Z angew Math Phys 45 (1994)
(ZAMP)

0044-2275/94/020312-24 \$ 1.50 + 0.20

Normal form for generalized Hopf bifurcation with non-semisimple 1 : 1 resonance*

By N. Sri Namachchivaya and Monica M. Doyle, Dept of Aeronautical and Astronautical Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61810, USA and William F. Langford and Nolan W. Evans, Dept of Mathematics and Statistics, University of Guelph, Guelph, Ontario, Canada, N1G 2W1

I. Introduction

When a multidegree-of-freedom dynamical system undergoes a bifurcation, it usually does so in only a few degrees of freedom. One simple example is the buckling of a column. If μ and μ_c represent the axial and Euler loads of a column, respectively, then, as μ is varied in the vicinity of μ_c, the temporal evolution of the motion is dominated by the critical mode which, in the first approximation, is governed by $\dot{x} = (\mu - \mu_c)x + ax^3$. A more complicated situation arises when several control parameters $\underline{\mu}$ are varied in such a way that several modes become marginally unstable simultaneously. In the latter case, the system is said to undergo a multiple bifurcation. The simplest and smallest number of equations which capture the essential dynamics of the original system in the vicinity of $\underline{\mu}_c$ are said to be in the normal form. The theory of normal forms is an important analytical tool for investigating the qualitative behavior of nonlinear dynamical systems.

The idea of normal forms for nonlinear systems dates back as far as Euler; however, Poincaré [16] and Birkhoff [3] were the first to bring forth the theory in a more definite form. Poincaré [16] considered the problem of reducing a system of nonlinear differential equations to a system of linear ones; namely,

$$\frac{dx}{dt} = Ax + f(x) \quad \text{to} \quad \frac{dy}{dt} = Ay, \quad x \in R^n, y \in R^n. \tag{1}$$

The formal solution of this problem entails finding near-identity coordinate transformations, $x = y + \Phi(y)$, which eliminate the analytic expressions of

* Partially supported by NSF through grant MSS 90-57437, AFOSR through grant 91-0041 and NSERC of Canada.

the nonlinear terms. It has been shown that such a formal solution exists provided the above system is hyperbolic and the eigenvalues λ_j of the diagonalizable matrix A satisfy the nonresonance condition

$$\lambda_i \neq \sum m_l \lambda_l \quad \text{for } i = 1, 2, \ldots, n, \qquad |m| = \sum m_l \geq 2 \tag{2}$$

where m is a vector of integers $m = (m_1, m_2, \ldots, m_n)$ with $m_l \geq 0$. Furthermore, it was proven that if, in addition to the above results, the eigenvalues lie strictly to one side of a line separating them from zero in the complex plane, then the formal series $\Phi(y)$ is convergent.

If the system is nonhyperbolic or condition (2) is violated, the analytic expressions of the nonlinear terms cannot be completely eliminated via a nonlinear change of coordinates. The remaining terms comprise the normal form of the system of equations given by (1). The normal form is dictated by the nature of the linear operator A. Thus, the nonlinear system in Eq. (1) can be reduced to

$$\frac{dy}{dt} = Ay + g(y), \qquad y \in R^n \tag{3}$$

where g is simpler than f. Such reductions have been widely used to study deterministic autonomous and nonautonomous systems (see Arnold [1]).

In bifurcation problems, the eigenvalues of the linear operator A are composed of two sets, one on the imaginary axis and the other with strictly negative real parts. The linear vector space E associated with A can also be divided accordingly as $E = E_c \oplus E_s$ such that $x_c \in E_c$ and $x_s \in E_s$ with $x = x_c + x_s$. There are two approaches to obtaining normal forms for deterministic systems. In the first, as shown in Guckenheimer and Holmes [12], one first computes the lower dimensional center manifold onto which the dynamics settle for large times. The dynamical system defined on the center manifold is then transformed to the normal form through a nonlinear change of coordinates. In the second method, one systematically expands the original vector field in powers of amplitudes of the critical modes to yield both the normal form and center manifold, simultaneously, as shown by Elphick et al. [7]. The approach adopted in this paper for the computation of the normal form assumes that the center manifold theorem has been applied to the original system and is based heavily on the work of Elphick et al. [7].

The aim of this paper is two-fold: first, to present an explicit formula for the normal form of a generalized Hopf bifurcation with non-semisimple 1 : 1 resonance and, second, to compare the results with those obtained via the method of averaging. The results for the corresponding semisimple case were obtained by Bajaj and Sethna [2] using center manifold theory and the method of integral averaging.

Recently, the normal form for a generalized Hopf bifurcation was expressed as a 4-dimensional real system by Cushman and Sanders [5] and as a 2-dimensional complex system by Elphick et al. [7] and Iooss and Adelmeyer [18]. Iooss et al. [19] employed the 2-dimensional normal form given in [7] to examine the steady bifurcating solutions in nonlinear hydrodynamic stability problems. However, there are no explicit formulas relating the coefficients of the original system to those of the normal form. This paper presents explicit formulas for the 4 leading constants in the complex normal form in terms of coefficients of the original nonlinear system with both quadratic and cubic nonlinearities. The complex normal form presented by Elphick et al. [7] has recently been analyzed by van Gils et al. [11]. It was shown that this co-dimension 3 bifurcation problem is more complicated than the closely related case of the non-resonant double Hopf bifurcation and contains three different types of co-dimension 1 singularities and 4 different types of co-dimension 2 singularities. Thus, with the help of the results presented in this paper, one can apply the analysis of van Giles et al. [11] to any physical problem exhibiting generalized Hopf bifurcation with non-semisimple 1 : 1 resonance. Furthermore, it has been shown by Hale [13] that, for systems with linear operators whose superdiagonal terms are equal to 1, an appropriate scaling can be used to obtain the averaged equations. In the final section, the averaged equations up to the second order approximation are obtained and compared with the normal form equations.

II. Background and notations

The problem of interest in this paper is a 4-dimensional one. However, we shall keep the analysis as general as possible for the time being. Consider a dynamical system governed by autonomous differential equations in C^n,

$$\dot{y} = A(\mu)y + f(y, \mu) \tag{4}$$

where $f: C^n \rightarrow C^n$ is a C^r vector field, $r \geq 2$, A is an $n \times n$ complex matrix, $x = 0$ is the trivial solution of Eq. (4) for all values of μ (i.e., $f(0, \mu) = 0$) and the nonlinear vector function can be represented as

$$f(y, \mu) = f^2(y, \mu) + f^3(y, \mu) + \cdots + f^k(y, \mu) + \cdots. \tag{5}$$

Here, we have expressed the nonlinear terms as a formal power series of homogeneous terms with degree denoted by the superscripts. We define H_n^k to be the linear space of homogeneous vector polynomials of degree k in n variables with range C^n. Let $(e_1, e_2, \ldots, e_n)$ denote the basis of C^n and $y = (y_1, y_2 \cdots y_n)$ be the coordinates with respect to this basis. Thus, an element $f^k(y, \mu)$ of H_n^k can be represented in the form of vector-valued monomials as

$$f^k(y,\mu) = \sum_s^n f_s^k(y,\mu)e_s = \sum_{s,m} f_{s;m}^k(\mu)y^m e_s, \qquad |m| = k$$
$$= \sum_s \sum_{|m|=k} f_{s:m_1,m_2,\ldots m_n}^k (y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n})e_s \tag{6}$$

with

$$\dim\{f_i^k(y,\mu)\} = (n+k-1)!/[(n-1)!k!]$$

and

$$\dim\{H_n^k\} = n \cdot \dim\{f_i^k(y,\mu)\}.$$

Now that a formal set-up for representing Eq. (4) has been obtained, we can consider the problem of reducing Eq. (4) to the normal form

$$\begin{aligned} \dot{x} &= A(\mu)x + g(x,\mu), \\ g(x,\mu) &= g^2(x,\mu) + g^3(x,\mu) + \cdots + g^k(x,\mu) + \cdots \end{aligned} \tag{7}$$

which, as stated previously, is in a simpler form than Eq. (4) and has all the essential features of the flow near the equilibrium point of the original system. The formal solution of this problem consists of determining near identity coordinate transformations

$$y = x + h(x), \qquad h(x) = h^2(x) + h^3(x) + \cdots + h^k(x) \tag{8}$$

where $x \in \Omega$, and Ω is a neighborhood of the origin of C^n, such that the analytic expressions of $f(y,\mu)$ are simplified to yield $g(x,\mu)$. Once again, f^k, g^k and h^k are homogeneous vector polynomials of degree k and belong to H_n^k. Assuming the normal form reduction up to order $k-1$ has been performed, differentiating Eq. (8) gives

$$\dot{y} = [I + D_x h^k(x)]\dot{x}$$

and substituting in Eq. (4) yields

$$\dot{x} = [I + D_x h^k(x)]^{-1}[A(x + h^k(x)) + f(x + h^k(x))].$$

Making use of the fact that, for $x \in \Omega$,

$$[I + D_x h^k(x)]^{-1} = I - D_x h^k(x) + O(|x|^{2(k-1)})$$

results in

$$\begin{aligned} \dot{x} = {} & Ax + f^2(x) + f^3(x) + \cdots + f^{k-1}(x) \\ & + \{f^k(x) + [Ah^k(x) - D_x h^k(x)Ax]\} + O(|x|^{k+1}). \end{aligned} \tag{9}$$

