Z angew Math Phys 46 (1995)
(ZAMP)

0044-2275/95/020289-06 \$ 1.50 + 0.20

A rigidity result for biharmonic functions clamped at a corner

By J. B. Boyling, Dept of Applied Mathematical Studies, University of Leeds, Leeds, LS2 9JT, England (Fax (0532) 429925)

1. Introduction

It is easily verified that the function

$$u_0(x, y) = \frac{1}{8\pi}(x^2 + y^2)[(\ln\sqrt{x^2+y^2}) - 1]$$

satisfies

$$\Delta^2 u_0 = \delta_0.$$

$G_0(x, y; \xi, \eta) = u_0(x - \xi, y - \eta)$ is therefore a Green's function for the biharmonic equation, which it is convenient to represent in the complex form

$$G_0(x, y; \xi, \eta) = \frac{1}{8\pi} \operatorname{Re}[|z - \zeta|^2\{\ln(z - \zeta) - 1\}],$$

where $z = x + iy$, $\zeta = \xi + i\eta$. Further Green's functions may be obtained, satisfying various boundary conditions [1]. For example

$$G_1(x, y; \xi, \eta) = \frac{1}{8\pi} \operatorname{Re}\left[|z - \zeta|^2 \ln\left(\frac{z^{1/2} - \zeta^{1/2}}{z^{1/2} + \bar{\zeta}^{1/2}}\right)\right] \tag{1}$$

vanishes together with its normal derivative on the negative real axis,

$$G_2(x, y; \xi, \eta) = \frac{1}{8\pi} \operatorname{Re}\left[|z - \zeta|^2 \ln\left(\frac{z - \zeta}{z + \bar{\zeta}}\right) + (\zeta + \bar{\zeta})z\right] \tag{2}$$

does so on the imaginary axis [2–5], and

$$G_3(x, y; \xi, \eta) = \frac{1}{8\pi} \operatorname{Re}\left[|z - \zeta|^2 \ln\left(\frac{z - \zeta}{1 - \bar{\zeta}z}\right) + \frac{(|\zeta|^2 - 1)(|z|^2 - 1)}{2}\right] \tag{3}$$

does so on the circle $|z| = 1$ [6–8], provided in each case ζ does not lie on the boundary in question. In none of these cases does the Green's function have a singularity on the (smooth) boundary.

The Green's functions (1)–(3) are strikingly similar in form to the corresponding Green's functions for Laplace's equation, easily obtained by the method of images and the use of conformal transformations for either Dirichlet or Neumann boundary conditions. This analogy might lead one to suppose that similarly simple Green's functions for the biharmonic equation could be constructed for more complicated regions such as sectors. In the Laplace case, the sector

$$z \neq 0, \qquad -\frac{1}{2}\alpha < \arg z < \frac{1}{2}\alpha \leq \pi$$

has the simple Green's function

$$\frac{1}{2\pi}\operatorname{Re}\left[\ln\left(\frac{z^{\pi/\alpha} - \zeta^{\pi/\alpha}}{z^{\pi/\alpha} + \bar{\zeta}^{\pi/\alpha}}\right)\right]$$

vanishing on the boundary of the sector, where ζ is inside the sector. The expression in square brackets is holomorphic in z at the corner $z = 0$ provided π/α is an integer, and otherwise has a logarithmic or finitely-sheeted branch point there. However, it seems most unlikely that any such simple Green's function can be constructed for a general sector in the biharmonic case. Indeed, Dean and Montagnon [9] have shown that, provided n (in general complex) satisfies

$$\sin n\alpha = \pm n \sin \alpha, \tag{4}$$

then the real and imaginary parts of the complex expression

$$(z^{n+1} \mp \bar{z}^{n+1})n \sin(n-1)\alpha + (-\bar{z}^n \bar{z} \pm z\bar{z}^n)(n+1) \sin n\alpha$$

are both biharmonic functions whose first order partial derivatives both vanish on the boundary of the sector. Any Green's function would be expected in general to contain terms with the above types of singular behaviour at $z = 0$ for the different roots n of the transcendental equations (4) [10]. It would therefore in general have a complicated singularity at $z = 0$ and could not be simply constructed from elementary functions, as in (1)–(3). Only in the special cases $\alpha = 2\pi$ and $\alpha = \pi$ are the solutions of (4) all integers, and these are precisely the cases already covered by (1) and (2) respectively.

We shall now prove a result which implies that the inhomogeneous biharmonic Green's function for a sector of angle α satisfying $0 < \alpha < \pi$ must be singular at $z = 0$. In fact we shall show that, if u is any non-constant biharmonic function whose first order partial derivatives both vanish on a curve which contains a conformal image of a pair of line segments meeting at an angle α (e.g. a pair of circular arcs meeting at this angle), then u must be singular at the corner. This result has relevance to numerical work in elastostatics [11–14] and steady viscous flow [2–5, 7, 8], where Green's functions are used extensively.

