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Coexistent phase mixtures in the anti-plane shear of an elastic tube

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Introduction

In 1949, Rivlin [10] applied the general theory of non-linear elasticity for an incompressible, isotropic, homogeneous body to study the helical shearing of a circular tube for a Mooney material. About 25 years later, Ogden, Chadwick & Haddon [9] reconsidered this problem in some detail for more general materials. In neither of these works was there an emphasis placed upon any detailed structure of the stored energy function for the material and how that might relate to the possible existence of equilibrium states other than those described. Fosdick & MacSithigh [5] re-examined this problem, in 1983, as one of energy minimization with emphasis placed on the situation when the stored energy function of the material is not convex. Earlier, based on work of Fosdick & Serrin [6], Fosdick & Kao [4] had already noted in the context of nonlinear elasticity theory that if the cross section of the tube is neither circular nor the annulus between concentric circles then an anti-plane shear field (2) is not possible unless the material is suitably restricted. More recently, Bauman & Phillips [2] have considered this anti-plane shear minimization problem for an elastic tube whose stored energy function is in this restricted class, and whose cross section is a convex ring. They investigated the issues concerning uniqueness and existence of solutions when the stored energy function is not convex.

For the elastic tube whose cross section is a convex ring and whose stored energy function is convex, there is a unique solution to the anti-plane shear minimization problem for any prescribed relative axial displacement of its lateral boundaries. The smoothness of the minimizer and its gradient depends only upon the smoothness of the stored energy function. For a material with a non-convex stored energy function, Fosdick & MacSithigh [5] have shown that, though the anti-plane shear minimization problem of an elastic, concentric, circular tube has a unique solution for any prescribed relative axial displacement of the lateral boundaries, the minimizer must possess a jump in its deformation gradient across a certain cylindrical surface when the prescribed relative axial displacement of the boundaries is given in a well-defined range. This range is determined by the Maxwell strains and the radii of the two concentric circles that form the cross section of the tube. For an elastic tube with a more general cross section consisting of a convex ring, the loss of convexity in the stored energy function implies that there is a range in the prescribed relative axial displacement of the lateral boundaries where the anti-plane shear minimization problem does not have a solution [2].

Our interest in this study is in the structure of the deformed configuration of an elastic tube whose cross section is a convex ring that is subjected to a prescribed relative axial displacement of its lateral boundaries. The material is assumed to have a non-convex stored energy function. Special attention is paid to the situation when there is no minimizer to this anti-plane shear minimization problem, but, nevertheless, the energy functional has an infimum.

The thrust of this investigation has several folds: First, let the outer lateral surface of the elastic tube be fixed and its inner lateral surface be displaced uniformly along its axial direction. A natural and practical question concerns the relationship between the force applied to the inner lateral surface of the tube and the distance displaced by the surface. Clearly, the determination of such a relationship requires an understanding of the structure of the deformed configuration, and moreover, this relationship may be experimentally investigated. Second, plasticity theory is commonly employed in the study of problems that involve the damage of materials and the localization of deformation (i.e., shear bands). By investigating the structure of the deformed configuration of an elastic tube with a non-convex stored energy function, we may provide a new perspective on the understanding of the mechanisms of material damage and the localization of severe deformations from an energetic point of view. Finally, the non-existence of a minimizer to the above anti-plane shear minimization problem for a certain range of prescribed relative axial displacements of the lateral boundaries implies that among all "admissible" deformations there is none for which the values of the stored energy function correspond to its convex points almost everywhere in the body. Because of this, we find that to reach the infimum the tube divides into three subdomains: one of high strain, one of low strain, and one of intermediate "mixed" strain. In the intermediate "mixed" strain subdomain, the field values of the stored energy correspond to convex combinations of convex, but not strictly convex, points of the stored energy function. The main variational problem then gives rise to a free boundary problem in which the subdomain where the strict convexity of the stored energy function breaks down must be determined as part of the solution. The characterization of this intermediate phase mixture region is one of the goals of this work.

We begin in Section 1 by describing the geometry of the cross section of the tube, the material constitutive assumptions, and the boundary conditions. For both simplicity and the ability to obtain more detailed information on the structure of the deformed configuration, we take the cross section to be bounded by two non-concentric circles, and we propose to consider a trilinear material ([1], [11]). The displacement boundary condition requires that the outer lateral surface of the tube be held fixed while its inner lateral surface is displaced uniformly a distance H along the axial direction. We, then, define the main anti-plane shear minimization *Problem P1* of this work in (10) and (11) and a corresponding relaxed minimization *Problem P2* in (16) and (11). Theorem 1 shows that a solution to the relaxed minimization problem is unique, and contains some properties that are useful in the construction of a minimizing sequence to the minimization *Problem P1.* These results are essentially taken from Bauman and Phillips [2]. In Theorem 2, the questions of uniqueness and existence of a solution to the minimization *Problem P1* are discussed. We determine a specific range (H_m, H_M) for the prescribed axial displacement H in which there is no solution to the minimization *Problem P1.* This range depends upon the radii of the circles, the eccentricity, and the Maxwell strains.

Section 2 is devoted to obtaining more detailed structure of the solution to the relaxed minimization *Problem P2* when the prescribed axial displacement H is in the range (H_m, H_M). The attention here is to the subdomain of the body (i.e., the phase mixture region) where the field values of the stored energy correspond to convex but not strictly convex points of the stored energy function. Proposition 2 shows that the contour curves of the solution to *Problem P2* must be straight lines in the phase mixture region. In Proposition 3, after dividing the cross section of the tube into symmetric halves, we prove that the magnitude of deformation gradient of the solution is monotone along the inner and outer half-boundaries of the cross sectional domain. In Proposition 5, we essentially demonstrate that at any point on the boundary of the phase mixture region where the normal to that boundary is parallel to the direction of the gradient of the solution, the thickness of the phase mixture region at that point must vanish. This happens on the axis of symmetry of the cross section. Finally, we show in Proposition 6 that the interior of the phase mixture region can not intersect the boundary of the tube. Thus, a picture that emerges from Section 2 of the level sets $u =$ const., which correspond to curves of constant axial displacement for *Problem P2* when $H \in (H_m, H_M)$, is illustrated (not to scale) in Figure 1. Only the top half of the whole region between the eccentric cylinders is shown since the figure is symmetric. Notice, in particular, the straight segments for a curve $u =$ const. as it runs through the phase mixture region.

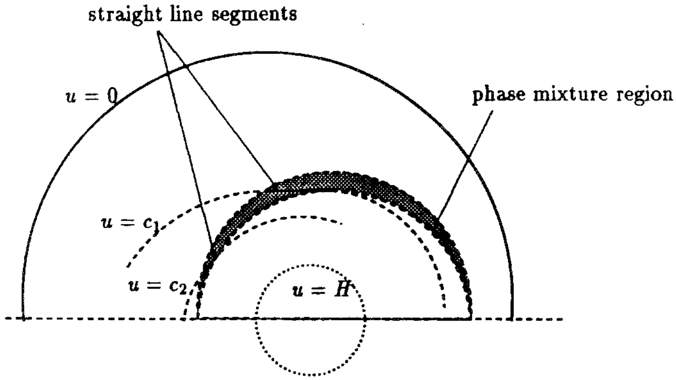


Figure 1
Anti-plane shear of eccentric cylinders. Illustration of $u = \text{const.}$ curves for $H \in (H_m, H_M)$; $0 < c_1 < c_2 < H$.

Section 3 contains the results of our numerical investigation, obtained by a finite element method as implemented on the CM 200. In Figures 5, 6, and 7 we show calculated constant displacement contours for *Problem P2* and the corresponding curves of constant displacement gradient magnitude (i.e., shear strain). Also, we exhibit the detailed shape of the phase mixture region for *Problem P1.* Our calculations, here, confirm much of our theoretical development in the previous section. In Figure 8 we show how the axial force per unit length and the prescribed axial displacement are related as the phase mixture region forms and advances through the body. Here, to calculate the force per unit length we have used the classical relations from nonlinear elasticity theory between stress and stored energy function.