It is worth noting that the transformation of degree k does not affect the normal form of order $(k-1)$ but does affect the terms of order k and higher. The task now is to select $h^k(x)$ so that the terms of degree k in the brackets are as simple as possible. Examining the terms of degree k in Eq. (9) and comparing with those of Eq. (8) yields

$$Ah^k(x) - D_x h^k(x)Ax + f^k(x) = g^k(x) \tag{10}$$

and $f^j(x) = g^j(x)$ for $j = 2, 3, \ldots k-1$. Introducing a linear operator L_A defined by

$$L_A h^k = [h^k, Ax] = Ah^k(x) - D_x h^k(x)Ax,$$

Eq. (10) can be rewritten as

$$-L_A h^k(x) = f^k(x) - g^k(x) = \eta^k(x). \tag{11}$$

The above equation is called a homological equation. $L_A: H_n^k \to H_n^k$ is called the homological operator and is linear in the space of homogeneous vector polynomials of degree k. Equation (11) is to be solved for $h^k(x)$.

Let us denote R_n^k as the range of L_A and let W_n^k be any complementary subspace to R_n^k in H_n^k. H_n^k can be decomposed as follows

$$H_n^k = R_n^k \oplus W_n^k, \qquad k \geq 2. \tag{12}$$

Thus, for each $f^k(x) \in H_n^k$ there exists $\eta^k(x) \in R_n^k$ and $g^k(x) \in W_n^k$ such that any given homogeneous polynomial of degree k can be written as

$$f^k(x) = g^k(x) + \eta^k(x)$$

and the suitable transformation $h^k(x)$ is obtained from

$$-L_A h^k(x) = \eta^k(x). \tag{13}$$

Since the choice of complementary space W_n^k is not unique, neither is the transformation $h^k(x)$ or the normal form $g^k(x)$. This nonuniqueness was resolved by Elphick et al. [7] through a particular choice of inner product. As in [7] (refer also to Helgason [14]), we can introduce an inner product in H_n^k. To this end, we introduce a differential operator associated with an arbitrary $f_i^k(x) \in H_n^k$ as

$$f_i^k(\partial)e_i = \sum_{|m|=k} f_{i,m}^k \left(\frac{\partial}{\partial x}\right)^m e_i, \qquad \left(\frac{\partial}{\partial x}\right)^m = \frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdot \frac{\partial^{m_2}}{\partial x_2^{m_2}} \cdots \frac{\partial^{m_n}}{\partial x_n^{m_n}}.$$

Then, for $f_i^k(x)$, $g_j^k(x)$ in H_n^k, the scalar product is given by

$$\langle f_i^k(x), g_j^k(x)\rangle = f_i^k(\partial)\bar{g}_j^k(x)\big|_{x=0}\delta_{ij} = \sum_{|\alpha|=k}\sum_{|\beta|=k} f_{i;\alpha}^k \bar{g}_{j;\beta}^k \cdot \frac{\partial^\alpha(x^\beta)}{\partial x^\alpha}\bigg|_{x=0} \delta_{ij}.$$

It is clear that the only terms that will survive are those for which α and β coincide, i.e.

$$\langle f_i^k(x), g_j^k(x)\rangle = \sum_{m=k} f_{i,m}^k \bar{g}_{j,m}^k m!\delta_{ij}, \qquad m! = m_1!m_2! \cdots m_n!.$$

Thus, the inner product in H_n^k is defined as

$$\langle f^k(x), g^k(x) \rangle_{H_n^k} = \sum_{i=1}^{n} \sum_{|m|=k} f_{i,m}^k \bar{g}_{i,m}^k m!. \tag{14}$$

Using this inner product, we can define the adjoint operator $(L_A)^*$ as

$$\langle L_A h^k(x), f^k(x) \rangle_{H_n^k} = \langle h^k(x), L_A^* f^k(x) \rangle_{H_n^k}$$

and making use of the fact

$$\langle h^k(Ax), f^k(x) \rangle_{H_n^k} = \langle h^k(x) m f^k(A^*x) \rangle_{H_n^k}$$

Elphick et al. [7] has shown that

$$\ker(L_{A^*}) = \ker(L_A)^*. \tag{15}$$

Since H_n^k is a finite dimensional space, $\ker(L_A)^*$ is an orthogonal complement of R_n^k the elements of which we are free to choose. Equation (12) may then be written as

$$H_n^k = R_n^k \oplus \ker(L_{A^*}). \tag{16}$$

Now, considering the linear equations in H_n^k, we have

$$-L_A h^k(x) = \eta^k(x), \qquad L_{A^*} g^k(x) = 0 \tag{17}$$

and the solvability condition

$$\langle \eta^k(x), g^k(x) \rangle_{H_n^k} = 0. \tag{18}$$

The normal form and explicit formulas for the coefficients can then be calculated using Eqs. (17) and (18). It is important to note that this normal form depends on the matrix A and the choice of complementary space W_n^k. Once the functions $f^k(x)$ are known, the above method can be applied to calculate both $h^k(x)$ and $g^k(x)$. A recursive algorithm, similar to that of Chow and Hale [4], can also be employed to compute the kth order nonlinearities $f^k(x)$ given all transformations $h(x)$ and normal forms $g(x)$ up to order $k-1$. Both methods have been employed independently herein to calculate the normal form coefficients which are given explicitly in the Appendix.

III. Normal form for non-semisimple case

For the non-semisimple case, the normal form calculations are not as easy as in the case of a diagonalizable linear operator. However, the calculations can be simplified using certain well known results in Lie algebra. These will be introduced as we proceed through the calculations of the normal form for the generalized Hopf bifurcation.

Given a finite dimensional vector space V over the complex numbers C and a space L of linear transformations of V onto itself, one can define the Lie bracket by the formula

$$[P, Q] = (P \cdot Q - Q \cdot P) \in L \qquad \text{for } P, Q \in L. \tag{19}$$

Then L becomes a Lie algebra and we say P commutes with Q iff $[P, Q] = 0$. The result that is of importance to us in the Jordan decomposition theorem which states that for any $A \in L$ there exist S and N such that

$$A = S + N \quad \text{and} \quad [S, N] = 0 \tag{20}$$

where S is semisimple (diagonalizable) and N is nilpotent. Moreover, these decompositions are unique and

$$\ker A = \ker S \cap \ker N. \tag{21}$$

In the calculation of normal forms for generalized Hopf bifurcation with non-semisimple 1 : 1 resonance, the linear operator of interest takes the form

$$\begin{aligned} A = \begin{bmatrix} i\omega & 1 & & \\ 0 & i\omega & \multicolumn{2}{c}{\mathbf{0}} \\ & & -i\omega & 1 \\ \multicolumn{2}{c}{\mathbf{0}} & 0 & -i\omega \end{bmatrix} &= \begin{bmatrix} i\omega & 0 & & \\ 0 & i\omega & \multicolumn{2}{c}{\mathbf{0}} \\ & & -i\omega & 0 \\ \multicolumn{2}{c}{\mathbf{0}} & 0 & -i\omega \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ 0 & 0 & \multicolumn{2}{c}{\mathbf{0}} \\ & & 0 & 1 \\ \multicolumn{2}{c}{\mathbf{0}} & 0 & 0 \end{bmatrix} \\ &= S + N \end{aligned} \tag{22}$$

and $[S, N] = 0$. In addition, the homological operator for any two matrices A and B also satisfies the relation $[L_A, L_B] = L_{[A,B]}$. This implies that the Lie brackets of L_S, L_N and L_{S^*}, L_{N^*} also commute. Thus, the $\ker(L_{A^*})$, which is needed for the calculation of the normal form, is given by

$$\ker(L_{A^*}) = \ker L_{S^*} \cap \ker L_{N^*}.$$

It is worth pointing out that the above results can also be obtained using the arguments given in Meyer [15]. Furthermore, the normal form $g(z)$, given in Eq. (17), commutes with elements of the Lie groups

$$G = \{e^{sA^*} \mid s \in R\} \quad \text{and} \quad S^1 = \{e^{sS} \mid s \in R\}$$

and the normal form is said to have G-equivariance and a simpler S^1-equivariance, respectively. Since the proofs of these results are similar, only that of S^1-equivariance, i.e.

