Navier's equations in terms of displacements

$$\nabla^2 u_i + \frac{1}{1-2v} u_{k,ik} + \frac{f_i}{\mu} = 0 \tag{25}$$

where $v$ is Poisson's ratio. To obtain a form of (25) which will be more amenable to later analytic solution it is usual to write the displacements in terms of the Galerkin vector $G_i$ [12] to yield

$$u_i = \nabla^2 G_i - \frac{1}{2(1-v)} G_{k,ik} \tag{26}$$

which upon substitution into (25) gives the following biharmonic equation for $G_i$

$$\nabla^4 G_i + \frac{f_i}{\mu} = 0 \tag{27}$$

Also, differentiating (27) and using (24) it can be shown that the stress tensor will be given by

$$\sigma_{ij} = \frac{\mu}{(1-v)}(vG_{k,nnk}\delta_{ij} - G_{k,ijk} + (1-v)(G_{i,jkk} + G_{j,ikk})) \tag{28}$$

The boundary integral formulation is well documented for elasticity problems [13, 14] and results in the following integral equation

$$c_{ij}(\boldsymbol{r}_0)u_j(\boldsymbol{r}_0) = \int_{\partial R} u_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)t_j(\boldsymbol{r})\,dS(\boldsymbol{r}) - \int_{\partial R} t_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)u_j(\boldsymbol{r})\,dS(\boldsymbol{r}) - \int_R u_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)f_j(\boldsymbol{r})\,dV(\boldsymbol{r}) \tag{29}$$

where $t_j$ represents the traction field, $u_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)$ is Kelvin's fundamental solution for displacement at $\boldsymbol{r}$ due to a system of unit point loads acting at $\boldsymbol{r}_0$ in an infinite elastic medium, $t_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)$ its associated traction and

$$c_{ij} = \begin{cases} \frac{1}{2}\delta_{ij} & \boldsymbol{r}_0 \in \partial R \text{ and } \partial R \text{ smooth at } \boldsymbol{r}_0 \\ \delta_{ij} & \boldsymbol{r}_0 \in R - \partial R \end{cases}$$

Proceeding as for Poisson's equation, the domain integral may be removed by finding a particular solution to (23) satisfying

$$\sigma_{ij,j}^p + f_i = 0$$

but not necessarily satisfying the boundary conditions for the problem. The remainder of the solution may be then obtained from

$$c_{ij}(\boldsymbol{r}_0)u_j^c(\boldsymbol{r}_0) = \int_{\partial R} u_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)t_j^c(\boldsymbol{r})\,dS(\boldsymbol{r}) - \int_{\partial R} t_{ij}^*(\boldsymbol{r}, \boldsymbol{r}_0)u_j^c(\boldsymbol{r})\,dS(\boldsymbol{r}) \tag{30}$$

where

$$u_i = u_i^c + u_i^p \tag{31}$$

$$t_i = t_i^c + t_i^p \tag{32}$$

Consider now an inhomogeneity approximated by a series of radial basis functions $\psi_i(\boldsymbol{r})$ ($i = 1$ to $N$)

$$\boldsymbol{f}(\boldsymbol{r}) \simeq \sum_{i=1}^{N} \boldsymbol{\alpha}_i \psi_i(\boldsymbol{r}) \tag{33}$$

where the $\boldsymbol{\alpha}_i$ are constants obtained from the fitting procedure. Writing $\phi(r) = G_i/\alpha_i$ and using (27) results in

$$\nabla^4\phi + \frac{1}{\mu}\psi(r) = 0 \tag{34}$$

Clearly the above equation is now in a suitable form for which the results of the biharmonic may be used to derive particular solutions for the displacement and traction fields. Using (2), the particular solution for the displacement may be written as

$$u_i^p = \alpha_i\phi_1 - \frac{1}{2(1-\nu)}(\alpha_i\phi_2 + r_{,i}r_{,k}\alpha_k\phi_3) \tag{35}$$