The effort in Section 4 is towards the construction of a minimizing sequence for the minimization *Problem P1.* For a prescribed axial displacement in the range (H_m, H_M), we give the detailed construction of a minimizing sequence which, in particular, ensures the continuity of each member of the sequence. In Theorem 3, we show that this sequence converges weakly to the solution of the relaxed minimization *Problem P2.* That is, the sequence converges to the solution of *Problem P2* pointwise, but in the phase mixture region the gradient of the sequence converges only weakly in L^p, $1 < p < \infty$, to the gradient of the solution of *Problem P2*; in a rough sense the limit takes on the form of a fine grained inhomogeneous mixture of two phases (i.e., gradients). Theorem 4 verifies that the total potential energy associated with this sequence converges to the infimum of the total potential energy within the admissible class of anti-plane shear deformations in $W^{1,2}(\Omega)$.

Finally, in Section 5, we interpret a few of our main conclusions and briefly discuss how our construction of a minimizing sequence carries over to more general materials. We also discuss the results (c.f., Figure 10) of a preliminary computational investigation for the anti-plane shear problem when the cross section of the tube is bounded by a fixed outer circle and an axially displaced inner concentric ellipse of small aspect ratio.

1. Preliminaries

Consider an elastic, incompressible, isotropic, and homogeneous cylindrical tube $\mathscr{B}$ of uniform cross section. Let Ω denote the cross section of $\mathscr{B}$ normal to its axis, and assume that Ω is bounded between two eccentric circular domains Ω_i, and Ω_o of radii R_i, and $R_o > R_i + |e|$, where

$$\Omega_i \equiv \{(x_1, x_2) \in \mathbb{R}^3 \colon (x_1 - e)^2 + x_2^2 < R_i^2\},$$

$$\Omega_o \equiv \{(x_1, x_2) \in \mathbb{R}^3 \colon x_1^2 + x_2^2 < R_o^2\}.$$

We shall assume that $\mathscr{B}$ is infinitely long with axis parallel to the x_3 coordinate direction of a rectangular Cartesian frame. Thus,

$$\mathscr{B} \equiv \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon (x_1, x_2) \in \Omega \text{ and } x_3 \in \mathbb{R})\},$$

where

$$\Omega \equiv \Omega_o \backslash \bar{\Omega}_i, \tag{1}$$

and $\bar{\Omega}_i$ is the closure of Ω_i. The eccentricity of the domain is supposed to satisfy $e \geq 0$, with $e = 0$ corresponding to a concentric circular ring.

The tube $\mathscr{B}$ is said to undergo an anti-plane shear deformation $\boldsymbol{\chi}$ if

$$\boldsymbol{\chi} \colon (x_1, x_2, x_3) \to (x_1, x_2, x_3 + u(x_1, x_2)) \in \mathbb{R}^3. \tag{2}$$

Because the displacement field u is independent of x_3, it will be convenient to set $\boldsymbol{x} \equiv (x_1, x_2)$, and to let $\boldsymbol{D} \equiv (\partial/\partial x_1, \partial/\partial x_2)$ denote the two dimensional gradient operator. It is straightforward to show that the deformation $\boldsymbol{\chi}$ of (2) is isochoric, and that the first two principal invariants of either the right or left Cauchy-Green strain tensor are equal and given by

$$I = II = 3 + [\kappa(\boldsymbol{x})]^2, \tag{3}$$

where the shear $\kappa(\boldsymbol{x})$ has the form

$$\kappa(\boldsymbol{x}) = |\boldsymbol{D}u(\boldsymbol{x})|. \tag{4}$$

On the lateral boundary of $\mathscr{B}$, i.e., on

$$\{(x_1, x_2, x_3) \colon \boldsymbol{x} \in \partial\Omega \text{ and } x_3 \in \mathbb{R}\} = \partial\Omega \cap \{x_3 \in \mathbb{R}\},$$

the axial displacement is prescribed so that the outer lateral surface of the tube is fixed and the inner lateral surface is displaced axially by a constant value $H > 0$. Thus, $u(\boldsymbol{x})$ is prescribed according to

$$u(\boldsymbol{x}) = u^* = \begin{cases} 0 & \forall \boldsymbol{x} \in \partial\Omega_o, \\ H & \forall \boldsymbol{x} \in \partial\Omega_i. \end{cases} \tag{5}$$

We let ω denote the specific stored energy per unit reference volume, and assume that ω depends upon the first principle Cauchy-Green strain invariant only. Employing (3), we then write ω as

$$\omega = \omega(I) = \omega(3 + \kappa^2) = \tilde{\omega}(\kappa). \tag{6}$$

More specifically, we shall consider a trilinear material defined by ([1], [11])

$$\tilde{\omega}'(\kappa) = \begin{cases} \mu^- \kappa & \text{if } \kappa < K^-, \\ \mu_h \kappa + b & \text{if } \kappa \in (K^-, K^+), \\ \mu^+ \kappa & \text{if } \kappa > K^+, \end{cases} \tag{7}$$

where $\tilde{\omega}' \equiv d\tilde{\omega}(\kappa)/d\kappa$,

$$\mu_h = \frac{\mu^+ K^+ - \mu^- K^-}{K^+ - K^-},$$

$$b = \frac{\mu^- - \mu^+}{K^+ - K^-} K^+ K^-,$$

and μ^+, μ^-, K^+, and K^- are positive constants with $K^- < K^+$ and $\mu^- K^- > \mu^+ K^+$. The specific stored energy associated with (7) then is given by

$$\tilde{\omega}(\kappa) = \begin{cases} \frac{1}{2}\mu^- \kappa^2 & \text{if } \kappa < K^-, \\ \frac{1}{2}[\mu_h(\kappa - K^-)^2 + \mu^- K^-(\kappa - K^-) + \mu^- K^- \kappa] & \text{if } \kappa \in (K^-, K^+), \\ \frac{1}{2}\mu^+ \kappa^2 + \tilde{\omega}_o & \text{if } \kappa > K^+, \end{cases} \tag{8}$$

where $\tilde{\omega}_o = \frac{1}{2}(\mu^- - \mu^+)K^+K^-$. It is straightforward to determine the unique values of $\kappa_1 \in (0, K^-)$ and $\kappa_2 \in (K^+, \infty)$ with $\tilde{\omega}'(\kappa_1) = \omega'(\kappa_2)$, so that when $\kappa \in (\kappa_1, \kappa_2)$ the specific stored energy function $\tilde{\omega}(\kappa)$ of (8) loses its convexity in κ. We shall call $\tilde{\omega}'(\kappa_1) = \tilde{\omega}'(\kappa_2) = \tilde{\omega}'_m$ the Maxwell stress, and κ_1 and κ_2 the associated Maxwell strains. Typical graphs of $\tilde{\omega}(\cdot)$ and $\tilde{\omega}'(\cdot)$ are depicted in Figures 2 and 3 respectively.

We shall not consider the effect of body force in this work. Consequently, the anti-plane deformation (2) permits us to write the total potential energy of $\mathscr{B}$ per unit length as

$$E[u] = \int_\Omega \tilde{\omega}(|\boldsymbol{D}u|)\, da, \tag{9}$$

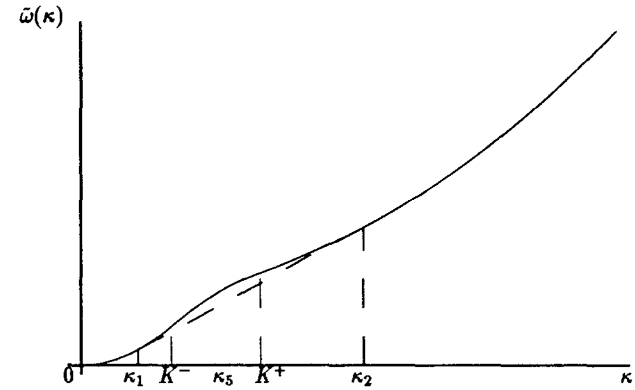


Figure 2
The specific stored energy.