$$g(e^{sS}\xi) = e^{sS} g(\xi)$$

will be given here. To this end, consider $z = e^{sS}\xi$ and $g(z) = g(e^{sS}\xi)$. Taking the total differential of $g(z)$ w.r.t. the variable s yields

$$\frac{dg(z)}{ds} = D_z g(z) S z.$$

Now, using the fact that the normal form is such that $g \in \ker(L_{A^*}) = \ker(L_{S^*}) \cap \ker(L_{N^*})$ and $S^* = -S$, we have

$$D_z g(z)Sz - Sg(z) = 0.$$

Combining the above two equations yields an O.D.E. for $g(z)$

$$\frac{dg(z)}{ds} = Sg(z), \qquad s \in R$$

whose solution can be written as

$$g(z) = e^{sS}g(z; s = 0) = e^{sS}g(\xi). \tag{23}$$

This proves the S^1-equivariance. The G-equivariance can be proven similarly by replacing S by A^* in the above steps.

1. Linear algebraic calculation of the normal form coefficients

Now we calculate the normal form and appropriate expressions for the coefficients of this normal form. To this end, consider the homological equation

$$-L_A h^k(x) = f^k(x).$$

It is easy to show that for the semisimple S with eigenvalues λ_i, $i = 1, 2, \ldots, n$, $L_A h^k(x)$ reduces to

$$L_S h^k(x) = \sum_{s,|m|=k} h^k_{s;m}[\langle m, \lambda\rangle - \lambda_s]x^m e_s$$

and

$$\langle m, \lambda\rangle - \lambda_s = 0; \qquad s = 1, 2, \ldots, n; \quad |m| \geq 2$$

is called the resonance condition. The $\ker(L_{S^*})$ is determined by the appropriate combination of m's which satisfy the above condition. The resonance condition for the problem under consideration can be expressed as

$$\left.\begin{aligned} &i\omega(m_1 + m_2 - m_3 - m_4 - 1) = 0, \\ &m_1 + m_2 + m_3 + m_4 = k \end{aligned}\right\} \quad \text{for } s = 1, 2$$

and

$$\left.\begin{aligned} &-i\omega(m_3 + m_4 - m_1 - m_2 - 1) = 0, \\ &m_1 + m_2 + m_3 + m_4 = k \end{aligned}\right\} \quad \text{for } s = 3, 4.$$

Since $m_i \geq 0$ and integer, it is obvious that k is always odd and the above conditions yield

$$(m_1 + m_2) = \frac{k+1}{2}, \qquad (m_3 + m_4) = \frac{k-1}{2} \quad \text{for } s = 1, 2$$

and

$$(m_1 + m_2) = \frac{k-1}{2}, \qquad (m_3 + m_4) = \frac{k+1}{2} \quad \text{for } s = 3, 4.$$

Thus, the non-zero nonlinear normal form exists only for $k = 3, 5, \ldots$. However, the original quadratic nonlinear terms can contribute to the cubic terms as a result of the nonlinear transformation as will be seen in the subsequent section. Calculation of the coefficients of the leading order normal form ($k = 3$) is of concern in this paper. Thus, in an 80-dimensional basis, only 24 vectors lie in the $\ker(L_{S^*})$ and can be written as

$$(x_1^2 x_3)e_s,\ (x_1 x_2 x_3)e_s,\ (x_2^2 x_3)e_s,\ (x_1^2 x_4)e_s,\ (x_1 x_2 x_4)e_s,\ (x_2^2 x_4)e_s \quad \text{for } s = 1, 2$$

$$(x_3^2 x_1)e_s,\ (x_3 x_4 x_1)e_s,\ (x_4^2 x_1)e_s,\ (x_3^2 x_2)e_s,\ (x_3 x_4 x_2)e_s,\ (x_4^2 x_2)e_s \quad \text{for } s = 3, 4.$$

The action of L_{N^*} on these bases can be represented by a 24×24 matrix of the form

$$\begin{bmatrix} C & 0 & 0 & 0 \\ -I & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & -I & C \end{bmatrix} \quad \text{where } C = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where I and 0 are 6×6 identity and zero matrices. The 8-dimensional null space of the above matrix can be easily computed. Making use of this, the basis of $\ker(L_{A^*})$ can be written as

$$\begin{Bmatrix} x_1(x_1 x_4 - x_2 x_3) \\ x_2(x_1 x_4 - x_2 x_3) \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1^2 x_3 \\ x_1^2 x_4 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1^2 x_3 \\ x_1 x_2 x_3 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ x_1^2 x_3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ x_3(x_2 x_3 - x_1 x_4) \\ x_4(x_2 x_3 - x_1 x_4) \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ 0 \\ x_3^2 x_1 \\ x_3^2 x_2 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ 0 \\ x_3^2 x_1 \\ x_3 x_4 x_1 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ 0 \\ 0 \\ x_3^2 x_1 \end{Bmatrix}.$$

It is worth noting that the first 4 basis vectors are complex conjugates of the last 4, as expected. Since any linear combination of these vectors spans the null space, we can manipulate the given basis such that the resulting normal form is as simple as possible. This manipulation is performed as follows: the second basis element is replaced by the vector obtained by subtracting the third basis element from the second, and the sixth basis element is replaced by the vector obtained by subtracting the seventh basis element from the sixth. This procedure yields the new second and sixth bases as

$$\left\{\begin{matrix} 0 \\ x_1(x_1x_4 - x_2x_3) \\ 0 \\ 0 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} 0 \\ 0 \\ 0 \\ x_3(x_2x_3 - x_1x_4) \end{matrix}\right\}.$$

Thus, the normal form for the generalized Hopf bifurcation with 1:1 resonance can be written as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} i\omega & 1 \\ 0 & i\omega \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \{a_1(z_1\bar{z}_1) + a_2(z_1\bar{z}_2 - \bar{z}_1z_2)\}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$+ \{b_1(z_1\bar{z}_1) + b_2(z_1\bar{z}_2 - \bar{z}_1z_2)\}\begin{pmatrix} 0 \\ z_1 \end{pmatrix}$$

where $a_j = c_j + id_j$, $b_j = e_j + if_j$, $j = 1, 2$. In the above equation, we have replaced (x_1, x_2, x_3, x_4) by $(z_1, z_2, \bar{z}_1, \bar{z}_2)$. Thus, the second and third equations can be obtained by conjugating the above equations.

While calculating the coefficients, we shall assume that the original system contains both quadratic and cubic nonlinearities. Thus, for the problem under consideration in this paper

$$f^2(y) = \sum_{s=1}^{4} \sum_{|m|=2} f^2_{s;m_1,m_2,m_3m_4}(y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4})e_s, \ \dim(H_4^2) = 40 \tag{25a}$$

$$f^3(y) = \sum_{s=1}^{4} \sum_{|m|=3} f^3_{s;m_1,m_2,m_3m_4}(y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4})e_s, \ \dim(H_2^3) = 80. \tag{25b}$$