2. The main result and its proof

Suppose the function $h(\zeta)$ is holomorphic, with $h'(\zeta) \neq 0$, on the open disc $U = \{\zeta \in \mathbb{C}; |\zeta| < \delta\}$. Let V be the image in the z-plane of U under the conformal transformation $z = h(\zeta)$, define

$$\Gamma = \Gamma_1 \cup \Gamma_2, \qquad \Gamma_1 = \{r\, e^{i\alpha/2}; 0 \leq r < \delta\}, \qquad \Gamma_2 = \{r\, e^{-i\alpha/2}; 0 \leq r < \delta\},$$

where $0 < \alpha < \pi$, and let C, C_1, C_2 ($\subset V$) be the images of Γ, Γ_1, Γ_2 ($\subset U$) under h. Then C is an analytic contour in the z-plane with a cusp of angle α at $z = h(0)$. It is the image under the transformation $z = h(Z^{\alpha/\pi})$ (conformal except at $Z = 0$) of the open segment $(-i\delta^{\pi/\alpha}, i\delta^{\pi/\alpha})$ of the imaginary axis of the Z-plane.

Theorem. Suppose $u(x, y)$ is a biharmonic function on V analytic in the real variable $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ (even at the point $z = h(0)$), such that $\partial u/\partial x = \partial u/\partial y = 0$ on C. Then $u(x \cdot y)$ is constant (on V).

Proof. Following Muskhelishvili [15], we express the biharmonic function u in the form

$$u(x, y) = \operatorname{Re} w(z), \tag{5}$$

where

$$w(z) = f(z) + \bar{z} g(z), \tag{6}$$

for some functions $f(z)$ and $g(z)$ holomorphic on V. It will be more convenient for our purposes to express the "complex bipotential" w in terms of ζ rather than z, writing

$$w(z) = W(\zeta), \tag{7}$$

where

$$W(\zeta) = F(\zeta) + \overline{h(\zeta)} G(\zeta), \tag{8}$$

and $F = f \circ h$ and $G = g \circ h$ are both holomorphic on U.

It follows from (5) that

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial z} = \frac{\partial w}{\partial z} + \overline{\left(\frac{\partial w}{\partial \bar{z}} \right)}, \tag{9}$$

where

$$\frac{\partial w}{\partial \bar{z}} = g(z) = G(\zeta)$$

by (6), and

$$\frac{\partial w}{\partial z} = \frac{1}{h'(\zeta)} \frac{\partial W}{\partial \zeta} = \frac{1}{h'(\zeta)} [F'(\zeta) + \overline{h(\zeta)} G'(\zeta)]$$

by (8). Thus

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{1}{h'(\zeta)} [F'(\zeta) + h^*(\bar{\zeta}) G'(\zeta)] + G^*(\bar{\zeta}), \tag{10}$$

where, for any holomorphic function $\phi(\zeta)$ on U, ϕ^* denotes the holomorphic function on U defined by

$$\phi^*(\zeta) = \overline{\phi(\bar{\zeta})}. \tag{11}$$

By hypothesis, (10) vanishes on Γ. It therefore vanishes on Γ_1, where $\bar{\zeta} = e^{-i\alpha}\zeta$, and so

$$F'(\zeta) + h^*(e^{-i\alpha}\zeta) G'(\zeta) = -h'(\zeta) G^*(e^{-i\alpha}\zeta) \tag{12}$$

on Γ_1. But both sides of (12) are holomorphic functions of ζ on U, so (12) must hold throughout U. Similarly (10) vanishes on Γ_2, where $\bar{\zeta} = e^{i\alpha}\zeta$, and so

$$F'(\zeta) + h^*(e^{i\alpha}\zeta) G'(\zeta) = -h'(\zeta) G^*(e^{i\alpha}\zeta) \tag{13}$$

throughout U. From (12) and (13), we deduce that

$$F'(\zeta) = \frac{h'(\zeta)[h^*(e^{-i\alpha}\zeta) G^*(e^{i\alpha}\zeta) - h^*(e^{i\alpha}\zeta) G^*(e^{-i\alpha}\zeta)]}{h^*(e^{i\alpha}\zeta) - h^*(e^{-i\alpha}\zeta)} \tag{14}$$

and

$$G'(\zeta) = -\frac{h'(\zeta)[G^*(e^{i\alpha}\zeta) - G^*(e^{-i\alpha}\zeta)]}{h^*(e^{i\alpha}\zeta) - h^*(e^{-i\alpha}\zeta)}, \tag{15}$$

and hence from (10) that

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{h^*(e^{-i\alpha}\zeta)G^*(e^{i\alpha}\zeta) - h^*(e^{i\alpha}\zeta)G^*(e^{-i\alpha}\zeta)}{h^*(e^{i\alpha}\zeta) - h^*(e^{-i\alpha}\zeta)} - \frac{h^*(\bar{\zeta})[G^*(e^{i\alpha}\zeta) - G^*(e^{-i\alpha}\zeta)]}{h^*(e^{i\alpha}\zeta) - h^*(e^{-i\alpha}\zeta)} + G^*(\bar{\zeta}). \tag{16}$$