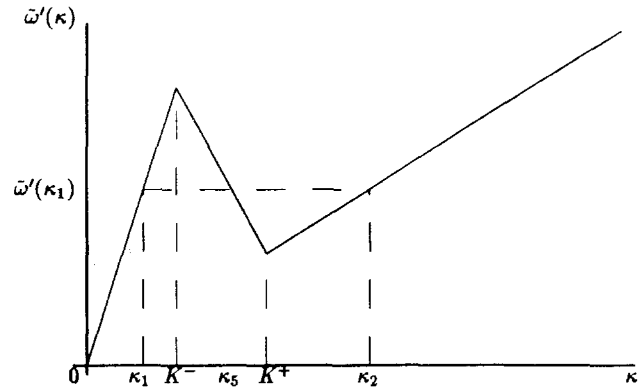


Figure 3
Gradient of the specific stored energy.

where da denotes an area element of Ω. We say that a stable deformation configuration corresponding to the given boundary condition (5) is one that minimizes the potential energy (9) in a given class of admissible fields. The fundamental minimization *Problem P1* then is to

$$\underset{v \in \mathscr{A}}{\text{Minimize}}\; E[v], \tag{10}$$

where the class $\mathscr{A}$ of admissible fields is defined as

$$\mathscr{A} \equiv \{u \in W^{1,p}(\Omega): p \geq 1, u = u^* \text{ on } \partial\Omega\}, \tag{11}$$

where u^* is the prescribed displacement boundary data of (5).

If we let $\tilde{g}(\kappa) \equiv \tilde{\omega}'(\kappa)/\kappa$, then the Euler-Lagrange equation and boundary conditions associated with (10) may be written as

$$
\begin{aligned}
\boldsymbol{D}\cdot(\tilde{g}(|\boldsymbol{D}u|)\boldsymbol{D}u) &= 0 && \text{in } \Omega \text{ (wherever } u \text{ is smooth)},\\
u &= 0 && \text{on } \partial\Omega_o,\\
u &= H && \text{on } \partial\Omega_i.
\end{aligned}
\tag{12}
$$

It is not hard to show that $|\boldsymbol{D}u(\boldsymbol{x})| \notin (\kappa_1, \kappa_2)$, $\forall \boldsymbol{x} \in \Omega$, for a stable configuration, so that (7) and (12) imply that whenever u is smooth it must be harmonic, i.e.,

$$
\begin{aligned}
\boldsymbol{D}\cdot\boldsymbol{D}u &= 0 && \text{in } \Omega \text{ (wherever } u \text{ is smooth)},\\
u &= 0 && \text{on } \partial\Omega_o,\\
u &= H && \text{on } \partial\Omega_i,
\end{aligned}
\tag{13}
$$

The class $\mathscr{A}$ allows the possibility of discontinuous deformation gradients, in which case a solution to *Problem P1* must satisfy an appropriate (force balance) jump condition. Let $C \subset \Omega$ be a curve across which a discontinuity in the deformation gradient occurs and let $\boldsymbol{n}$ be a unit normal to C. Suppose $\boldsymbol{x}_o \in C$. We shall say that $\boldsymbol{D}u(\boldsymbol{x}) \to \boldsymbol{D}u(\boldsymbol{x}_o)^+$ when the limit $\boldsymbol{x} \to \boldsymbol{x}_o$ is taken from the side of C into which the curve normal $\boldsymbol{n}(\boldsymbol{x}_o)$ points. Similarly, $\boldsymbol{D}u(\boldsymbol{x}) \to \boldsymbol{D}u(\boldsymbol{x}_o)^-$ when the limit $\boldsymbol{x} \to \boldsymbol{x}_o$ is taken from the side of C opposite to which $\boldsymbol{n}(\boldsymbol{x}_o)$ points. The jump condition across C then may be expressed as

$$
[\![\tilde{g}(|\boldsymbol{D}u|)\boldsymbol{D}u(\boldsymbol{x})]\!]\cdot\boldsymbol{n}(\boldsymbol{x}) = 0 \tag{14}
$$

for all $\boldsymbol{x} \in C$, where $[\![\,\cdot\,]\!] \equiv (\,\cdot\,)^+ - (\,\cdot\,)^-$.

Because the specific stored energy function $\tilde{\omega}(\kappa)$ is not convex, the minimization *Problem P1* may not have a solution. To discuss this situation, it is convenient to introduce an associated relaxed minimization problem. Let $\tilde{\omega}^*(\kappa)$ be the lower convex envelope of $\tilde{\omega}(\kappa)$. For the trilinear material (8) of Figures 2 and 3,

$$
\tilde{\omega}^*(\kappa) = \begin{cases} \tilde{\omega}(\kappa) & \text{if } \kappa \notin (\kappa_1, \kappa_2),\\ \tilde{\omega}(\kappa_1) + \tilde{\omega}_m'(\kappa - \kappa_1) & \text{otherwise.} \end{cases} \tag{15}
$$

The associated relaxed minimization *Problem P2* then has the form:

$$
\underset{v \in \mathscr{A}}{\text{Minimize}}\; E^*[v], \tag{16}
$$

where

$$
E^*[u] \equiv \int_\Omega \tilde{\omega}^*(|\boldsymbol{D}u|)\, da.
$$

It is well known that the minimization *Problem P2* always has a solution due to the convexity of $\tilde{\omega}^*(\kappa)$. Also, it is not difficult to show that a solution in $\mathscr{A}$ of *Problem P2* must satisfy the Laplace equation $\boldsymbol{D} \cdot \boldsymbol{D}u(\boldsymbol{x}) = 0$ for any $\boldsymbol{x} \in \Omega$ as long as $|\boldsymbol{D}u(\boldsymbol{x})| \notin (\kappa_1, \kappa_2)$, and

$$\boldsymbol{D} \cdot \boldsymbol{e}(\boldsymbol{x}) = 0 \tag{17}$$

for any $\boldsymbol{x} \in \Omega$ whenever $|\boldsymbol{D}u(\boldsymbol{x})| \in (\kappa_1, \kappa_2)$, where

$$\boldsymbol{e}(\boldsymbol{x}) \equiv -\frac{\boldsymbol{D}u(\boldsymbol{x})}{|\boldsymbol{D}u(\boldsymbol{x})|}. \tag{18}$$

In addition, if there is a discontinuity in deformation gradients across a curve $C \subset \Omega$, the jump condition

$$[\![\tilde{g}^*(|\boldsymbol{D}u|)\boldsymbol{D}u]\!] \cdot \boldsymbol{n} = 0 \tag{19}$$

must be satisfied over the curve C, where $\tilde{g}^*(\kappa) \equiv \tilde{\omega}^{*\prime}(\kappa)/\kappa$.

Bauman and Phillips [2] have investigated the issues concerning existence and uniqueness of solutions to the minimization *Problem P1* for the anti-plane shear of a tube whose cross section is a convex ring, for a general class of materials of the form (6). They showed that there exists a unique solution to the associated relaxed convex minimization *Problem P2.* Because the trilinear material (8) is included in their class of materials, we know that there is a unique solution to *Problem P2* for any given e and H. For later reference, we summarize some of the results of Bauman & Phillips in

Theorem 1 (Bauman & Phillips). Let Ω be a convex ring. Then there exists a solution u to *Problem P2* with the following properties:

(i) u is unique,
(ii) the level curves $0 \leq u = l \leq H$ are of class C^1;
(iii) the unit vector $\boldsymbol{e}(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$, of (18) is continuous;
(iv) the set

$$\Omega_l \equiv \{\boldsymbol{x} \in \Omega\colon u(\boldsymbol{x}) > l \text{ for } 0 < l < H\} \cup \Omega_i \tag{20}$$

is convex;
(v) $\boldsymbol{D}u \cdot \boldsymbol{n} < 0$ and $(\boldsymbol{DD}u)\boldsymbol{n} \cdot \boldsymbol{n} \geq 0$ on $\partial\Omega_l$, where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega_l$, $l \in (0, H)$.