We have shown in the previous section that $\ker(L_{A^*}) = \{\varnothing\}$ for $k = 2$ since $\ker(L_{S^*}) = \{\varnothing\}$. Thus, all the quadratic terms given by expression (25a) can be eliminated and the transformation which performs this reduction, obtained by the matrix representation of $\ker(L_{A^*})$, is given by

$$\begin{bmatrix} B & I & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & \bar{B} & I \\ 0 & 0 & 0 & \bar{B} \end{bmatrix} \begin{bmatrix} h^2_{1;m} \\ h^2_{2;m} \\ h^2_{3;m} \\ h^2_{4;m} \end{bmatrix} = \begin{bmatrix} f^2_{1;m} \\ f^2_{2;m} \\ f^2_{3;m} \\ f^2_{4;m} \end{bmatrix}$$

where

$$B = \begin{bmatrix} -i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & i\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 3i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3i\omega \end{bmatrix}$$

and $h^2_{i;m}$ and $f^2_{i;m}$ are vectors of dimension 10. Since B is nonsingular, it is easy to calculate

$$h^2_{2;m} = B^{-1} f^2_{2;m}, \qquad h^2_{1;m} = B^{-1}(f^2_{1;m} - h^2_{2;m})$$

and $h^2_{3;m}$ and $h^2_{4;m}$ are the conjugates of $h^2_{1;m}$ and $h^2_{2;m}$, respectively. The complete expressions for $h^2_{1;m}$ and $h^2_{2;m}$, are given explicitly in the Appendix. As these transformations annihilate all of the quadratic nonlinearities in the given system, they alter the terms of order 3 and above. We denote the new coefficients of the cubic nonlinearities as

$$p^3_{s;m} = f^3_{s;m} + \bar{f}^3_{s;m} \quad \text{and} \quad \bar{f}^3_{s;m} = F(f^2_{1;m}, f^2_{2;m}, f^2_{3;m}, f^2_{4;m})$$

where $f^3_{s;m}$ are the original coefficients of the cubic nonlinearities, and $\bar{f}^3_{s;m}$ are the coefficients of the new cubic terms generated while eliminating the original quadratic nonlinearities. The coefficients are indeed functions of the coefficients of the original quadratic nonlinearities as one would expect. Now, the normal form for the leading nonlinearity is given by Eq. (23) and is defined in the space complementary to R^3_4. The coefficients a_1, a_2, b_1, b_2 and their conjugates are calculated using the solvability condition of Eq. (18). The first 4 coefficients are

$$a_1 = \frac{1}{4}\{3f^3_{1;2010} + f^3_{2;1110} + f^3_{2;2001}\} + \tilde{a}_1$$

$$a_2 = \frac{1}{6}(2f^3_{1;2001} - 2f^3_{2;0210} - f^3_{1;1110} + f^3_{2;1101}) + \tilde{a}_2$$

$$b_1 = f^3_{2;2010} + \tilde{b}_1$$

$$b_2 = \frac{1}{4}(f^3_{1;2010} - f^3_{2;1110} + 3f^3_{2;2001}) + \tilde{b}_2$$

where the expressions for $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$ and $\tilde{b}_2$ in terms of the coefficients of the quadratic nonlinearities are given in the Appendix. The remaining four coefficients are obtained by conjugation of the above expressions, i.e., $a_3 = \bar{a}_1$, $a_4 = \bar{a}_2$, $b_3 = \bar{b}_1$ and $b_4 = \bar{b}_2$.

2. Recursive calculation of normal form coefficients

This approach is based on a series of papers by Ponce, Gamero and Freire [8, 9, 10, 17] which are, in turn, implementations of a method of Chow and Hale [4, Chap. 12] which employs a technique developed by Deprit [6] using Lie transforms to determine the normal form.

In order to remain consistent with the literature, the following notation will be used: define $F^k, U^k, G^k \in H_n^k$ by

$$F^k = (k-1)!f^k(y), \quad U^k = (k-1)!h^k(x), \quad G^k(k-1)!g^k(x).$$

The first step is a rescaling to isolate the homogeneous terms of degree k. Letting $x \to \varepsilon x$ and $y \to \varepsilon y$ for $\varepsilon \in R$, the original system of Eq. (4) becomes

$$\dot{y} = Ay + \sum_{k \geq 2} \frac{\varepsilon^{k-1}}{(k-1)!} F^k(y) \tag{26}$$

the near identity transformation, (8), becomes

$$y = x + \sum_{k \geq 2} \frac{\varepsilon^{k-1}}{(k-1)!} U^k(x) \tag{27}$$

and the system in normal form, (7), becomes

$$\dot{x} = Ax + \sum_{k \geq 2} \frac{\varepsilon^{k-1}}{(k-1)!} G^k(x). \tag{28}$$

Following Chow and Hale [4], the sequence $\{F_l^k\}$ is defined by the recursion relation

$$F_l^k = F_{l-1}^k + \sum_{j=2}^{k-l+2} \binom{k-l}{j-2} F_{l-1}^{k-j+1} \times U^j, \qquad l = 2, \dots, k, \quad k = 2, \dots \tag{29}$$

where $F_1^k = F^k$, $F_1^1 = Ax$ and

$$P \times Q = \frac{\partial P}{\partial x} Q - \frac{\partial Q}{\partial x} P.$$

It can be shown (see Chow and Hale [4]) that

$$F_k^k = G^k.$$

This recursion can be represented by a Lie triangle,

$$
\begin{array}{llllll}
[(F_1^1)] & & & & & \\
(F_1^2) & [F_2^2] & & & & \\
(F_1^3) & F_2^3 & [F_3^3] & & & \\
(F_1^4) & F_2^4 & F_3^4 & [F_4^4] & & \\
(F_1^5) & F_2^5 & F_3^5 & F_4^5 & [F_5^5] & \\
(\vdots) & \vdots & \vdots & \vdots & \cdots & [\vdots].
\end{array}
$$

The terms in round brackets are from the original system and those in square brackets are the final normal form. Each term in the triangle depends on those immediately to the left and above. The indexing scheme used here is different from that in the above references; the superscript k refers to both the order of the monomials in the vector and the row in which it appears in the Lie triangle and the subscript refers to the column in which it appears in the Lie triangle.

The recursion operates across rows of the Lie triangle from left to right. As an example of what occurs during a recursion, consider the fifth row. F_1^5 is a vector containing the order 5 terms in the original system. To generate F_2^5, F_1^5 is added to the sum of the terms in column 1 above F_1^5 combined with the appropriate U^j's; $F_1^4 \times U^2, F_1^3 \times U^3, F_1^2 \times U^4, F_1^1 \times U^5$. To generate F_3^5, F_2^5 is added to the sum of the terms in column 2 above F_2^5 combined with the appropriate U^j's; $F_2^4 \times U^2, F_2^3 \times U^3, F_2^2 \times U^4$. This process is continued until F_5^5 is reached at which time the normal form has been obtained. What is happening as the recursion moves across the Lie triangle is the accumulation of the order 5 contributions of the near identity transformations of orders 2 to 5. In column 2, the contributions from substituting the transformations into the original equations are collected. In succeeding columns, the contributions from the interaction of new terms generated by the transformation and the subsequent transformations are collected until finally, in column 5, one has the order 5 terms of the normal form. The coefficient $\binom{k-l}{j-2}$ which appears in the sum is a counting term analogous to the binomial coefficient in the binomial theorem.

Now rewrite Eq. (11) using the new notation

$$-L_A U^k = \tilde{F}^k - G^k$$

where $\tilde{F}^k$ is a vector of the order k monomials resulting from the near identity transformations up to order $k-1$. If Eq. (11) is rewritten as

$$\tilde{F}^k = G^k - L_A U^k$$

then

$$\mathrm{proj}_{\ker L_{A^*}} \tilde{F}^k = G^k \tag{30}$$

and

$$\mathrm{proj}_{R_n^k} \tilde{F}^k = -L_A U^k. \tag{31}$$

However, if (11) is written as

$$G^k = \tilde{F}^k + L_A U^k$$

and it is noted that $F_1^1 \times U^k = (Ax) \times U^k = L_A U^k$, then

$$G^k = \tilde{F}^k + F_1^1 \times U^k.$$

Now consider the recursion (29). The only time U^k will appear is when $l = 2$, in which case (29) can be written

$$F_2^k = F_1^k + \sum_{j=2}^{k-1} \binom{k-2}{j-2} F_1^{k-j+1} \times U^j + F_1^1 \times U^k$$

or

$$F_2^k = \tilde{F}_2^k + F_1^1 \times U^k.$$

For $l = 3$, Eq. (29) can be written

$$F_3^k = \tilde{F}_2^k + F_1^1 \times U^k + \sum_{j=2}^{k-1} \binom{k-2}{j-2} F_2^{k-j+1} \times U^j$$

or

$$F_3^k = \tilde{F}_3^k + F_1^1 \times U^k.$$

So, for any l, $2 \le l \le k$,

$$F_l^k = \tilde{F}_l^k + F_1^1 \times U^k.$$

It is easy to see that the F_l^k obey the recursion relations

$$\tilde{F}_2^k = \tilde{F}_1^k + \sum_{j=2}^{k-1} \binom{k-2}{j-2} F_1^{k-j+1} \times U^j$$

$$\tilde{F}_l^k = \tilde{F}_{l-1}^k + \sum_{j=2}^{k-l+2} \binom{k-l}{j-2} F_{l-1}^{k-j+1} \times U^j, \qquad l = 3, \ldots, k.$$

This recursion is identical to Eq. (29) except for $\tilde{F}_2^k$, there the $F_1^1 \times U^k$ term was left out (and thus will not appear in any of the subsequent F_l^k). Thus

$$\tilde{F}_k^k = \tilde{F}^k = G^k - L_A U^k.$$

So Eq. (30) can be used to determine the order k normal form, and if Eq. (31) is written

$$U^k = -L_A^{-1} \operatorname{proj}_{R_n^k} \tilde{F}^k$$

the order k near identity transformation can be obtained.