where

$$\phi_1 = \phi'' + \frac{2}{r}\phi' \tag{36}$$

$$\phi_2 = \frac{1}{r}\phi' \tag{37}$$

$$\phi_3 = \phi'' - \frac{1}{r}\phi' \tag{38}$$

From (28) the particular solution for the stress will be

$$\sigma^p_{ij} = \frac{\mu}{(1-\nu)}(\nu r_{,k}\alpha_k\phi_4\delta_{ij} - (r_{,i}\delta_{jk} + r_{,j}\delta_{ik} + r_{,k}\delta_{ij})\alpha_k\phi_5 - r_{,i}r_{,j}r_{,k}\alpha_k\phi_6 + (1-\nu)(\alpha_i r_{,j} + \alpha_j r_{,i})\phi_4) \tag{39}$$

where

$$\phi_4 = \phi''' + \frac{2}{r}\phi'' - \frac{2}{r^2}\phi' \tag{40}$$

$$\phi_5 = \frac{1}{r}\phi'' - \frac{1}{r^2}\phi' \tag{41}$$

$$\phi_6 = \phi''' - \frac{3}{r}\phi'' + \frac{3}{r^2}\phi' \tag{42}$$

from which the traction may be readily derived.

### *3.2.1. Example: Plane strain rotating disk*

As a particular example, consider the rotation of a hollow shaft inner radius $a$ outer radius $b$ rotating with angular velocity $\omega$ with ends constrained between two frictionless platters. The analytical solutions for this problem are given by [15]

$$\sigma_{rr} = A + \frac{B}{r^2} - \frac{1}{8}\frac{3-2\nu}{1-\nu}\varrho\omega^2 r^2 \tag{43}$$

$$\sigma_{\theta\theta} = A - \frac{B}{r^2} - \frac{1}{8}\frac{1+2\nu}{1-\nu}\varrho\omega^2 r^2 \tag{44}$$

$$\sigma_{zz} = 2\nu A - \frac{1}{2}\frac{\nu}{1-\nu}\varrho\omega^2 r^2 \tag{45}$$

$$u_r = \frac{r}{2\mu}\left((1-2\nu)A - \frac{B}{r^2} - \frac{1}{8}\frac{1-2\nu}{1-\nu}\varrho\omega^2 r^2\right) \tag{46}$$

where

$$A = \tfrac{1}{2}\varrho\omega^2(a^2 + b^2) \tag{47}$$

$$B = -\tfrac{1}{2}\varrho\omega^2 a^2 b^2, \tag{48}$$

$\varrho$ is the density, $\mu$ is the shear modulus and $\nu$ Poisson's ratio. The numerical values employed are as follows: $a = 1$, $b = 2$, $\varrho = 1$, $\mu = 1$, $\nu = 0.5$, and $\omega = 1$.

An existing three-dimensional boundary element program was modified to implement the method. The program uses linear or quadratic isoparametric elements to represent the boundary. Simulations using quadratic elements were carried out for a quarter of the domain using the mesh depicted in Figure 1. The $\beta_i$ for each point $\boldsymbol{r}_i$ was set to be the average distance from its immediate neighbours multiplied by a weighting factor $w$. Numerical experiments using weighting factors ranging from 0.5 to 2.5 showed that good solutions could be obtained with $w$ from 1.0 to 2.0. For smaller values of $w$ ($w < 1.0$), the body force was clearly underestimated as was evident from the lower stresses and displacements. For larger values of $w$ ($w > 2.0$), the approximation procedure became more ill-conditioned resulting in a deterioration of the computed solution. Table 5 showing the net force (the exact value being (0, 0.583333, 0.583333)) due to the approximation to the body force and the condition number CN of system of equations used in the fitting procedure for the Gaussian basis function clearly backs up these observations. Similar behavior was observed for basis functions of the form $\psi(r) = (1 + r^2)^n$. Table 6 summarizes the radial displacements for a range of $w$ also for the Gaussian basis function. Figures 2 to 5 and Table 7 are a comparison of the results and maximum absolute errors for several different basis functions with $w = 1.5$—corresponding to about the optimum $w$ for

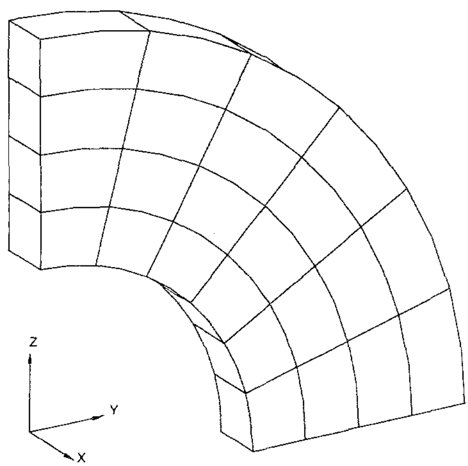


Figure 1
Boundary discretisation for rotating shaft.