Proof. The detailed proofs of these results can be found in [2]. Specifically, (i) is Theorem 2.10; (ii) is Lemma 3.1; (iii) is part of Corollary 3.3; (iv) is Theorem 2.8; and (v) is the combination of Lemmas 2.4 and 2.6 together with an appropriate limit. □

For the trilinear material $\tilde{\omega}(\kappa)$ of (8) and the convex ring of (1), the solution to the minimization *Problem P1* has its own special properties which we begin to discuss in

Theorem 2. If the eccentricity $e = 0$, *Problem P1* has a unique solution for any H. If $e \neq 0$, *Problem P1* has a solution if and only if $H \notin (H_m, H_M)$, where

$$H_m \equiv \frac{\ln \tilde{R} \left| [a(R_i + e) - R_o][R_i + e - R_o a] \right|}{R_o(a^2 - 1)} \kappa_1, \tag{21}$$

$$H_M \equiv \frac{\ln \tilde{R}(a+1)}{a-1} R_o \kappa_2, \tag{22}$$

in which

$$\tilde{R} \equiv \frac{R_o^2 + e^2 - R_i^2 + \sqrt{[R_o^2 - (R_i + e)^2][R_o^2 - (R_i - e)^2]}}{2R_o R_i},$$

$$a \equiv \frac{R_o^2 + R_i^2 - e^2 + \sqrt{[R_o^2 - (R_i + e)^2][R_o^2 - (R_i - e)^2]}}{2R_o e}.$$

Proof. When $e = 0$, Ω is a concentric circular ring. Fosdick and MacSithigh [5] have investigated the solution to this problem for a fairly general class of stored energy functions which includes the trilinear material (8). The conclusion here follows from their results [5].

When $e \neq 0$, suppose first that $H \notin (H_m, H_M)$. It is straightforward to show that the field

$$u(\boldsymbol{x}) = \frac{H}{\ln \tilde{R}} \ln R(\boldsymbol{x}), \tag{23}$$

where

$$R(\boldsymbol{x}) = \frac{\sqrt{R_o^2 x_2^2 (1 - a^2)^2 + [(ax_1 - R_o)(x_1 - aR_o) + ax_2^2]^2}}{(ax_1 - R_o)^2 + a^2 x_2^2}.$$

satisfies the Laplace equation $\boldsymbol{D} \cdot (\boldsymbol{D}u(\boldsymbol{x})) = 0$, and $|\boldsymbol{D}u(\boldsymbol{x})| \notin (\kappa_1, \kappa_2)$ for any $\boldsymbol{x} \in \Omega$. In fact, when $0 < H < H_m$, it follows that $0 < |\boldsymbol{D}u(\boldsymbol{x})| < \kappa_1$, $\forall \boldsymbol{x} \in \Omega$, and when $H > H_M$, we have $|\boldsymbol{D}u(\boldsymbol{x})| > \kappa_2$, $\forall \boldsymbol{x} \in \Omega$. Thus, (23) is the unique solution to *Problem P1* in this case.

Conversely, suppose that $u(\boldsymbol{x})$ is a solution to *Problem P1*. We must have either $0 < |\boldsymbol{D}u(\boldsymbol{x})| \leq \kappa_1$, or $|\boldsymbol{D}u(\boldsymbol{x})| \geq \kappa_2$ for all $\boldsymbol{x} \in \Omega$ by Theorem 5.3 of [2]. In this case, the Euler equation for $u(\boldsymbol{x})$ in Ω is the Laplace equation, and therefore $u(\boldsymbol{x})$ must take the form of (23) because of the boundary conditions and the standard uniqueness property. Consequently, $H \notin (H_m, H_M)$. □

To present our next result, we shall call

$$\Omega_m \equiv \{\boldsymbol{x} \in \Omega : |\boldsymbol{D}u(\boldsymbol{x})| \in [\kappa_1, \kappa_2]\}, \tag{24}$$

and denote $\overset{o}{\Omega}_m$ as the interior of Ω_m. We have

Proposition 1. Let u be a solution to *Problem P2.* Suppose the eccentricity e is not zero. Let H_m and H_M be given by (21) and (22), respectively. When $H \in (H_m, H_M)$, the cross section Ω is partitioned into three sets Ω_1, Ω_2, and Ω_m such that

(i) Ω_1 and Ω_2 are open and u is of class $C^{2,\alpha}$ on $\Omega_1 \cup \Omega_2$, with

$$0 < |\boldsymbol{D}u| < \kappa_1 \quad \text{on } \Omega_1,$$

$$\kappa_2 < |\boldsymbol{D}u| \quad \text{on } \Omega_2;$$

(ii) $\kappa_1 \le |\boldsymbol{D}u| \le \kappa_2$ almost everywhere on Ω_m, with $\overset{o}{\Omega}_m \neq \emptyset$.

Proof. We refer readers to Corollary 3.5 of Bauman and Phillips [2] for the proof of (i) and the first part of (ii). To show $\overset{o}{\Omega}_m \neq \emptyset$, we note that if $\overset{o}{\Omega}_m = \emptyset$, then $u(\boldsymbol{x})$ is a solution to *Problem P1*, which is impossible for $H \in (H_m, H_M)$ because of Theorem 2. □

According to Proposition 1 and Theorem 2, when $\overset{o}{\Omega}_m = \emptyset$, *Problem P1* has a solution. Otherwise, *Problem P1* has no solution. We observe from Proposition 1 that Ω_m separates the region Ω_1 where $|\boldsymbol{D}u| \in (0, \kappa_1)$ from that of Ω_2 where $|\boldsymbol{D}u| \in (\kappa_2, \infty)$. When there is no solution to *Problem P1*, we shall construct a minimizing sequence of deformation fields in $\mathscr{A}$ which drives the total potential energy (9) to its infimum. First, however, we discuss some of the properties concerning Ω_m in the next section. An understanding of the character and form of Ω_m is key to the construction of our minimizing sequence.

2. On the domain of phase mixture

As noted in Theorem 2, when the prescribed displacement boundary data H is in the range (H_m, H_M), *Problem P1* has no solution. In this situation, $\overset{o}{\Omega}_m \neq \emptyset$ and the magnitude of the deformation gradient field associated with the solution to *Problem P2* is bounded below by κ_1 and above by κ_2 for any points in Ω_m. In a later section, we shall show that for *Problem P1* there is a minimizing sequence of deformation gradient fields in $\mathscr{A}$ which at each $\boldsymbol{x} \in \Omega_m$ converges weakly to a particular convex combination of two special deformation gradients whose magnitudes are κ_1 and κ_2. For this reason, we call Ω_m the domain of phase mixture.

In order to characterize the shape and form of the phase mixture domain Ω_m, we first give the following

Proposition 2. Let u be the solution to *Problem P2* for a given H. Suppose $\overset{o}{\Omega}_m \neq \emptyset$. Then, the following holds:

(i) Each contour $\{x \in \overset{o}{\Omega}_m : u(x) = l\}$, for $0 < l < H$, is a straight line;
(ii) Let $x_o, x \in \Omega$ be given so that $|Du(x_o)| = \kappa_1$, $x \in \overset{o}{\Omega}_m$, and $y(\beta) = \beta x_o + (1-\beta)x \in \overset{o}{\Omega}_m$ for any $\beta \in (0, 1)$. Then, $u(y) = u(x_o)$ for all $\beta \in (0, 1)$ if and only if $x = x_o + |x - x_o| t(x_o)$, where $t(x_o) \cdot e(x_o) = 0$, and where $e(x_o)$ is given by (18).