In order to continue on to higher order terms in the normal form, it is necessary to convert the $\tilde{F}_l^k$'s into F_l^k's. This is accomplished using the following correction

$$F_l^k = \tilde{F}_l^k - \operatorname{proj}_{R_n^k} \tilde{F}^k, \qquad l = 2, \ldots, k.$$

IV. Dominant normal form

In order to study perturbations of a vector field with linear part given by the non-semisimple matrix A, we consider the universal unfolding of the linear vector field Ax used in van Gils et al. [11]

$$A(\lambda) = \begin{pmatrix} i+\alpha & 1 \\ \mu & i+\alpha \end{pmatrix}, \quad \lambda = (\alpha, \mu_1, \mu_2), \quad \alpha \in R, \quad \mu = \mu_1 + i\mu_2 \in C. \quad (32)$$

This may be calculated explicitly using the homological equation (10) applied to first degree polynomials $h^1(x)$, in the same manner as (23) for cubic polynomials. The unfolding parameters λ are found in terms of the original linear coefficients to be

$$\alpha = \frac{1}{2}\{f^1_{1;1000} + f^1_{2;0100}\} \quad \text{and} \quad \mu = f^1_{2;1000}$$

The above unfolding of $A(\lambda)$ may also be found from the viewpoint of versal deformations of matrices, as in Arnold [1], allowing for rescaling of time.

Now, making the observation that $z_1 = 0$ implies $z_2 = 0$ and the normal form commutes with S, we choose a transformation as in van Gils et al. [11]

$$z_1 = r\,e^{i\Phi}, \qquad z_2 = r\,e^{i\Phi}w, \qquad w = u + iv, \qquad \Phi = \omega t + \theta$$

which yields three real equations independent of the phase variable θ

$$\dot{r} = r[\alpha + u + r^2(c_1 + 2d_2 v)]$$

$$\dot{u} = \mu_1 - u^2 + v^2 + r^2(e_1 + 2f_2 v)$$

$$\dot{v} = \mu_2 - 2uv + r^2(f_1 - 2e_2 v)$$

and

$$\dot{\theta} = v + r^2(d_1 - 2c_2 v).$$

In order to "blow up" the dominant terms, we rescale the above variables as $r = \varepsilon\hat{r}$, $u = \varepsilon\hat{u}$, $v = \varepsilon\hat{v}$, $\varepsilon t = \hat{t}$, $\alpha = \varepsilon\hat{\alpha}$, $\mu_1 = \varepsilon^2\hat{\mu}_1$, $\mu_2 = \varepsilon^2\hat{\mu}_2$. Introducing $\hat{r}^2 = \hat{\varrho}$ and dropping the hats, we have, in new time,

$$\begin{aligned}
\varrho' &= 2\varrho(\alpha + u) + \varepsilon 2c_1\varrho^2 + O(\varepsilon^2) \\
u' &= \mu_1 - u^2 + v^2 + e_1\varrho + \varepsilon 2f_2\varrho v + O(\varepsilon^2) \\
v' &= \mu_2 - 2uv + f_1\varrho - \varepsilon 2e_2\varrho v + O(\varepsilon^2)
\end{aligned} \tag{33}$$

and

$$\theta' = v + \varepsilon d_1\varrho + O(\varepsilon^2)$$

where

$$\begin{aligned}
c_1 &= \mathrm{Re}(a_1), & e_1 &= \mathrm{Re}(b_1), & e_2 &= \mathrm{Re}(b_2) \\
d_1 &= \mathrm{Im}(a_1), & f_1 &= \mathrm{Im}(b_1), & f_2 &= \mathrm{Im}(b_2).
\end{aligned}$$

V. Averaged equations

In this section, we shall demonstrate the relationship between second order averaging and normal forms for the nilpotent case under consideration. To this end, we make use of the scaling suggested by Hale [13] for linear operators whose superdiagonals are equal to 1. In order to make the calculations less cumbersome, we only consider cubic nonlinearities and the nonlinear system can be written as

$$\dot{y} = A(\mu)y + F^0(y_1, y_3) + F^1(y) + F^2(y) + F^3(y_2, y_4) \tag{34}$$

where A is as given in Eq. (22) and the nonlinearities of degree 3 can be written in terms of the original notation as

$$F^0 = \sum_{s=1}^{4} \{f^3_{s;3000}y_1^3 + f^3_{s;2010}y_1^2y_3 + f^3_{s;1020}y_1y_3^2 + f^3_{s;0030}y_3^3\}e_s$$

$$\begin{aligned}
F^1 = \sum_{s=1}^{4} \{&f^3_{s;2100}y_1^2y_2 + f^3_{s;2001}y_1^2y_4 + f^3_{s;1110}y_1y_2y_3 + f^3_{s;1011}y_1y_3y_4 \\
&+ f^3_{s;0120}y_2y_3^2 + f^3_{s;0021}y_3^2y_4\}e_s
\end{aligned}$$

$$\begin{aligned}
F^2 = \sum_{s=1}^{4} \{&f^3_{s;1200}y_1y_2^2 + f^3_{s;1002}y_1y_4^2 + f^3_{s;1101}y_1y_2y_4 + f^3_{s;0111}y_2y_3y_4 \\
&+ f^3_{s;0210}y_2^2y_3 + f^3_{s;0012}y_3y_4^2\}e_s
\end{aligned}$$

$$F^3 = \sum_{s=1}^{4} \{f^3_{s;0300}y_2^3 + f^3_{s;0201}y_2^2y_4 + f^3_{s;0102}y_2y_4^2 + f^3_{s;0003}y_4^3\}e_s.$$

In order to bring the above equations into "standard form", we make use of the scaling suggested by Hale [13], which is in line with that of van Gils et al. [11],

$$y_1 = \varepsilon x_1, \qquad y_2 = \varepsilon^2 x_2, \qquad y_3 = \varepsilon x_3; \qquad y_4 = \varepsilon^2 x_4$$

and transform Eq. (34) to new variables z by means of the transformation

$$x_j = z_j e^{i\omega t}, \qquad x_{j+2} = \bar{z}_j e^{-i\omega t}, \qquad j = 1, 2.$$

This procedure yields a set of equations in standard form to $O(\varepsilon^2)$ as

$$\dot{z} = \varepsilon X^0(z, \bar{z}, t) + \varepsilon^2 X^1(z, \bar{z}, t), \qquad z = (z_1, z_2) \tag{35}$$

where

$$X^0 = \begin{bmatrix} z_2 \\ e^{-i\omega t} F_2^0(z_1, \bar{z}_1, t) \end{bmatrix}, \qquad X^1 = \begin{bmatrix} e^{-i\omega t} F_1^0(z_1, \bar{z}_1, t) \\ e^{-i\omega t} F_2^1(z, \bar{z}, t) \end{bmatrix} \tag{36}$$

and the $\dot{\bar{z}}$ equations are obtained by conjugating Eq. (35). Now, applying the averaging procedure up to the second order yields

$$\dot{z} = \varepsilon \underset{t}{M}\{X^0(z, \bar{z}, t)\} + \varepsilon^2 \underset{t}{M} \times \left\{ \frac{\partial X^0}{\partial z} W + \frac{\partial X^0}{\partial \bar{z}} \bar{W} - \frac{\partial W}{\partial z} \tilde{X}^0 - \frac{\partial W}{\partial \bar{z}} \bar{\tilde{X}}^0 + X^1(z, \bar{z}, t) \right\} \tag{37}$$

where $\underset{t}{M}$ is the averaging operator defined as

$$\underset{t}{M}(\;\cdot\;) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (\;\cdot\;)\, dt$$

and

$$W(z, \bar{z}, t) = \int_0^t \tilde{X}^0\, dt + c(z, \bar{z}),$$

$$\tilde{X}^0(z, \bar{z}, t) = X^0(z, \bar{z}, t) - \underset{t}{M}\{X^0(z, \bar{z}, t)\}$$

i.e.