Table 5
Net force on disc and condition number of fitting matrix corresponding to different values of $w$

| $w$ | $f_x$ | $f_y$ | $f_z$ | CN |
|---|---|---|---|---|
| 0.5 | $O\ (10^{-9})$ | 0.461253 | 0.461253 | $O\ (10^{0})$ |
| 1.0 | $O\ (10^{-10})$ | 0.591482 | 0.591482 | $O\ (10^{3})$ |
| 1.5 | $O\ (10^{-10})$ | 0.586814 | 0.586814 | $O\ (10^{7})$ |
| 2.0 | $O\ (10^{-8})$ | 0.587221 | 0.587221 | $O\ (10^{11})$ |
| 2.5 | $O\ (10^{-6})$ | 0.683724 | 0.684923 | $O\ (10^{15})$ |

Table 6
Radial displacement for a range of $w$ for the Gaussian basis function

| R | Radial displacement | | | | | |
|---|---|---|---|---|---|---|
| | exact | $w = 0.5$ | $w = 1.0$ | $w = 1.5$ | $w = 2.0$ | $w = 2.5$ |
| 1 | 1 | 0.76935 | 1.0205 | 1.0103 | 1.0088 | 0.92795 |
| 1.125 | 0.888889 | 0.67741 | 0.90825 | 0.89907 | 0.8976 | 0.82828 |
| 1.25 | 0.8 | 0.60516 | 0.81818 | 0.80995 | 0.80853 | 0.74769 |
| 1.375 | 0.727273 | 0.54644 | 0.74443 | 0.73707 | 0.73575 | 0.68275 |
| 1.5 | 0.666667 | 0.49775 | 0.68298 | 0.67641 | 0.67526 | 0.62987 |
| 1.625 | 0.615385 | 0.45692 | 0.63096 | 0.62512 | 0.6242 | 0.58585 |
| 1.75 | 0.571429 | 0.42256 | 0.58631 | 0.58111 | 0.58046 | 0.54972 |
| 1.875 | 0.533333 | 0.3975 | 0.54743 | 0.54273 | 0.54225 | 0.51706 |
| 2 | 0.5 | 0.36959 | 0.5129 | 0.50845 | 0.50775 | 0.48511 |

Table 7
Maximum absolute errors

| Basis function | Maximum absolute error | | | |
|---|---|---|---|---|
| | $u_r$ | $\sigma_{rr}$ | $\sigma_{\theta\theta}$ | $\sigma_{zz}$ |
| $\exp(-r^2)$ | 0.010300 | 0.60460 | 0.26050 | 0.023650 |
| $(1+r^2)^{1/2}$ | 0.000174 | 0.058750 | 0.013700 | 0.014090 |
| $(1+r^2)^{-1/2}$ | 0.001874 | 0.060787 | 0.014830 | 0.015080 |
| $(1+r^2)^{-3/2}$ | 0.008860 | 0.060529 | 0.026730 | 0.022360 |

each. Again, the best performance was given by the basis function of the form $(1+r^2)^{1/2}$. However, since this doesn't have the required decay properties it would not be expected to perform as well when the body is not such a simple form. The basis function $(1+r^2)^{-1/2}$ gave the best results from the group having the required decay properties. It should be noted that the poorer results for $\sigma_{rr}$ are due to their derivation from the displacement field, whereas the other results could be obtained directly from the boundary tractions and displacements.

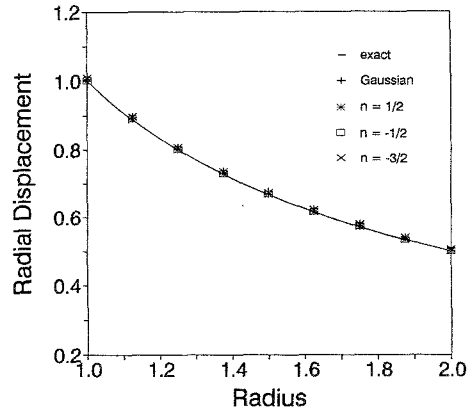


Figure 2
Radial displacement $u_r$ with $w = 1.5$ for basis functions $\exp(-r^2)$, $(1+r^2)^{1/2}$, $(1+r^2)^{-1/2}$, and $(1+r^2)^{-3/2}$.