Proof. (i) Because u is the solution to *Problem P2*, when $\overset{o}{\Omega}_m \neq \varnothing$ Proposition 1 implies that $|Du(x)| \in (\kappa_1, \kappa_3)$, $\forall x \in \overset{o}{\Omega}_m$. Thus, $u(x)$ satisfies

$$D \cdot \frac{Du(x)}{|Du(x)|} = 0 \tag{25}$$

for all $x \in \overset{o}{\Omega}_m$. We rewrite (25) as

$$u_{x_2}^2 u_{x_1 x_1} - 2u_{x_1} u_{x_2} u_{x_1 x_2} + u_{x_1}^2 u_{x_2 x_2} = 0. \tag{26}$$

Suppose the generic level set $u = \text{const.}$ has a parametric representation

$$x_1 = x_1(\alpha) \quad \text{and} \quad x_2 = x_2(\alpha) \tag{27}$$

with x_1, x_2 twice continuously differentiable and $x_2'(\alpha) \neq 0$. Upon eliminating u from (26) with the help of

$$u' \equiv \frac{du}{d\alpha} = 0,$$

which holds true along the generic level set of u, we derive the differential equation

$$x_1'' x_2' - x_2'' x_1' = 0. \tag{28}$$

Then

$$\frac{d}{d\alpha}\left(\frac{x_1'(\alpha)}{x_2'(\alpha)}\right) = 0,$$

from which we have

$$x_1(\alpha) + b x_2(\alpha) = c,$$

where b and c are two constants.

(ii) Let x_o, $x \in \Omega$, and $y(\beta)$ be given as stated in the hypotheses. Suppose first that $u(y(\beta)) = u(x_o)$, for all $\beta \in (0, 1)$. Then

$$\frac{du(y(\beta))}{d\beta} = 0,$$

and by the chain rule

$$Du(y(\beta)) \cdot (x - x_o) = 0 \tag{29}$$

for all $\beta \in (0, 1)$. Because $\boldsymbol{y}(1) = \boldsymbol{x}_o$, we then have

$$(\boldsymbol{x} - \boldsymbol{x}_o) \cdot \boldsymbol{D}u(\boldsymbol{y}(1)) = (\boldsymbol{x} - \boldsymbol{x}_o) \cdot \boldsymbol{e}(\boldsymbol{x}_o) = 0,$$

which implies that

$$\boldsymbol{x} - \boldsymbol{x}_o = |\boldsymbol{x} - \boldsymbol{x}_o| \boldsymbol{t}(\boldsymbol{x}_o)$$

since $\boldsymbol{x} - \boldsymbol{x}_o \neq 0$.

Conversely, suppose $\boldsymbol{x} = \boldsymbol{x}_o + |\boldsymbol{x} - \boldsymbol{x}_o| \boldsymbol{t}(\boldsymbol{x}_o) \in \overset{o}{\Omega}_m$ and $\boldsymbol{y}(\beta) = \beta \boldsymbol{x}_o + (1 - \beta)\boldsymbol{x}$, $\beta \in (0, 1)$. Then,

$$\boldsymbol{y}(\beta) = (1 - \beta)|\boldsymbol{x} - \boldsymbol{x}_o| \boldsymbol{t}(\boldsymbol{x}_o) + \boldsymbol{x}_o,$$

and

$$\frac{du(\boldsymbol{y}(\beta))}{d\beta} = -|\boldsymbol{x} - \boldsymbol{x}_o| \boldsymbol{D}u(\boldsymbol{y}(\beta)) \cdot \boldsymbol{t}(\boldsymbol{x}_o)$$

for all $\beta \in (0, 1)$. Since $\boldsymbol{y}(1) = \boldsymbol{x}_o$ and $\boldsymbol{t}(\boldsymbol{x}_o) \cdot \boldsymbol{D}u((\boldsymbol{y}(1)) = 0$ by assumption, then the straight line $\boldsymbol{y}(\beta)$, $\beta \in (0, 1)$ is perpendicular to $\boldsymbol{e}(\boldsymbol{x}_o)$. Thus, from part (i) of this proposition, we conclude that $\boldsymbol{y}(\beta)$, $\beta \in (0, 1)$, is the contour of the level set $u = u(\boldsymbol{x}_o)$. □

Let $\boldsymbol{x}_l(s)$ be a parametric representation of the level set $u = l$, $0 \leq l \leq H$, where s is the arc length. We shall denote $2s^*$ as the total arc length of the level set $u = l$. Because this level set is symmetric about a line through the centers of the eccentric circular regions Ω_i and Ω_o, and Ω_l is convex by Theorem 1, the representation $\boldsymbol{x}_l(s)$ will intersect the axis of symmetry at two distinct points. Without loss of generality, we assume that the arc length s increases as one moves clockwise along $\boldsymbol{x}_l(s)$, and that the direction of $\boldsymbol{x}_l(0) - \boldsymbol{x}_l(s^*)$ coincides with the negative direction of the axis of symmetry of Ω. With the above notational agreements, we are now ready to investigate how $|\boldsymbol{D}u|$ varies along the boundaries of Ω in

Proposition 3. Let u be the solution to *Problem P2.* Then, $|\boldsymbol{D}u|$ is monotone along $\partial\Omega_i$ and $\partial\Omega_o$ for $s \in (0, s^*)$. Moreover, let $\boldsymbol{x}_1 \in \partial\Omega_i$ and $\boldsymbol{x}_2 \in \partial\Omega_o$ be such that

$$dist(\boldsymbol{x}_1, \partial\Omega_o) = \min\{|\boldsymbol{x} - \boldsymbol{y}|: \boldsymbol{x} \in \partial\Omega_i \text{ and } \boldsymbol{y} \in \partial\Omega_o\}$$

and

$$dist(\boldsymbol{x}_2, \partial\Omega_i) = \max\{|\boldsymbol{x} - \boldsymbol{y}|: \boldsymbol{x} \in \partial\Omega_o \text{ and } \boldsymbol{y} \in \partial\Omega_i\}.$$

Then,

$$|\boldsymbol{D}u(\boldsymbol{x}_1)| = \max\{|\boldsymbol{D}u(\boldsymbol{x})|: \boldsymbol{x} \in \Omega\}$$

and

$$|Du(x_2)| = \min\{|Du(x)|: x \in \Omega\}.$$

Proof. Without loss of generality, we shall assume the eccentricity $e > 0$. Let x_o and o be the centers of Ω_i and Ω_o, respectively, and let T_o be a straight line passing through x_o with unit normal t such that $t \cdot (x_o - o) \geq 0$. For any fixed t, let the cap cut off from Ω by T_o, into which t is directed, be denoted by Σ_o and its reflection through T_o by Σ'_o so that

$$\Sigma_o \equiv \{x \in \Omega: (x - x_o) \cdot t \geq 0\}.$$

It is clear from the construction that $\Sigma'_o \subset \Omega$ (Figure 4).

We claim that

$$Du \cdot t \leq 0 \tag{30}$$

for any $x \in T_o \cap \Omega$, where the equality holds for all $x \in T_o \cap \Omega$ if t is perpendicular to the symmetry axis of Ω. We shall establish this claim in a moment, but first let us suppose that it holds. Let $\tilde{x} \in T_o \cap \partial\Omega_i$ and $x \in T_o \cap \Omega$ so that $|x - \tilde{x}| = \delta$. We observe that T_o is perpendicular to $\partial\Omega_i$ at $\tilde{x}$, and that, since u is the solution to *Problem P2*, $Du(\tilde{x}) \cdot t = -(\partial u/\partial s)(\tilde{x}) = 0$. Because

$$\frac{\partial}{\partial s}(Du(\tilde{x}) \cdot n) = D\left(\frac{\partial u}{\partial s}(\tilde{x})\right) \cdot n + \varrho(\tilde{x})\frac{\partial u}{\partial s}(\tilde{x}),$$

where $n(\tilde{x})$ is the unit outer normal to Ω_i at $\tilde{x}$ and $\varrho(\tilde{x})$ is the curvature of $\partial\Omega_i$ at $\tilde{x}$, and because

$$D\left(\frac{\partial u}{\partial s}(\tilde{x})\right) \cdot n \equiv \lim_{\delta \to 0} \frac{Du(x) \cdot t - Du(\tilde{x}) \cdot t}{\delta} \geq 0,$$

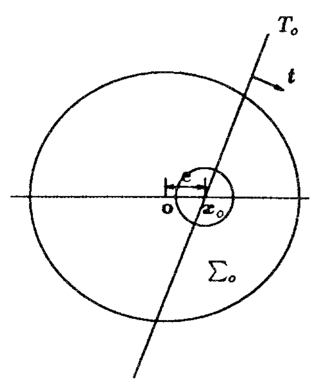


Figure 4
Construction for Proposition 3.

we then have

$$\frac{\partial}{\partial s}(\boldsymbol{D}u(\tilde{\boldsymbol{x}}) \cdot \boldsymbol{n}) \geq 0. \tag{31}$$

The equality holds for $\tilde{\boldsymbol{x}}$ on the symmetry axis of Ω since by hypothesis $\boldsymbol{D}u(\boldsymbol{x}) \cdot \boldsymbol{t} \equiv 0$ for all $\boldsymbol{x} \in T_o \cap \Omega$ when $\boldsymbol{t}$ is perpendicular to the axis of symmetry of Ω. Since (31) holds for any $\boldsymbol{t}$, then $\boldsymbol{D}u(\tilde{\boldsymbol{x}}) \cdot \boldsymbol{n}$ must be monotone along $\partial\Omega_i$.

