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Higher-degree reference field in the generalized Stokes-Helmert scheme for geoid computation

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Abstract. In this paper we formulate two corrections that have to be applied to the higher-degree reference spheroid if one wants to use it in conjunction with the Stokes-Helmert scheme for geoid determination. We show that in a precise geoid determination one has to apply the correction for the residual topographical potential and the correction for the earth ellipticity. Both these corrections may reach several decimetres; we show how their magnitudes vary within Canada and we give their global ranges.

Introduction

In Vaníček and Martinec, [1994] we explain the idea behind the Stokes-Helmert scheme for precise geoid computation. We define the "Helmert potential" W^h as the difference between the real gravity potential W and the residual topographical potential, V. Neglecting the atmospheric attraction effects, the residual topographical potential in the Stokes-Helmert approach is defined as

$$V = V^t - V^c \ , \qquad (1)$$

where V^t is the potential of topographical masses trapped between the earth's surface and the geoid, and V^c is the potential of the "Helmert condensation layer" on the geoid; the use of Helmert's second method of condensation is thus implied.

In Vaníček and Sjoberg [1991] we show how the classical Stokes theory for geoid determination, conceived for computing geoidal height above a reference ellipsoid, can be reformulated for a higher-degree reference field. We argue that a higher-degree reference field, with a higher-degree reference spheroid generated as an equipotential surface of the reference field (cf., with the standard Stokes technique), accords a better utilization of local gravity data. In particular, this generalization gives a smaller truncation error when the Stokes convolution integral, applied to the observed gravity reduced for the reference field, (cf., with the standard Stokes technique) is truncated to a spherical cap of small radius.

In this paper, we look at the possibility of generalizing the Stokes-Helmert scheme in a way that would take advantage of using a higher-degree reference field. We assume that the higher-order field is described by a set of potential coefficients and investigate the steps to be taken before the coefficients can generate the reference field. We show that the potential coefficients have to undergo several corrections before they can be used to generate the reference spheroid for the Stokes-Helmert scheme and before they can generate the reference gravity to be subtracted from the gravity observed at the earth's surface and reduced to the geoid.

Throughout our derivations, we assume that the potential coefficients we are interested in using were obtained from satellite orbit analyses. The reasons for limiting ourselves to only a relatively low-degree reference field and to solely satellite-determined potential coefficients are explained in Vaníček and Sjoberg [1991]. The optimal degree of reference field we have settled on is 20, and we have used this cutoff degree extensively (cf., Vaníček and Kleusberg [1987] and Vaníček et al. [1990]). We use this cutoff degree L=20 also in this paper; the expressions for the corrections are given for an arbitrary degree L but the numerical results are obtained for L=20.

Transformation of the Reference Field into the Helmert model

Let us begin by writing the usual spectral expression for the gravity potential; namely,

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$$W(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=-n}^{n} W_{nm} Y_{nm}(\Omega), \quad (2)$$

where GM is the geocentric constant, W_{nm} are the potential coefficients, r is the length of the radius vector from the center of mass of the earth, a is the major semi-axis of the reference ellipsoid used, Y_{nm} is the normalized spherical function of degree n and order m, and Ω is the geocentric direction defined by geocentric co-latitude θ and longitude λ. The reference potential consists of the first L degrees (for L read typically 20) of the above series and it is this potential that we are interested in transforming into the Helmert model (space) by subtracting, from it, the direct topographical effect.

The residual topographical potential (eqn. (1)) has been treated in detail by Martinec and Vaníček [1994]. Here we briefly show the derivation of the expression for the effect in a spectral form. The spectral form is more convenient here because the effect can then be easily subtracted from the reference field. We begin by writing the standard Newton integral for the potential of topographical masses in spherical approximation

$$V^t(r,\Omega) \approx G\rho_0 \int_{\Omega'} \int_{z=0}^{H'} \frac{dz}{\ell(r,\Omega,R+z,\Omega')} (R+z)^2 d\Omega' \quad , \quad (3)$$

where G is Newton's universal constant, R is the mean radius of the earth, ρ_0 is the mean topographical density, H' is the terrain height reckoned along the radius vector equal to a sufficient accuracy to the orthometric height, see (Vaníček and Martinec, 1994), and ℓ is the spatial distance between points (r,Ω) and $(R+z,\Omega')$. We know that the representation of actual topographical density ρ by its mean value ρ_0 may be too coarse an approximation and we shall discuss this point later on.