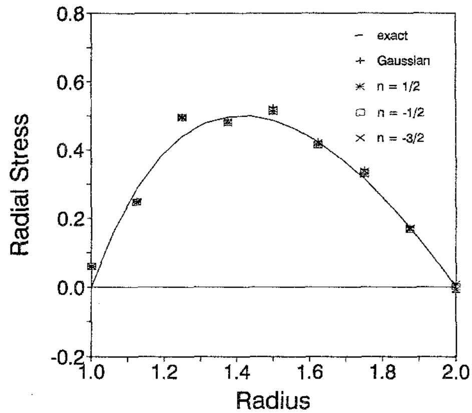


Figure 3
Radial stress $\sigma_{rr}$ with $w = 1.5$ for basis functions $\exp(-r^2)$, $(1+r^2)^{1/2}$, $(1+r^2)^{-1/2}$, and $(1+r^2)^{-3/2}$.

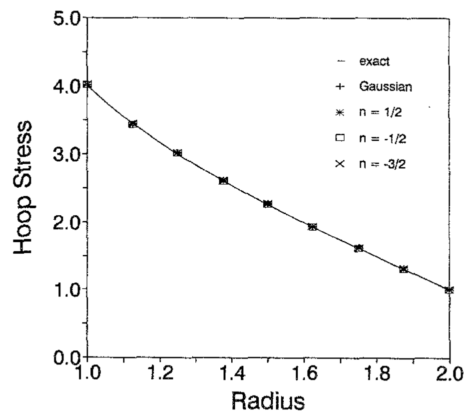


Figure 4
Hoop stress $\sigma_{\theta\theta}$ with $w = 1.5$ for basis functions $\exp(-r^2)$, $(1+r^2)^{1/2}$, $(1+r^2)^{-1/2}$, and $(1+r^2)^{-3/2}$.

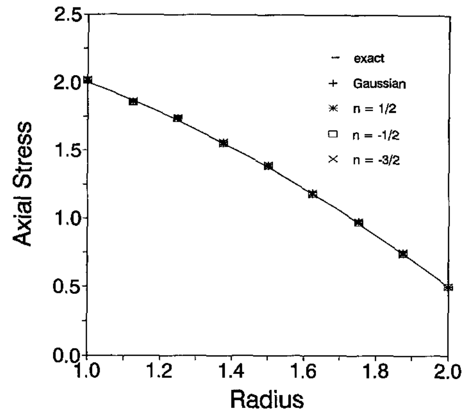


Figure 5
Axial stress $\sigma_{zz}$ with $w = 1.5$ for basis functions $\exp(-r^2)$, $(1+r^2)^{1/2}$, $(1+r^2)^{-1/2}$, and $(1+r^2)^{-3/2}$.

## 4. Concluding remarks

A method has been developed which effectively removes the domain integral arising in inhomogeneous linear PDE's by approximating the inhomogeneity in terms of radial basis functions and thus resulting in an integral equation involving boundary values alone. The method is completely general, and unlike previous methods using trigonometric or Chebyshev polynomials it adapts to irregular regions easily. The method is also particularly suited to problems in which the inhomogeneity has localised anomalies such as peaks, since, an approximation using radial basis functions is effectively piecewise and may concentrate sample points around the anomaly with no effect on the approximation elsewhere. This contrasts with polynomial approximations in which many terms are required to accurately model an anomaly, and as such, often lead to a deterioration in the approximation.

Example inhomogeneous potential and elasticity problems have been presented and accurate solutions have been achieved in both cases. The method may be extended to more complex problems where the inhomogeneity is unknown *a priori*. Such is the case for modelling flows of visco-elastic fluids and flows with inertia. This will be the subject of future investigations.

## 5. Acknowledgements

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**Abstract**

A method for removing the domain or volume integral arising in boundary integral formulations for linear inhomogeneous partial differential equations is presented. The technique removes the integral by considering a particular solution to the homogeneous partial differential equation which approximates the inhomogeneity in terms of radial basis functions. The remainder of the solution will then satisfy a homogeneous partial differential equation and hence lead to an integral equation with only boundary contributions. Some results for the inhomogeneous Poisson equation and for linear elastostatics with known body forces are presented.

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