$$W_1(z, \bar{z}, t) = c_1(z, \bar{z}) \quad \text{and} \quad W_2(z, \bar{z}, t) = k(z, \bar{z}, t) + c_2(z, \bar{z})$$

with $k(z, \bar{z}, t)$ defined as

$$k(z, \bar{z}, t) = \frac{1}{2i\omega} f^3_{2;3000} z_1^3 e^{2i\omega t} - \frac{1}{2i\omega} f^3_{2;1020} z_1 \bar{z}_1^2 e^{-2i\omega t} - \frac{1}{4i\omega} f^3_{2;0030} \bar{z}_1^3 e^{-4i\omega t}$$

where c is an arbitrary vector function of z and $\bar{z}$. The choice of c is made such that the normal form coincides with the resulting second order averaged equations. We have made two observations concerning the product terms within the second curly bracket of Eq. (37). Note, in Eq. (36), that X_2^0 is only a function of z_1 and $\bar{z}_1$ and $\tilde{X}_1^0 = 0$, $\bar{\tilde{X}}_1^0 = 0$. Thus, the second order contribution from $k(z, \bar{z}, t)$ is identically zero. The second order contributions to the averaged equations are

$$\varepsilon^2 \underset{t}{M}\{c_2 - L(c_1) + X_1^1(z, \bar{z}, t)\}$$

$$\varepsilon^2 \underset{t}{M}\{2f^3_{2;2010} c_1 z_1 \bar{z}_1 + f^3_{2;2010} \bar{c}_1 z_1^2 - L(c_2) + X_2^1(z, \bar{z}, t)\}$$

where

$$L(\cdot) = \frac{\partial(\cdot)}{\partial z_1} z_2 + \frac{\partial(\cdot)}{\partial z_2} f^3_{2;2010} z_1^2 \bar{z}_1 - \frac{\partial(\cdot)}{\partial \bar{z}_1} \bar{z}_2 - \frac{\partial(\cdot)}{\partial \bar{z}_2} f^3_{4;1020} z_1 \bar{z}_1^2.$$

Comparing terms of like order in the averaged and normal form equations, the appropriate choice of the vector c is given by

$$c_1(z, \bar{z}) = \alpha_1 z_2, \qquad c_2(z, \bar{z}) = \alpha_2 z_1^2 \bar{z}_1.$$

Equating coefficients yields

$$\alpha_1 f^3_{2;2010} - \alpha_2 = \frac{1}{4}[f^3_{1;2010} - f^3_{2;2001} - f^3_{2;1110}],$$

$$\bar{\alpha}_1 f^3_{2;2010} - \alpha_2 = \frac{1}{4}[f^3_{1;2010} - f^3_{2;2001} - f^3_{2;1110}].$$

It is obvious that α_1 must be real. Choosing α_1 to be identically zero, α_2 is obtained as

$$\alpha_2 = -\frac{1}{4}[f^3_{1;2010} - f^3_{2;2001} - f^3_{2;1110}].$$

Thus, the averaged equations are

$$\dot{z}_1 = \varepsilon z_2 + \varepsilon^2 a_1(z_1 \bar{z}_1) z_1 + O(\varepsilon^3)$$

$$\dot{z}_2 = \varepsilon b_1(z_1 \bar{z}_1) z_1 + \varepsilon^2 [b_2(z_1 \bar{z}_2) + (a_1 - b_2)(\bar{z}_1 z_2)] z_1 + O(\varepsilon^3). \tag{38}$$

The second pair of equations are obtained by conjugating Eq. (38). As before, we introduce the universal unfolding defined by matrix $A(\lambda)$ (see Eq. (32)) into Eq. (38), use the transformation

$$z_1 = r\, e^{i\theta}, \qquad z_2 = r\, e^{i\theta} w, \qquad w = u + iv \tag{39}$$

and rescale the variables as

$$\varepsilon t = \hat{t}, \qquad \mu_1 = \varepsilon^2 \hat{\mu}_1, \qquad \mu_2 = \varepsilon^2 \hat{\mu}_2, \qquad \alpha = \varepsilon \hat{\alpha}. \tag{40}$$

After substituting Eqs. (39) and (40) into Eq. (38) and dropping the hats, we have the averaged equations in terms of $\varrho = r^2$ as expressed in Eq. (33). Thus, one can conclude that the dominant terms of the scaled normal form equations (33) agree completely with those of the averaged equations.

Appendix

The transformations $h^2_{i;m}$, $i = 1, 2$, which eliminate the quadratic terms are:

$$h^2_{1;2000} = \frac{1}{\omega^2}(f^2_{2;2000} + i\omega f^2_{1;2000})$$

$$h^2_{1;1100} = \frac{1}{\omega^3}[\omega(-2f^2_{1;2000} + f^2_{2;1100}) + i(4f^2_{2;2000} + \omega^2 f^2_{1;1100})]$$

$$h^2_{1;0200} = \frac{1}{\omega^4}[-6f^2_{2;2000} + \omega^2(-f^2_{1;1100} + f^2_{2;0200}) + i\omega(-2f^2_{1;2000} + 2f^2_{2;1100} + \omega^2 f^2_{1;0200})]$$

$$h^2_{1;1010} = \frac{1}{\omega^2}(f^2_{2;1010} - i\omega f^2_{1;1010})$$

$$h^2_{1;0110} = \frac{1}{\omega^3}[\omega(-f^2_{1;1010} + f^2_{2;0110}) + i(-2f^2_{2;1010} - \omega^2 f^2_{1;0110})]$$

$$h^2_{1;0020} = \frac{1}{3\omega^2}\left(\frac{1}{3}f^2_{2;0020} - i\omega f^2_{1;0020}\right)$$

$$h^2_{1;1001} = \frac{1}{\omega^3}[\omega(-f^2_{1;1010} + f^2_{2;1001}) + i(-2f^2_{2;1010} - \omega^2 f^2_{1;1001})]$$

$$h^2_{1;0101} = \frac{1}{\omega^4}[-6f^2_{2;1010} + \omega^2(-f^2_{1;0110} - f^2_{1;1001} + f^2_{2;0101}) + i\omega(2f^2_{1;1010} - \omega^2 f^2_{1;0101})]$$

$$h^2_{1;0011} = \frac{1}{3\omega^3}\left[\frac{1}{3}\omega(-2f^2_{1;0020} + f^2_{2;0011}) + i\left(-\frac{4}{9}f^2_{2;0020} - \omega^2 f^2_{1;0011}\right)\right]$$

$$h^2_{1;0002} = \frac{1}{3\omega^4}\left[-\frac{2}{9}f^2_{2;0020} + \frac{1}{3}(-f^2_{1;0011} + f^2_{2;0002}) + i\omega\left(\frac{2}{9}f^2_{1;0020} - \omega^2 f^2_{1;0002}\right)\right]$$

$$h^2_{2;2000} = \frac{1}{\omega}f^2_{2;2000}$$

$$h^2_{2;1100} = \frac{1}{\omega^2}(-2f^2_{2;2000} + i\omega f^2_{2;1100})$$

$$h^2_{2;0200} = \frac{1}{\omega^3}[-\omega f^2_{2;1100} + i(-2f^2_{2;2000} + \omega^2 f^2_{2;0200})]$$

$$h^2_{2;1010} = \frac{-i}{\omega} f^2_{2;1010}$$

$$h^2_{2;0110} = \frac{-1}{\omega^2} (f^2_{2;1010} + i f^2_{2;0110})$$

$$h^2_{2;0020} = \frac{-i}{3\omega} f^2_{2;0020}$$

$$h^2_{2;1001} = \frac{-1}{\omega^2} (f^2_{2;1010} + i\omega f^2_{2;1001})$$

$$h^2_{2;0101} = \frac{1}{\omega^3} [-\omega(f^2_{2;0110} + f^2_{2;1001}) + i(2f^2_{2;1010} - \omega^2 f^2_{2;0101})]$$

$$h^2_{2;0011} = \frac{-1}{3\omega^3} \left(\frac{2}{3} f^2_{2;0020} + i\omega f^2_{2;0011}\right)$$

$$h^2_{2;0002} = \frac{1}{3\omega^3} \left[-\frac{1}{3} f^2_{2;0011} + i\left(\frac{2}{9} f^2_{2;0020} - \omega^2 f^2_{2;0002}\right)\right].$$