To show the claim of (30), let $\tilde{\omega}_\varepsilon(\kappa)$ be a sequence of class $C^3(\mathbb{R}^+)$ approximations to $\tilde{\omega}^*(\kappa)$ such that $\tilde{\omega}'_\varepsilon(\kappa) \geq 0$, $\tilde{\omega}'_\varepsilon(0) = 0$, $\tilde{\omega}''_\varepsilon(\kappa) > 0$, $|\tilde{\omega}''_\varepsilon(0) - \tilde{\omega}^{*\prime\prime}(0)| \leq (\varepsilon/2)\tilde{\omega}^{*\prime\prime}(0)$, and $|\tilde{\omega}_\varepsilon(\kappa) - \tilde{\omega}^*(\kappa)| \leq \varepsilon(1 + \kappa^p)$, where $p > 1$. For each $\varepsilon \in (0, 1)$, consider the following minimization *Problem* P_ε:

$$\underset{v \in \mathscr{A}}{\text{Minimize}} \left\{ E_\varepsilon[v] = \int_\Omega \tilde{\omega}_\varepsilon(|\boldsymbol{D}v|)\, da \right\}.$$

The corresponding Dirichlet problem that is associated with *Problem* P_ε is given by

$$\begin{aligned} \boldsymbol{D} \cdot (\tilde{g}_\varepsilon(|\boldsymbol{D}u|)\boldsymbol{D}u) &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega_o, \\ u &= H && \text{on } \partial\Omega_i, \end{aligned} \tag{32}$$

where $\tilde{g}_\varepsilon(\kappa) \equiv \tilde{\omega}_\varepsilon(\kappa)/\kappa$, and it has a unique solution u_ε of class $C^{2,\alpha}(\bar{\Omega})$ (see Theorem 15.11 of [8]).

Let $v_\varepsilon(\boldsymbol{x}) = u_\varepsilon(\boldsymbol{x}')$ for any $\boldsymbol{x} \in \Sigma'_o$, where $\boldsymbol{x}' \in \Sigma_o$ is the mirror reflection of $\boldsymbol{x}$ through T_o. For any $\varepsilon \in (0, 1)$, v_ε then satisfies

$$\begin{aligned} \boldsymbol{D} \cdot (\tilde{g}_\varepsilon(|\boldsymbol{D}v_\varepsilon|)\boldsymbol{D}v_\varepsilon) &= 0 && \text{in } \Sigma'_o, \\ v_\varepsilon &= u_\varepsilon && \text{on } (T_o \cap \Omega) \cup (\partial\Sigma'_o \cap \partial\Omega_i), \\ v_\varepsilon &= 0 \leq u_\varepsilon && \text{on } \partial\Sigma'_o \cap \Omega. \end{aligned} \tag{33}$$

Thus, upon employing a standard comparison principle for elliptic equations (see Theorem 10.1 of [8]), we see that

$$v_\varepsilon(\boldsymbol{x}) \leq u_\varepsilon(\boldsymbol{x}) \tag{34}$$

for any $\boldsymbol{x} \in \Sigma'_o$. Since the very definition of $v_\varepsilon(\boldsymbol{x})$ requires

$$\boldsymbol{D}v_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{t} = -\boldsymbol{D}u_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{t}$$

for any $\boldsymbol{x} \in T_o \cap \Omega$, and since (34) and the boundary condition on $T_o \cap \Omega$ as given in (33) requires $\boldsymbol{D}v_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{t} \geq \boldsymbol{D}u_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{t}$, we readily conclude that

$$\boldsymbol{D}u_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{t} \leq 0 \qquad \forall \boldsymbol{x} \text{ on } T_o \cap \Omega$$

for any $\varepsilon \in (0, 1)$. Because a sequence of minimizers to *Problem* P_ε has a subsequence such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$, where u solves *Problem P2* (see Lemma 2.1 of [2]), we have

$$\boldsymbol{D}u \cdot \boldsymbol{t} \le 0$$

on $T_o \cap \Omega$. Moreover, when $\boldsymbol{t}$ is perpendicular to the axis of symmetry of Ω we have $v_\varepsilon(\boldsymbol{x}) \equiv u_\varepsilon(\boldsymbol{x})$ in Σ'_o, and consequently, $\boldsymbol{D}u(\boldsymbol{x}) \cdot \boldsymbol{t} \equiv 0$ for any $\boldsymbol{x} \in T_o \cap \Omega$. Thus, (30) holds.

Now let T_o be a straight line passing through the center of Ω_o, and as before, let $\boldsymbol{t}$ be the unit normal to T_o so that $\boldsymbol{t} \cdot (\boldsymbol{x}_o - \boldsymbol{o}) \ge 0$. For any fixed direction $\boldsymbol{t}$, let the cap cut off from Ω by T_o in the direction of $\boldsymbol{t}$ be denoted by Σ_o and its reflection through T_o by Σ'_o so that

$$\Sigma_o \equiv \{\boldsymbol{x} \in \Omega \colon (\boldsymbol{x} - \boldsymbol{x}_o) \cdot \boldsymbol{t} \ge 0\}.$$

Upon employing the above argument, we may conclude that $\boldsymbol{D}u \cdot \boldsymbol{n}$ is monotone along $\partial\Omega_o$.

Since $\boldsymbol{D}u \cdot \boldsymbol{n} < 0$ on $\partial\Omega_i$ according to Theorem 1, and $\boldsymbol{x}_1 \in \partial\Omega_i$ is on the axis of symmetry of Ω, (31) implies that $\boldsymbol{D}u \cdot \boldsymbol{n}(\boldsymbol{x})$ reaches its minimum along $\partial\Omega_i$ at $\boldsymbol{x}_1$. Similarly, $\boldsymbol{D}u \cdot \boldsymbol{n}(\boldsymbol{x})$ is negative and reaches its maximum along $\partial\Omega_o$ at $\boldsymbol{x}_2$. Since, according to Theorem 1, $(\boldsymbol{DD}u)\boldsymbol{n} \cdot \boldsymbol{n} \ge 0$, we may conclude that $|\boldsymbol{D}u(\boldsymbol{x}_1)|$ is the maximum of $|\boldsymbol{D}u(\boldsymbol{x})|$ amd $|\boldsymbol{D}u(\boldsymbol{x}_2)|$ is the minimum of $|\boldsymbol{D}u(\boldsymbol{x})|$ for any $\boldsymbol{x} \in \Omega$. □

In what follows, we shall define

$$D(\kappa) \equiv \{\boldsymbol{x} \in \Omega \colon |\boldsymbol{D}u(\boldsymbol{x})| > \kappa\} \cup \Omega_i, \tag{35}$$

and denote by $\boldsymbol{v}$ the outer unit normal to $\partial D(\kappa)$. Thus, by (24) we have

$$\Omega_m \equiv \bar{D}(\kappa_1) \backslash D(\kappa_2),$$

where $\bar{D}(\kappa_1)$ is the closure of $D(\kappa_1)$. Let $\boldsymbol{d}^*(\boldsymbol{x})$ denote the distance from $\partial D(\kappa_2)$ to a given point $\boldsymbol{x} \in \partial D(\kappa_1)$. Then, $\boldsymbol{d}^*(\boldsymbol{x})$ is given by

$$\boldsymbol{d}^*(\boldsymbol{x}) \equiv \min\{|\boldsymbol{z} - \boldsymbol{x}| \colon \boldsymbol{z} \in \partial D(\kappa_2)\}. \tag{36}$$

For a given $\boldsymbol{x}_o \in \partial D(\kappa_1)$, $\boldsymbol{d}^*(\boldsymbol{x}_o) = 0$ if and only if it coincidently happens that $\boldsymbol{x}_o \in \partial D(\kappa_2)$. If $\boldsymbol{d}^*(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \partial D(\kappa_1)$, then $D(\kappa_1) = D(\kappa_2)$ and $\overset{o}{\Omega}_m = \varnothing$. When the eccentricity $e \neq 0$ and $H \in (H_m, H_M)$, Proposition 1 shows that this possibility can not happen. However, when $e \neq 0$ it can happen that $\boldsymbol{d}^*(\boldsymbol{x}_o) = 0$ for some given $\boldsymbol{x}_0 \in \partial D(\kappa_1)$, and in this case it is of interest to understand how the boundaries $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$ will touch one another. The remaining two lemmas and three propositions in this section address certain aspects of this issue as well as other special properties of the subdomains $D(\kappa_1)$ and $D(\kappa_2)$.