Suppose now that the holomorphic functions h and G have in U the Taylor expansions

$$h(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n, \qquad G(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^n. \tag{17}$$

Substituting these into (15) gives

$$\left[\sum_{k=1}^{\infty} kb_k\zeta^{k-1}\right]\left[\sum_{m=1}^{\infty} \bar{a}_m(e^{im\alpha} - e^{-im\alpha})\zeta^m\right] + \left[\sum_{k=1}^{\infty} ka_k\zeta^{k-1}\right]\left[\sum_{m=1}^{\infty} \bar{b}_m(e^{im\alpha} - e^{-im\alpha})\zeta^m\right] = 0. \tag{18}$$

By considering the coefficient of ζ^n, we deduce that

$$\sum_{k=1}^{n} k[e^{i(n+1-k)\alpha} - e^{-i(n+1-k)\alpha}](b_k\bar{a}_{n+1-k} + a_k\bar{b}_{n+1-k}) = 0 \qquad \text{for } n \geq 1. \tag{19}$$

Putting $n = 1$ in (19) gives

$$(e^{i\alpha} - e^{-i\alpha})(b_1\bar{a}_1 + a_1\bar{b}_1) = 0.$$

Since $a_1 = h'(0) \neq 0$ by hypothesis, we must have

$$\frac{b_1}{a_1} = -\frac{\bar{b}_1}{\bar{a}_1} = ir \tag{20}$$

for some real number r. We now prove by induction on n that

$$b_n = ira_n \tag{21}$$

for all positive integers n. Suppose that $b_k = ira_k$ for $k = 1, \ldots, n-1$, where $n \geq 2$. Then the summand in (19) vanishes for $k = 2, \ldots, n-1$, and (19) reduces to

$$(e^{in\alpha} - e^{-in\alpha})a_1(\bar{b}_n + ir\bar{a}_n) + n(e^{i\alpha} - e^{i\alpha})\bar{a}_1(b_n - ira_n) = 0,$$

i.e.

$$b_n - ira_n = -\left(\frac{\sin n\alpha}{n\sin\alpha}\right)\left(\frac{a_1}{\bar{a}_1}\right)\overline{(b_n - ira_n)}. \tag{22}$$

Since $0 < \alpha < \pi$, the quantity

$$\frac{\sin n\alpha}{n\sin\alpha} = \frac{e^{i(n-1)\alpha} + e^{i(n-3)\alpha} + \cdots + e^{-i(n-1)\alpha}}{n}$$

has modulus strictly less than 1 (implying that (4) has no integer solutions n of modulus greater than 1), and we conclude from (22) that $b_n - ira_n = 0$, thus completing the proof of (21).

From (17) and (21), we now deduce that

$$G(\zeta) = irh(\zeta) + c, \tag{23}$$

where

$$c = b_0 - ira_0.$$

Substituting (23) into (16) now shows that $\partial u/\partial x - i(\partial u/\partial y) = 0$ throughout V, i.e. that $u = \text{constant}$ in V.

3. The special case of a sector

This special case corresponds to putting

$$z = h(\zeta) = \zeta, \qquad F = f, \qquad G = g.$$

In this case, the conclusion of the theorem holds under the weaker assumption that the functions $f(z)$ and $g(z)$ have at worst isolated singularities at $z = 0$.

To prove this, note first that (14) and (15) reduce to

$$f'(z) = \frac{[e^{-i\alpha}g^*(e^{i\alpha}z) - e^{i\alpha}g^*(e^{-i\alpha}z)]}{(e^{i\alpha} - e^{-i\alpha})}, \tag{24}$$

$$g'(z) = -\frac{[g^*(e^{i\alpha}z) - g^*(e^{-i\alpha}z)]}{(e^{i\alpha} - e^{-i\alpha})z}. \tag{25}$$