We now turn to the mathematical details. Let us first develop the reciprocal distance ℓ^{-1} into an infinite series in Legendre polynomials $P_n(\cos\psi)$

$$\ell^{-1} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{R+z}{r}\right)^n P_n(\cos\psi), \quad (4)$$

convergent for $r > R+z$. Here, ψ is the geocentric angular distance between Ω and Ω'. Substitution of eqn. (4) into eqn.(3) yields

$$V^t(r,\Omega) \approx G\rho_0 \times \int_{\Omega'} \int_{z=0}^{H'} \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{R+z}{r}\right)^n P_n(\cos\psi) \times (R+z)^2 dz d\Omega', \quad (5)$$

which, for $r > R+z$, i.e., for points outside the Brillouin sphere (minimal geocentric sphere containing all the earth mass), can be rewritten as

$$V^t(r,\Omega) \approx G\rho_0 \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n+1} \times \int_{\Omega'} P_n(\cos\psi) \int_{z=0}^{H'} (R+z)^{n+2} dz d\Omega' \quad . \quad (6)$$

Since the series is convergent, the integral in z may be easily evaluated using the binomial theorem, giving

$$\int_{z=0}^{H'} (R+z)^{n+2} dz = \frac{R^{n+3}}{n+3} \sum_{k=1}^{n+3} \binom{n+3}{k} \left(\frac{H'}{R}\right)^k \quad (7)$$

so that the final expression for the topographical potential at points outside the Brillouin sphere becomes

$$V^t(r,\Omega) \approx G\rho_0 R^2 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{1}{n+3} \times \sum_{k=1}^{n+3} \binom{n+3}{k} \int_{\Omega'} \left(\frac{H'}{R}\right)^k P_n(\cos\psi)\, d\Omega'. \quad (8)$$

The potential of Helmert's condensation layer on the geoid, implying the use of Helmert's second method, is now to be subtracted from V^t. When defining the condensation layer, we first decide what form of condensation we want to employ. Here we choose to use the condensation scheme that preserves the position of the centre of mass. The reason for this choice is that when the direct effect is expressed in the spectral form, the terms of degree 1 are identically equal to zero. The spectral form of the reference field in Helmert's space then does not contain terms of degree 1 either, and the field is properly expressed in geocentric coordinates as required in the Stokes theory [Vaníček and Krakiwsky 1986]. Wichiencharoen [1982] and Martinec [1993] show that the following condensation

$$\sigma(\Omega) = \bar{\rho}(\Omega)\, H(\Omega) \left[1 + \frac{3H(\Omega)}{2R} + \frac{H^2(\Omega)}{R^2} + \frac{H^3(\Omega)}{4R^3}\right] \quad (9)$$

rigorously preserves the centre of mass. Here σ stands for the areal density of the condensation layer, and $\bar{\rho}$ is the mean density of the topographical column.

Now, the potential of the condensation layer can be expressed as

$$V^c(r,\Omega) \approx GR^2 \int_{\Omega'} \frac{\sigma(\Omega')}{\ell(r,\Omega,R,\Omega')} \, d\Omega'. \qquad (10)$$

After substitution from eqn. (9) and taking the mean value of topographical density, we get

$$V^c(r,\Omega) \approx GR^2\rho_0 \times$$
$$\times \int_{\Omega'} \left(H' + \frac{3H'^2}{2R} + \frac{H'^3}{R^2} + \frac{H'^4}{4R^3} \right) \ell^{-1} \, d\Omega'. \qquad (11)$$

As above, we can develop the reciprocal distance

$$\ell^{-1} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n P_n(\cos\psi) \qquad (12)$$

into a series of Legendre polynomials, and for r > R we finally obtain

$$V^c(r,\Omega) \approx GR^2\rho_0 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \times$$
$$\times \int_{\Omega'} \left[\frac{H'}{R} + \frac{3}{2}\left(\frac{H'}{R}\right)^2 + \left(\frac{H'}{R}\right)^3 + \frac{1}{4}\left(\frac{H'}{R}\right)^4 \right] \times$$
$$\times P_n(\cos\psi)\, d\Omega', \qquad (13)$$

since the series in eqn. (12) properly converges. Subtracting the condensation layer potential from the topographical potential, we arrive at the residual topographical potential

$$V(r,\Omega) \approx G\rho_0 R^2 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \times$$
$$\times \left\{ \frac{1}{n+3} \sum_{k=1}^{n+3} \binom{n+3}{k} \int_{\Omega'} \left(\frac{H'}{R}\right)^k P_n(\cos\psi)\, d\Omega' - \right.$$
$$- \int_{\Omega'} \left[\frac{H'}{R} + \frac{3}{2}\left(\frac{H'}{R}\right)^2 + \left(\frac{H'}{R}\right)^3 + \frac{1}{4}\left(\frac{H'}{R}\right)^4 \right] \times$$
$$\left. \times P_n(\cos\psi)\, d\Omega' \right\}. \qquad (14)$$