Lemma 1. The subdomain $D(\kappa)$, $\kappa > 0$, is simply connected.

Proof. This result follows immediately from the facts that (i) $|\boldsymbol{Du}|$ is monotone on $\partial\Omega$ in the sense described in Proposition 3, (ii) Ω_l, $0 < l < H$, is convex (Theorem 1), and (iii) $\boldsymbol{Du} \cdot \boldsymbol{n} < 0$ and $(\boldsymbol{DDu})\boldsymbol{n} \cdot \boldsymbol{n} \geq 0$, where $\boldsymbol{n}$ is the unit outer normal to Ω_l, $0 < l < H$ (Theorem 1). □

Since $D(\kappa_2) \subset D(\kappa_1)$, then Lemma 1 ensures that the boundaries $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$ cannot cross, though they can have points in common. Thus, in the case that these boundaries are smooth and touch one another, we have the elementary

Lemma 2. Let u be the solution to *Problem P2* and suppose that the boundaries $\partial D(\kappa_i)$, $i = 1, 2$, are differentiable curves.[1] Suppose $\boldsymbol{x}_o \in \partial D(\kappa_1) \cap \Omega$ and let $\boldsymbol{v}_1$ denote the unit outer normal to $\partial D(\kappa_1)$ at $\boldsymbol{x}_o$. Then, either $\boldsymbol{x}_o \notin \partial D(\kappa_2)$ or $\boldsymbol{x}_o \in \partial D(\kappa_2)$ and $\boldsymbol{v}_2 = \boldsymbol{v}_1$, where $\boldsymbol{v}_2$ is the unit outer normal to $\partial D(\kappa_2)$ at $\boldsymbol{x}_o$.

In the following Proposition, we show that for any $\boldsymbol{x}_o \in \overset{o}{\Omega}_m$ and for a sufficiently regular boundary $\partial\Omega_m$, the straight line contour $\{u(\boldsymbol{x}) = u(\boldsymbol{x}_o)\}$ can meet neither $\partial D(\kappa_1)$ nor $\partial D(\kappa_2)$ twice. In particular, we have

Proposition 4. Let u be the solution to *Problem P2*, and let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in \partial\Omega_m$ be two distinct points such that

(i) $(\boldsymbol{x}_2 - \boldsymbol{x}_1) \cdot \boldsymbol{e}(\boldsymbol{x}_1) = 0$, where $\boldsymbol{e}(\cdot)$ is the unit normal field of (18), and
(ii) $\beta\boldsymbol{x}_2 + (1 - \beta)\boldsymbol{x}_1 \equiv \boldsymbol{x}(\beta) \in \Omega_m$ for all $\beta \in (0, 1)$.

Let $\partial\Omega_m$ be piecewise differentiable.[2] Then $\boldsymbol{x}_1 \in \partial D(\kappa_i) \cap \Omega$ and $\boldsymbol{x}_2 \in \partial D(\kappa_j) \cap \Omega$, where $i \neq j$, and $i, j = 1$ or 2.

Proof. Consider, without loss of generality, $\boldsymbol{x}_1 \in \partial D(\kappa_1) \cap \Omega$. We need only to show that $\boldsymbol{x}_2 \notin \partial D(\kappa_1) \cap \Omega$. For contradiction, suppose $\boldsymbol{x}_2 \in \partial D(\kappa_1) \cap \Omega$.

First, we note immediately, from Proposition 2, that $\boldsymbol{x}(\beta)$ is the level contour

$$u(\boldsymbol{x}(\beta)) = u(\boldsymbol{x}_1)$$

for all $\beta \in (0, 1)$, and, from the hypotheses (i) and (ii), that

$$\boldsymbol{Du}(\boldsymbol{x}_2) = \boldsymbol{Du}(\boldsymbol{x}_1) = -\kappa_1 \boldsymbol{e}(\boldsymbol{x}_1).$$

[1] In general, the degree of regularity of the free boundary $\partial\Omega_m$ has not been resolved, and is an open question which P. Bauman and D. Phillips have been considering for anti-plane shear problems. Our computational results in Section 3 indicate that for the problem of this investigation $\partial\Omega_m$ is smooth. Also, these computations support the view that $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$ are tangent at points of contact, which is the result of this lemma.

[2] See footnote 1.

Let

$$\Sigma \equiv \{x \in \overset{o}{\Omega}_m : (x - x_1) \cdot e(x_1) > 0\},$$

and

$$\partial\Sigma_1 \equiv \{x \in \partial D(\kappa_1) : (x - x_1) \cdot e(x_1) \geq 0\}.$$

We shall conclude, later in this proof, that the level contour $x(\beta): \beta \in (0, 1) \subset \partial\Sigma_1$. However, first let us suppose that this claim holds and show that the proposition is valid under this assumption. To begin, let $N(x^*)$ be an open neighborhood of $x^* \in \{x(\beta): \beta \in (0, 1)\}$ in $\mathbb{R}^2$, and let $N^+ \equiv N(x^*) \cap (\Omega \backslash D(\kappa_1))$. Clearly, then, $N^+ \neq \emptyset$ is simply connected, and $\{x(\beta): \beta \in (0, 1)\} \cap N(x^*)$ separates N^+ and its complementary set $N(x^*) \backslash N^+$. Thus, we have

$$\boldsymbol{D} \cdot \boldsymbol{D}u = 0$$

in N^+, and also

$$u(x) = u(x_1) \quad \text{and} \quad |\boldsymbol{D}u(x)| = \kappa_1$$

for all $x \in \{x(\beta): \beta \in (0, 1)\} \cap N(x^*)$. Because the level contour $\{x(\beta): \beta \in (0, 1)\} \cap N(x^*)$ is a straight line, and because u is harmonic and all the level contours of u are convex in the neighborhood N^+, upon employing an argument similar to that of Lemma 2.9 in [2], we find that all of the level contours in N^+ must have zero curvature. Since $(\boldsymbol{DD}u)\boldsymbol{n} \cdot \boldsymbol{n} \geq 0$ and $\boldsymbol{D}u \cdot \boldsymbol{n} < 0$ for any points in Ω (Theorem 1), the neighborhood N^+ may be extended to $\partial\Omega_o$, the particular contour $\{u = 0\}$, which is not a straight line. Thus, we have a contradiction, and so $x_2 \notin \partial D(\kappa_1) \cap \Omega$.