Substituting the Laurent expansion $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ into (25) gives

$$\sum_{n=-\infty}^{\infty} nb_n z^{n-1} = -\sum_{n=-\infty}^{\infty} \bar{b}_n \left(\frac{\sin n\alpha}{\sin \alpha}\right) z^{n-1},$$

from which we infer that $b_n = 0$ for $|n| > 1$, and that b_1 and b_{-1} are both purely imaginary. Thus

$$g(z) = c + i(rz + r'z^{-1}), \qquad g^*(z) = \bar{c} - i(rz + r'z^{-1}), \tag{26}$$

where $c \in \mathbb{C}$, r, $r' \in \mathbb{R}$. Substituting (26) into (24) yields

$$f'(z) = -\bar{c} + (2ir' \cos \alpha)z^{-1},$$

whence $r' \cos \alpha = 0$, since $f(z)$ cannot have a logarithmic singularity at $z = 0$. Thus

$$f(z) = -\bar{c}z + \text{constant}. \tag{27}$$

Substituting (26) and (27) into (6), we see that

$$w(z) = -\bar{c}z + c\bar{z} + ir|z|^2 + i\bar{r}'\bar{z}z^{-1} + \text{constant},$$

and hence by (5) that

$$u(x, y) = -r' \operatorname{Im}(\bar{z}z^{-1}) + \text{constant}.$$

But $u(x, y)$ must have the same constant value on C_1, where $\bar{z} = e^{-i\alpha}z$, as it has on C_2, where $\bar{z} = e^{i\alpha}z$. Hence $r' \sin \alpha = 0$, $r' = 0$, and $u(x, y) = \text{constant}$.

Thus we have shown that a non-constant biharmonic function u on the sector $z \neq 0$, $-\frac{1}{2}\alpha < \arg z < \frac{1}{2}\alpha$ must be of the form given by (5) and (6), where f or g has a non-isolated singularity at $z = 0$.

We conjecture that the conclusion of the theorem of Section 2 for general h will likewise continue to hold under the weaker assumption that $f(z)$ and $g(z)$ have at worst isolated singularities at $z = h(0)$. Unfortunately the method of proof used in Section 2 breaks down in this case. This is because there is no longer a lower bound to the summation over k in the first term of (18) or the summation over m in its second term, so that there is no way to start the induction.

Acknowledgement

I wish to thank Dr. Mark Kelmanson for arousing my interest in this subject and for his subsequent help and encouragement.

References

[1] C. Pozrikidis, *Boundary Integral and Singular Methods for Linearized Viscous Flow*, Cambridge University Press, Cambridge 1992.
[2] E. B. Hansen, *Stokes flow down a wall into an infinite pool*, J. Fluid Mech., *178*, 243–256 (1987).
[3] E. B. Hansen and M. A. Kelmanson, *An integral equation justification of the boundary conditions of the driven-cavity problem*, Computer & Fluids, *23*, 225–240 (1994).
[4] M. A. Kelmanson and B. Lonsdale, *Annihilation of boundary singularities via suitable Green's functions*, Computational Mathematics with Applications, in press (1995).
[5] M. A. Kelmanson and B. Lonsdale, *A singularity annihilation method for steady viscous flow in the double-lid cavity*, J. Comp. Phys., submitted (1994).
[6] P. R. Garabedian, *Partial Differential Equations*, Wiley, New York 1964.
[7] M. A. Kelmanson, *Theoretical and numerical investigation of steady, viscous, free-surface flow*, Appl. Math. Lett., *7*, 81–85 (1994).
[8] E. B. Hansen and M. A. Kelmanson, *Steady, viscous, free-surface flow on a rotating cylinder*, J. Fluid Mech., *272*, 91–107 (1994).
[9] W. R. Dean and P. E. Montagnon, *On the steady motion of a viscous liquid in a corner*, Proc. Camb. Phil. Soc., *45*, 389–394 (1949).
[10] M. A. Kelmanson, *On the solution structure of a nonlinear transcendental eigenvalue equation*, Dynamics & Stability of Systems, *4*, 245–257 (1989).
[11] M. Maiti and S. K. Chakrabarty, *Integral equation solutions for simply supported polygonal plates*, Int. J. Engng. Sci., *12*, 793–806 (1974).
[12] A. Mir-Mohamad-Sadegh and K. R. Rajagopal, *The flow of a non-Newtonian fluid past projections and depressions*, J. Appl. Mech., *47*, 485–488 (1980).
[13] L. N. McCartney, *A new boundary element technique for solving plane problems of linear elasticity. I. Theory*, Appl. Math. Modelling, *7*, 441–451 (1983).
[14] L. N. McCartney, *A new boundary element technique for solving plane problems of linear elasticity: improved theory*, Appl. Math. Modelling, *8*, 243–250 (1984).
[15] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen 1963.

Abstract

Let C be a curve which is conformal to a pair of line segments meeting at an angle α strictly between 0 and π, u a biharmonic function analytic in a neighbourhood of C, whose gradient vanishes on C. It is shown that $u(x, y)$ must be constant.

(Received: July 4, 1994; revised: October 28, 1994)