The summation over k converges very quickly, since $H' \ll R$, and we can safely truncate it at degree 3. Then eqn (14) can be rewritten in a more transparent form as

$$V(r,\Omega) \approx G\rho_0 R^2 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \times$$
$$\times \left\{ \frac{n-1}{2} \int_{\Omega'} \left(\frac{H'}{R}\right)^2 P_n(\cos\psi)\, d\Omega' + \right.$$
$$\left. + \frac{n^2+3n-4}{6} \int_{\Omega'} \left(\frac{H'}{R}\right)^3 P_n(\cos\psi)\, d\Omega' \right\}. \qquad (15)$$

It is now easy to see that the first degree term is equal to 0, as expected, and neglecting over the 3rd degree term, the residual topographical potential becomes

$$V(r,\Omega) \approx -\frac{G\rho_0 R}{2r} \int_{\Omega'} H'^2 d\Omega' +$$
$$+ \frac{G\rho_0}{2} \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} (n-1) \int_{\Omega'} H'^2 P_n(\cos\psi)\, d\Omega', \qquad (16)$$

where the first term is the zero-degree residual topographical potential V_0. Expressing the Legendre polynomials as sums of products of harmonic functions,

$$P_n(\cos\psi) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y^*_{nm}(\Omega')\, Y_{nm}(\Omega), \qquad (17)$$

we finally obtain

$$V(r,\Omega) \approx V_0 + 2\pi G\rho_0 \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} \left(\frac{n-1}{2n+1}\right) \times$$
$$\times \sum_{m=-n}^{n} (H^2)_{nm}\, Y_{nm}(\Omega), \qquad (18)$$

where the higher than second degree terms in H were left out. This is possible because for the first L degrees, the error caused by this approximation is smaller than $(L+1)/2 \cdot 10^{-3}$, which, for L = 20, amounts to about 1%. The symbol $(H^2)_{nm}$ denotes, of course, the harmonic coefficients of squared topography. We note that the value of V_0 at the earth's surface $r \doteq R$ can be evaluated from a global topographical model. Using the TUG87 model [Wieser, 1987], we obtain the value for the corresponding displacement of the equipotential surface

$$N_0 = \frac{V_0}{\gamma_0} \doteq -4 \text{ cm}. \tag{19}$$

The transformation of the potential coefficients into the Helmert space (to get the W^h_{nm} coefficients), which we set out to do, is carried out by subtracting the coefficients of the residual topographical potential

$$\forall n \geq 2, m: \; V_{nm} \approx 2\pi G \rho_0 \frac{n-1}{2n+1} (H^2)_{nm} \tag{20}$$

from the corresponding potential coefficients W_{nm} of the reference field. When computing the reference spheroid of degree L in the Helmert space, the residual topographical potential above is replaced by the residual topographical potential on the geoid and divide by normal gravity γ_0. The zero-degree correction above, N_0, must then be subtracted from Helmert's reference spheroid.

We have evaluated the residual topographical potential V on the geoid globally from the TUG87 model, for L = 20 (including the zero-degree term), divided it by γ_0, and found it to be between -13 cm and +18 cm. A plot of the effect for the territory of Canada is shown in Figure 1.

A comment on the use of mean topographic density ρ_0 is in order. From eqn.(20) it seems that to achieve the one-centimetre accuracy everywhere in the world, regional density anomalies should be considered. In most parts of the world, including Canada, however, the mean lithospheric density of 2.67 g/cm^3 used in our calculations above will be good enough.