The contributions from the quadratic non-linearities to the normal form are given by the coefficients $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$ and $\tilde{b}_2$ and are expressed as follows:

$$\begin{aligned}
\tilde{a}_1 = \frac{1}{\omega^2} \bigg\{ & \frac{1}{4} f^2_{1;1010} f^2_{2;2000} - f^2_{1;2000} f^2_{2;1010} - \frac{3}{4} f^2_{2;1001} f^2_{2;2000} + \frac{1}{2} f^2_{2;1010} f^2_{3;1010} \\
& + \frac{1}{9} f^2_{2;0020} f^2_{3;2000} - \frac{1}{4} f^2_{2;1010} f^2_{4;0110} - \frac{1}{4} f^2_{2;1010} f^2_{4;1001} - \frac{3}{4} f^2_{1;1010} f^2_{4;1010} \\
& - \frac{1}{4} f^2_{2;0110} f^2_{4;1010} + \frac{1}{2} f^2_{2;1001} f^2_{4;1010} - \frac{1}{18} f^2_{2;0020} f^2_{4;1100} - \frac{1}{6} f^2_{1;0020} f^2_{4;2000} \\
& + \frac{1}{36} f^2_{2;0011} f^2_{4;2000} \bigg\} + \frac{i}{\omega^3} \bigg\{ f^2_{2;1010} f^2_{2;2000} - f^2_{2;1010} f^2_{4;1010} - \frac{2}{27} f^2_{2;0020} f^2_{4;2000} \\
& + \omega^2 \bigg[\frac{3}{4} f^2_{1;1010} f^2_{1;2000} - \frac{1}{4} f^2_{1;2000} f^2_{2;1001} + \frac{1}{2} f^2_{1;1100} f^2_{2;1010} + \frac{1}{2} f^2_{2;0200} f^2_{2;1010} \\
& + \frac{1}{4} f^2_{1;1010} f^2_{2;1100} + \frac{1}{4} f^2_{2;1001} f^2_{2;1100} - \frac{1}{4} f^2_{1;0110} f^2_{2;2000} + \frac{1}{2} f^2_{1;1001} f^2_{2;2000} \\
& - \frac{1}{4} f^2_{2;0101} f^2_{2;2000} - \frac{1}{4} f^2_{2;1010} f^2_{3;0110} - \frac{1}{4} f^2_{2;1010} f^2_{3;1001} - \frac{3}{4} f^2_{1;1010} f^2_{3;1010} \\
& - \frac{1}{4} f^2_{2;0110} f^2_{3;1010} - \frac{1}{6} f^2_{2;0020} f^2_{3;1100} - \frac{1}{2} f^2_{1;0020} f^2_{3;2000} - \frac{1}{12} f^2_{2;0011} f^2_{3;2000} \\
& - \frac{1}{4} f^2_{2;1001} f^2_{4;0110} - \frac{1}{4} f^2_{2;1001} f^2_{4;1001} - \frac{3}{4} f^2_{1;1001} f^2_{4;1010} - \frac{1}{4} f^2_{2;0101} f^2_{4;1010} \\
& - \frac{1}{12} f^2_{2;0011} f^2_{4;1100} - \frac{1}{4} f^2_{1;0011} f^2_{4;2000} - \frac{1}{6} f^2_{2;0002} f^2_{4;2000} \bigg] \bigg\}
\end{aligned}$$

$$
\begin{aligned}
\tilde{a}_2 = \frac{1}{\omega^4}\Big\{ & f^2_{2;1010}f^2_{4;1010} - \frac{4}{81}f^2_{2;0020}f^2_{4;2000} + \omega^2\Big[\frac{1}{3}f^2_{1;2000}f^2_{2;0110} \\
& - \frac{1}{3}f^2_{1;2000}f^2_{2;1001} + \frac{1}{6}f^2_{2;0110}f^2_{2;1100} - \frac{1}{6}f^2_{2;1001}f^2_{2;1100} + \frac{1}{6}f^2_{2;1010}f^2_{3;0110} \\
& + \frac{1}{6}f^2_{2;1010}f^2_{3;1001} + \frac{1}{6}f^2_{1;1010}f^2_{3;1010} - \frac{1}{6}f^2_{2;0110}f^2_{3;1010} - \frac{2}{27}f^2_{2;0020}f^2_{3;1100} \\
& - \frac{2}{27}f^2_{1;0020}f^2_{3;2000} + \frac{1}{27}f^2_{2;0011}f^2_{3;2000} - \frac{1}{6}f^2_{2;1010}f^2_{4;0101} + \frac{1}{6}f^2_{1;1010}f^2_{4;0110} \\
& + \frac{1}{3}f^2_{2;0110}f^2_{4;0110} + \frac{1}{6}f^2_{2;1001}f^2_{4;0110} + \frac{2}{27}f^2_{2;0020}f^2_{4;0200} - \frac{1}{3}f^2_{1;1010}f^2_{4;1001} \\
& - \frac{1}{6}f^2_{2;0110}f^2_{4;1001} + \frac{1}{6}f^2_{2;1001}f^2_{4;1001} + \frac{1}{6}f^2_{1;0110}f^2_{4;1010} + \frac{1}{6}f^2_{1;1001}f^2_{4;1010} \\
& - \frac{1}{6}f^2_{2;0101}f^2_{4;1010} + \frac{1}{27}f^2_{1;0020}f^2_{4;1100} - \frac{1}{18}f^2_{2;0011}f^2_{4;1100} \\
& - \frac{2}{27}f^2_{1;0011}f^2_{4;2000} + \frac{2}{27}f^2_{2;0002}f^2_{4;2000}\Big]\Big\} \\
& + \frac{i}{\omega^3}\Big\{\frac{1}{3}f^2_{2;1010}f^2_{3;1010} - \frac{4}{81}f^2_{2;0020}f^2_{3;2000} - \frac{1}{3}f^2_{2;1010}f^2_{4;0110} \\
& - \frac{1}{3}f^2_{2;1010}f^2_{4;1001} - \frac{1}{3}f^2_{1;1010}f^2_{4;1010} + \frac{1}{3}f^2_{2;0110}f^2_{4;1010} + \frac{1}{3}f^2_{2;1001}f^2_{4;1010} \\
& + \frac{4}{81}f^2_{2;0020}f^2_{4;1100} + \frac{4}{81}f^2_{1;0020}f^2_{4;2000} - \frac{4}{81}f^2_{2;0011}f^2_{4;2000} \\
& + \omega^2\Big[\frac{1}{3}f^2_{1;1001}f^2_{1;2000} - \frac{1}{3}f^2_{1;0110}f^2_{1;2000} \\
& - \frac{1}{6}f^2_{1;1100}f^2_{2;0110} - \frac{1}{3}f^2_{2;0110}f^2_{2;0200} + \frac{1}{6}f^2_{1;1100}f^2_{2;1001} + \frac{1}{3}f^2_{2;0200}f^2_{2;1001} \\
& - \frac{1}{6}f^2_{1;0110}f^2_{2;1100} + \frac{1}{6}f^2_{1;1001}f^2_{2;1100} - \frac{1}{6}f^2_{2;1010}f^2_{3;0101} + \frac{1}{6}f^2_{1;1010}f^2_{3;0110} \\
& + \frac{1}{3}f^2_{2;0110}f^2_{3;0110} + \frac{2}{9}f^2_{2;0020}f^2_{3;0200} - \frac{1}{3}f^2_{1;1010}f^2_{3;1001} - \frac{1}{6}f^2_{2;0110}f^2_{3;1001} \\
& + \frac{1}{6}f^2_{1;0110}f^2_{3;1010} + \frac{1}{9}f^2_{1;0020}f^2_{3;1100} - \frac{1}{18}f^2_{2;0011}f^2_{3;1100} - \frac{1}{9}f^2_{1;0011}f^2_{3;2000} \\
& - \frac{1}{6}f^2_{2;1001}f^2_{4;0101} + \frac{1}{6}f^2_{1;1001}f^2_{4;0110} + \frac{1}{3}f^2_{2;0101}f^2_{4;0110} + \frac{1}{9}f^2_{2;0011}f^2_{4;0200}
\end{aligned}
$$

$$-\frac{1}{3}f^2_{1;1001}f^2_{4;1001}-\frac{1}{6}f^2_{2;0101}f^2_{4;1001}+\frac{1}{6}f^2_{1;0101}f^2_{4;1010}+\frac{1}{18}f^2_{1;0011}f^2_{4;1100}$$