Evaluation of Helmert's Reference Spheroid

Let us now assume that the potential coefficients of the reference field have been corrected for the residual topographical potential and thus transformed into the Helmert space. The Helmert reference potential can then be written as

$$W^h(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{L} \left(\frac{a}{r}\right)^{n+1} \sum_{m=-n}^{n} W^h_{nm} Y_{nm}(\Omega). \tag{21}$$

We note that because the summation is finite, the validity of this expression is no longer limited to the outside of the Brillouin sphere; we can use this expression anywhere on and above the geoid. To compute the reference spheroid in Helmert's space, we have to evaluate this series for

$$r(\Omega) = r_g(\Omega), \tag{22}$$

where r_g denotes the distance of the geoid from the centre of the earth; this is now clearly permissible.

To do so, let us write r_g for the time being as

$$r_g(\Omega) = a\,(1 + \delta(\Omega)), \tag{23}$$

where $\delta(\Omega)$ is a quantity of the order of flattening of the reference ellipsoid, i.e., much smaller than 1. Then, for $n \leq 20$, we obtain

$$\left(\frac{a}{r_g}\right)^{n+1} = 1 - (n+1)\,\delta + 0\,(\delta^2). \tag{24}$$

To an accuracy better than $5*10^{-3}$, δ may be taken as the leading term in the expression for the length of the radius vector of the reference ellipsoid, i.e.,

$$\delta(\Omega) = -f \cos^2\theta \tag{25}$$

see, for instance Bomford [1980, p. 432]. The equation for the Helmert reference potential on the geoid becomes

$$W^h(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{L} \left[1 + f\,\frac{n+1}{2}(1 + \cos 2\theta)\right] \times \\ \times \sum_{m=-n}^{n} W^h_{nm} Y_{nm}(\Omega). \tag{26}$$

It might be expedient to express this "flattening effect" in the form of a correction δW^h to the Helmert reference potential W^h. From eqn.(26) we easily get

$$\delta W^h(\Omega) \approx -\frac{f}{2}\,(1 + \cos 2\theta) \sum_{n=2}^{L} (n+1) \times \\ \times \sum_{m=-n}^{n} W^h_{nm} Y_{nm}(\Omega). \tag{27}$$

We have evaluated this correction for the territory of Canada and the results, expressed in values of this correction are: -88 cm and +65 cm. The potential coefficients (up to 20,20) used in this evaluation are taken from the GRIM4-S4P [Gruber and Anzenhofer, 1993] global solution.

The last task to be performed is to transform the total Helmert potential W^h to the Helmert disturbing potential T^h. This is done simply by subtracting the appropriate normal potential from the Helmert potential. Division by an appropriate value of normal gravity, finally yields the desired reference spheroid. We note that it is the gradient of

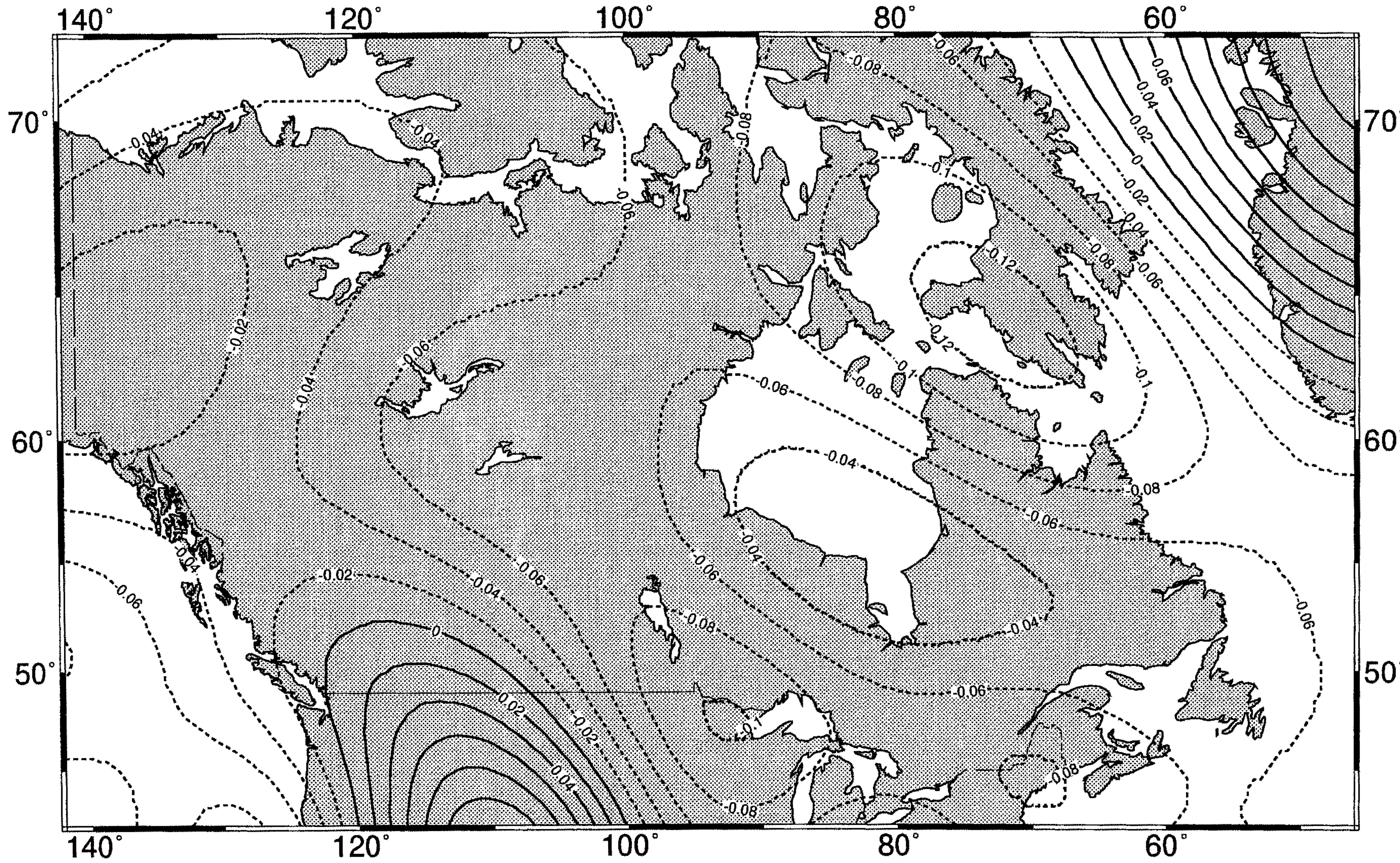

Figure 1: Residual topographical potential effect (including the zero-degree term) on the spheroid of degree 20 in Canada, computed from the TUG87 global topographical model. The minimum and maximum values are - 0.13m and 0.18m respectively, contour interval is 0.02m.