$$-\frac{1}{9}f^2_{2;0002}f^2_{4;1100}-\frac{2}{9}f^2_{1;0002}f^2_{4;2000}\Bigg]\Bigg\}$$

$$\tilde{b}_1=-\frac{1}{\omega^2}\left\{3f^2_{2;1010}f^2_{2;2000}+f^2_{2;1010}f^2_{4;1010}+\frac{2}{9}f^2_{2;0020}f^2_{4;2000}\right\}$$

$$+\frac{i}{\omega}\Bigg\{f^2_{2;1010}f^2_{2;1100}-f^2_{1;2000}f^2_{2;1010}+2f^2_{1;1010}f^2_{2;2000}-f^2_{2;0110}f^2_{2;2000}$$

$$-f^2_{2;1010}f^2_{3;1010}-\frac{2}{3}f^2_{2;0020}f^2_{3;2000}-f^2_{2;1001}f^2_{4;1010}-\frac{1}{3}f^2_{2;0011}f^2_{4;2000}\Bigg\}$$

$$\tilde{b}_2=\frac{1}{\omega^2}\Bigg\{f^2_{2;1010}f^2_{2;1100}-f^2_{1;2000}f^2_{2;1010}+\frac{3}{4}f^2_{1;1010}f^2_{2;2000}-\frac{9}{4}f^2_{2;1001}f^2_{2;2000}$$

$$+\frac{1}{2}f^2_{2;1010}f^2_{3;1010}-\frac{1}{9}f^2_{2;0020}f^2_{3;2000}+\frac{1}{4}f^2_{2;1010}f^2_{4;0110}-\frac{3}{4}f^2_{2;1010}f^2_{4;1001}$$

$$-\frac{1}{4}f^2_{1;1010}f^2_{4;1010}+\frac{1}{4}f^2_{2;0110}f^2_{4;1010}+\frac{1}{2}f^2_{2;1001}f^2_{4;1010}+\frac{1}{18}f^2_{2;0020}f^2_{4;1100}$$

$$-\frac{1}{18}f^2_{1;0020}f^2_{4;2000}-\frac{5}{36}f^2_{2;0011}f^2_{4;2000}\Bigg\}$$

$$+\frac{i}{\omega^3}\Bigg\{3f^2_{2;1010}f^2_{2;2000}-f^2_{2;1010}f^2_{4;1010}+\frac{2}{27}f^2_{2;0020}f^2_{4;2000}$$

$$+\omega^2\Bigg[\frac{1}{4}f^2_{1;1010}f^2_{1;2000}-\frac{3}{4}f^2_{1;2000}f^2_{2;1001}+\frac{1}{2}f^2_{1;1100}f^2_{2;1010}$$

$$-\frac{1}{2}f^2_{2;0200}f^2_{2;1010}-\frac{1}{4}f^2_{1;1010}f^2_{2;1100}+\frac{3}{4}f^2_{2;1001}f^2_{2;1100}-\frac{3}{4}f^2_{1;0110}f^2_{2;2000}$$

$$+\frac{3}{2}f^2_{1;1001}f^2_{2;2000}-\frac{3}{4}f^2_{2;0101}f^2_{2;2000}+\frac{1}{4}f^2_{2;1010}f^2_{3;0110}-\frac{3}{4}f^2_{2;1010}f^2_{3;1001}$$

$$-\frac{1}{4}f^2_{1;1010}f^2_{3;1010}+\frac{1}{4}f^2_{2;0110}f^2_{3;1010}+\frac{1}{6}f^2_{2;0020}f^2_{3;1100}-\frac{1}{6}f^2_{1;0020}f^2_{3;2000}$$

$$-\frac{1}{4}f^2_{2;0011}f^2_{3;2000}+\frac{1}{4}f^2_{2;1001}f^2_{4;0110}-\frac{3}{4}f^2_{2;1001}f^2_{4;1001}-\frac{1}{4}f^2_{1;1001}f^2_{4;1010}$$

$$+\frac{1}{4}f^2_{2;0101}f^2_{4;1010}+\frac{1}{12}f^2_{2;0011}f^2_{4;1100}$$

$$-\frac{1}{12}f^2_{1;0011}f^2_{4;2000}-\frac{1}{2}f^2_{2;0002}f^2_{4;2000}\Bigg]\Bigg\}.$$

Acknowledgements

This research was partially supported by the National Science Foundation through Grant MSS 90-57437-PYI, the Air Force Office of Scientific Research under grant 91-0041, and the National Science and Research Council of Canada. The first author would like to thank Prof. S. T. Ariaratnam of the University of Waterloo for pointing out the scaling given in Hale [13] and Prof. Ranga Rao of the University of Illinois for his valuable discussions.

References

[1] V. I. Arnold, *Geometric Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York 1983.
[2] A. K. Bajaj and P. R. Sethna, *Flow induced bifurcations to three-dimensional oscillatory motions in continuous tubes*, SIAM J. Appl. Math. *44*(2), 270–286 (1984).
[3] G. D. Birkhoff, *Dynamical Systems*, AMS Publ., Providence 1927.
[4] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York 1982.
[5] R. Cushman and J. A. Sanders, *Nilpotent normal forms and representation theory of sl (2, R)*, in *Multiparameter Bifurcation Theory*, ed. M. Golubitski and J. Guckenheimer, Contemp. Math. 56, AMS Publ., Providence 1986, pp. 31–51.
[6] A. Deprit, *Canonical transformations depending on a small parameter*, Celest. Mech. *1*, 12–32 (1969).
[7] C. Elphick, E. Tirapequi, M. E. Brachet, P. Coullet and G. Iooss, *A simple global characterization for normal forms of singular vector fields*, Physica *29D*, 95–127 (1987).
[8] E. Freire, E. Gamero and E. Ponce, *An algorithm for symbolic computation of Hopf bifurcation*, in *Computers and Mathematics*, eds. E. Kaltofen and S. M. Watt, Springer-Verlag, New York 1989, pp. 109–118.
[9] E. Freire, E. Gamero, E. Ponce and L. G. Franquelo, *An algorithm for symbolic computation of centre manifolds*, in *Symbolic and Algebraic Computation*, ed. P. Gianni, Lect. Notes in Comp. Sci. 358, Springer-Verlag, New York 1989, pp. 218–230.
[10] E. Gamero, E. Freire and E. Ponce, *Normal forms for planar systems with nilpotent linear part*, Int. Series of Num. Math. *97*, 123–127 (1991).
[11] S. A. van Gils, M. Krupa and W. F. Langford, *Hopf bifurcation with non-semisimple 1 : 1 resonance*, Nonlinearity *3*, 825–850 (1990).
[12] J. Guckenheimer and P. Holmes, *Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York 1983.
[13] J. K. Hale, *Oscillations in Nonlinear Systems*, McGraw-Hill, New York 1963.
[14] S. Helgason, *Groups and Geometric Analysis*, Academic Press, Inc., New York 1984.
[15] K. R. Meyer, *Normal forms for the general equilibrium*, Funkcialaj Ekvacioj *27*, 261–271 (1984).
[16] H. Poincaré, *Mémoire sur les courbes définis par une equation différentielle IV*, J. Math. Pures Appl. *1*, 167–244 (1885).
[17] E. J. Ponce-Nunez and E. Gamero, *Generating Hopf bifurcation formulae with MAPLE*, Int. Series of Num. Math. *97*, 295–299 (1991).
[18] G. Iooss and M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore 1992.
[19] G. Iooss, A. Mielke and Y. Demay, *Theory of steady Ginsburg-Landau equations in hydrodynamic stability problems*, Eur. J. Mech., B/Fluids. *8*(3), 229–268 (1989).

Abstract

The primary result of this research is the derivation of an explicit formula for the Poincaré-Birkhoff normal form of the generalized Hopf bifurcation with non-semisimple 1 : 1 resonance. The classical nonuniqueness of the normal form is resolved by the choice of complementary space which yields a unique equivariant normal form. The 4 leading complex constants in the normal form are calculated in terms of the original coefficients of both the quadratic and cubic nonlinearities by two different algorithms. In addition, the universal unfolding of the degenerate linear operator is explicitly determined. The dominant normal forms are then obtained by rescaling the variables. Finally, the methods of averaging and normal forms are compared. It is shown that the dominant terms of the equivariant normal form are, indeed, the same as those of the averaged equations with a particular choice for the constant of integration.

(Received: April 23, 1992; revised: April 23, 1993)