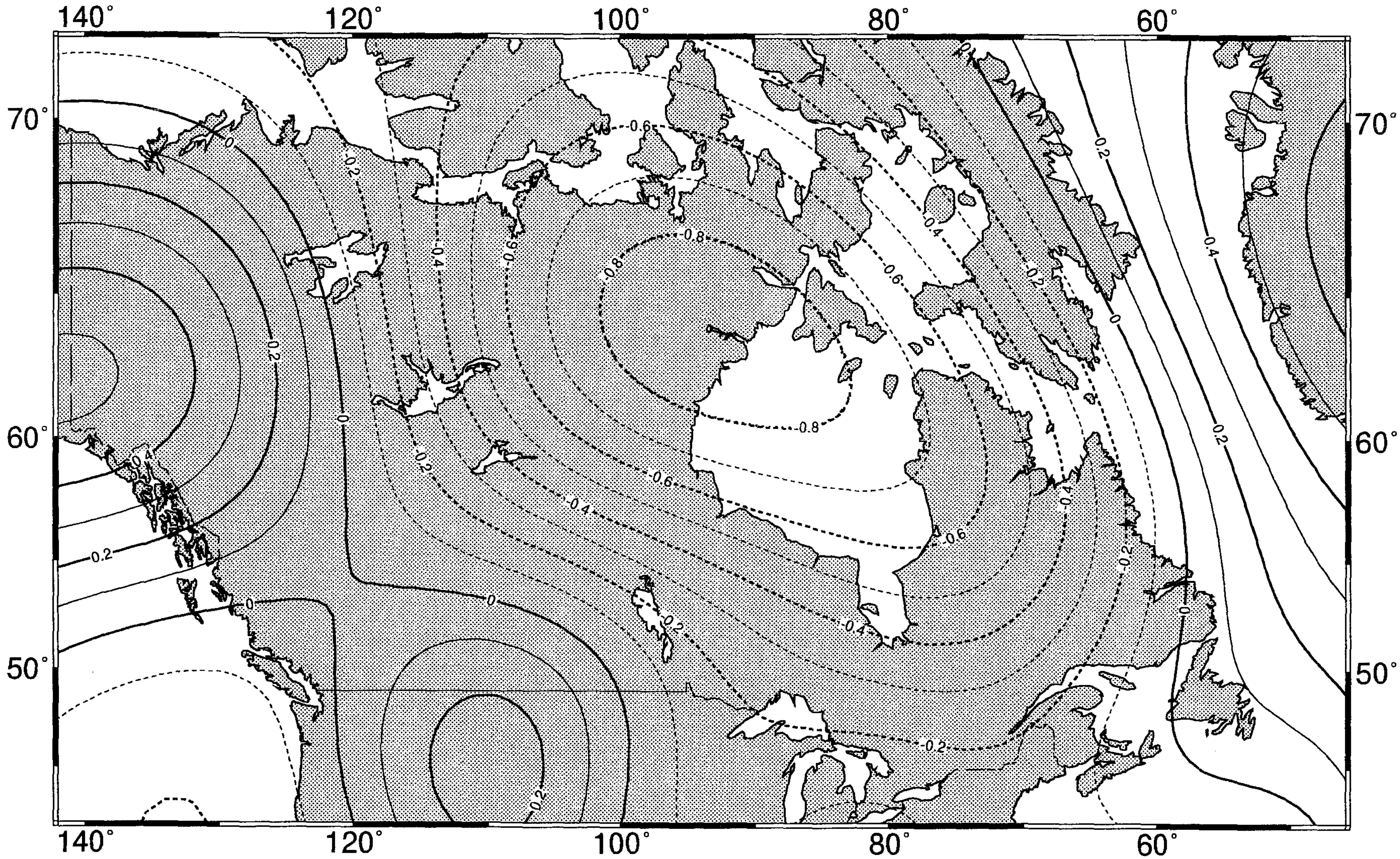

Figure 1: Ellipsoidal correction to the spheroid of degree 20 in Canada, computed from the GRIM4-s4p global geopotential model. The minimum and maximum values are - 0.88m and 0.65m respectively, contour interval is 0.10m.

this potential that has to be subtracted from the observed gravity anomalies before they are convolved with Stokes's spheroidal kernel, see for instance Vaníček and Kleusberg, [1987]. It has to be mentioned that for a precise geoid determination the low degree secondary indirect topographical effect [Vaníček and Martinec, 1994], called sometimes the Bowie correction, also has to be added here. The mathematical treatment of this effect is considered beyond the scope of this paper.

To conclude, the (Helmert) reference spheroid in the Helmert space plays the same role as its counterpart in the real space [Vaníček and Sjoberg, 1991]. The only difference is that when the contribution coming from the local gravity data is added it to, we obtain not the geoid but the Helmert co-geoid, which has to be transformed into the geoid by adding the primary indirect topographical effect [Vaníček and Martinec, 1994].

Conclusions

The reference spheroid in Helmert's model plays much the same role as its real counterpart plays in the real space; the Helmert co-geoid is referred to it, thus reducing the magnitude of the quantities one works with and serving as a linerization tool. To compute the reference spheroid in Helmert's model, we use potential coefficients derived from satellite orbit analyses up to a degree and order where there is still meaningful information in these coefficients, i.e., around 20.

We have shown that to compute the Helmert reference spheroid, the residual topographical potential has to be added to the field generated by the potential coefficients. For the degree of the spheroid equal to 20, this effect may reach several decimetres and has to be considered if a high accuracy geoid is to be produced. A non-trivial constant correction of a magnitude of 4 cm, arising from the implied transfer of masses in the transformation from the real space to Helmert's space, should as well be considered.

When computing the reference spheroid, the generating potential coefficients must be corrected for the effect of the flattening of the geoid (co-geoid). In absolute value, this correction is almost around 1 metre and, once again, has to be considered when an accurate geoid is to be produced. We note that the irregularities of the geoid, other than its ellipticity, also contribute to this correction. These other contributions are below the "magic" 1 cm level, however, and are not considered at present.

The numerical results for both corrections shown in the paper were obtained for the selected degree of the reference field L=20. If a different degree is considered, the value of the corrections will change. We have not investigated the nature of such changes since it is our belief that the optimum degree (20) is unlikely to change in the foreseeable future.

Before the suitably corrected reference potential can be turned into the reference spheroid, an appropriate normal field has to be subtracted from it. This is a well-known operation and is mentioned here only for the sake of completeness. Finally, the reader is reminded of the fact that once the Helmert co-geoid is computed (in the Helmert model) with the help of the reference spheroid, it must be converted into the geoid by applying the indirect topographical effect correction. This step is outside the scope of this contribution.

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