Now, in order to show that the level contour $x(\beta) \subset \partial\Sigma_1$ for all $\beta \in (0, 1)$, we first observe that if such is not the case there will be a sub-interval $(\beta_1, \beta_2) \subset (0, 1)$ with $x(\beta_i) \in \partial\Sigma_1$, $i = 1, 2$, such that $x(\beta) \not\subset \partial\Sigma_1$ for all $\beta \in (\beta_1, \beta_2)$. We would then have $|\boldsymbol{D}u(x(\beta))| \geq \kappa_1$ for all $\beta \in (\beta_1, \beta_2)$ with strict inequality holding for some $\beta \in (\beta_1, \beta_2)$. In the following, it is notationally convenient and without loss of generality to rescale temporarily the level contour $x(\beta)$ so that β_1 and β_2 correspond to 0 and 1, respectively. Now, to proceed with this counter-possibility, let N_1 and N_2 be sufficiently small open disjoint neighborhoods of x_1 and x_2, respectively, in $\mathbb{R}^2$, and let $y(\alpha)$, $\alpha \in (0, 1)$, be a parametric representation for a segment of the curve $\partial\Sigma_1$ in $N_1 \cap \partial\Sigma_1$ with $y(0) = x_1$. Then, because $\partial\Sigma_1$ is piecewise differentiable and the level contours in Ω_m are straight lines, there is a parameterization $z(\alpha) \in N_2 \cap \partial\Sigma_1$ for each $\alpha \in (0, 1)$, with $z(0) = x_2$, of the form

$$z(\alpha) \equiv y(\alpha) + s(y(\alpha))t(y(\alpha)), \tag{37}$$

with $s(y(\alpha)) \neq 0$, $|t(y(\alpha))| = 1$, and $t(y(\alpha)) \cdot e(y(\alpha)) = 0$, such that

$$u(z(\alpha)) = u(y(\alpha)) \tag{38}$$

and

$$Du(z(\alpha)) = Du(y(\alpha)) = -\kappa_1 e(y(\alpha)). \tag{39}$$

It follows by differentiation of (38) and use of (39) that

$$(z'(\alpha) - y'(\alpha)) \cdot e(y(\alpha)) = 0.$$

Further, by differentiation of (37) we then find that

$$\frac{dt(y(\alpha))}{d\alpha} \cdot e(y(\alpha)) = 0,$$

which, since $t(y(\alpha))$ is a unit vector perpendicular to $e(y(\alpha))$, implies that

$$\frac{dt(y(\alpha))}{d\alpha} = 0.$$

Since this is supposed to hold for all $\alpha \in (0, 1)$, we see that

$$t(y) = t(x_1), \qquad e(y) = e(x_1) \tag{40}$$

for all $y \in N_1 \cap \partial\Sigma_1$. Then, by differentiation and use of (39) and (40) we get

$$\frac{du(y(\alpha))}{d\alpha} = Du(y(\alpha)) \cdot \frac{dy(\alpha)}{d\alpha} = -\kappa_1 e(x_1) \cdot \frac{dy(\alpha)}{d\alpha},$$

which, by integration, gives

$$u(y) = u(x_1) - \kappa_1 e(x_1) \cdot (y - x_1) \tag{41}$$

for all $y \in N_1 \cap \partial\Sigma_1$.

With the above agreement we may conclude that the straight level contour lines in Σ which connect the corresponding points $y(\alpha) \in N_1 \cap \partial\Sigma_1$ and $z(\alpha) \in N_2 \cap \partial\Sigma_1$ for $\alpha \in (0, 1)$ are all *parallel* to the straight line $x(\beta)$ which connects x_1 and x_2. Then, for any $x \in \Sigma$ and corresponding $y \in N_1 \cap \partial\Sigma_1$ such that $(x - x_1) \cdot e(x_1) = (y - x_1) \cdot e(x_1)$, we have $u(x) = u(y)$, and from (41) we see that

$$u(x) = u(x_1) - \kappa_1 e(x_1) \cdot (x - x_1). \tag{42}$$

Since (42) yields $Du(x) = -\kappa_1 e(x_1)$, we find, by limiting $x \to x(\beta)$ for any $\beta \in (0, 1)$, that $|Du(x(\beta))| = \kappa_1$ which contradicts the original working hypothesis that $|Du(x(\beta))| \geq \kappa_1$ for all $\beta \in (0, 1)$ with strict inequality holding for some $\beta \in (0, 1)$. Thus, $x(\beta) \subset \partial\Sigma_1$ for all $\beta \in (0, 1)$, which shows that every sub-interval of the original (before rescaling) level contour $x(\beta)$ must lie in $\partial\Sigma_1$. □

Because of this last proposition we now see that the low strain and high strain boundaries of the phase mixture region, $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$, must intersect on the axis of symmetry of Ω. This is the content of

Proposition 5. Let $\overset{o}{\Omega}_m \neq \varnothing$. Suppose $\boldsymbol{x}_o \in \Omega_m$ and is on the axis of symmetry of Ω. Then $\boldsymbol{x}_o \in \partial D(\kappa_1) \cap \partial D(\kappa_2) \cap \Omega$.

Proof. Let u be the solution to *Problem P2*, and consider, without loss of generality, a point $\boldsymbol{x}_o \in \Omega \cap \partial D(\kappa_1)$ that is on the axis of symmetry of Ω. We need only to show that $d^*(\boldsymbol{x}_o) = 0$.

For convenience, we let (ξ, η) denote the coordinates of a local rectangular coordinate system whose origin is at $\boldsymbol{x}_o$ and orientation is such that the positive direction of the ξ axis is $\boldsymbol{e}(\boldsymbol{x}_o)$, where $\boldsymbol{e}(\boldsymbol{x}_o)$ is given by (18) and, therefore, is parallel to the direction of the axis of symmetry. Because $\boldsymbol{x}_o$ is on the axis of symmetry of Ω and $\eta = 0$ represents the axis of symmetry, we have

$$u(\boldsymbol{x}(\xi, \eta)) = u(\boldsymbol{x}(\xi, -\eta)),$$

and

$$|\boldsymbol{D}u(\boldsymbol{x}(\xi, \eta))| = |\boldsymbol{D}u(\boldsymbol{x}(\xi, -\eta))|.$$

Suppose, for contradiction, that $d^*(\boldsymbol{x}_0) \neq 0$. Because $D(\kappa_2) \subset D(\kappa_1)$ and $\boldsymbol{D}u(\boldsymbol{x}_0) \cdot \boldsymbol{e}(\boldsymbol{x}_0) < 0$, we note that $\boldsymbol{x}(\xi, 0) \in \overset{o}{\Omega}_m$ for any $\xi \in (-d^*(\boldsymbol{x}_0), 0)$. Recall from Proposition 2 that the contours $u(\boldsymbol{x}(\xi, \eta)) = u(\boldsymbol{x}(\xi, 0))$ are straight lines in $\overset{o}{\Omega}_m$ for each $\xi \in (-d^*, 0)$. Then, for any $\xi \in (-d^*, 0)$, there is a $\eta^* \neq 0$ such that one of the following holds:

(i) $\boldsymbol{x}(\xi, \eta^*)$ and $\boldsymbol{x}(\xi, -\eta^*) \in \partial D(\kappa_1) \cap \{u(\boldsymbol{x}(\xi, \eta)) = u(\boldsymbol{x}(\xi, 0))\} \cap \Omega$,
(ii) $\boldsymbol{x}(\xi, \eta^*)$ and $\boldsymbol{x}(\xi, -\eta^*) \in \partial D(\kappa_2) \cap \{u(\boldsymbol{x}(\xi, \eta)) = u(\boldsymbol{x}(\xi, 0))\} \cap \Omega$,
(iii) $\boldsymbol{x}(\xi, \eta^*)$ and $\boldsymbol{x}(\xi, -\eta^*) \in \{u(\boldsymbol{x}(\xi, \eta)) = u(\boldsymbol{x}(\xi, 0))\} \cap \partial\Omega_o$.

We observe from Proposition 4 that the first two conditions are impossible. The last condition implies that there is a subdomain in $\overset{o}{\Omega}_m$ where $u = \text{const.}$, which contradicts the condition $|\boldsymbol{D}u| \geq \kappa_1 > 0$ in Ω_m. □

As noted in Proposition 1, when $H \in (H_m, H_M)$, we have $\overset{o}{\Omega}_m \neq \varnothing$. Since, according to Proposition 3, the maximum of $|\boldsymbol{D}u|$ occurs on the boundary $\partial\Omega_i$, it is inevitable that for some $H \in (H_m, H_M)$ it will happen that $\Omega_m \cap \partial\Omega \neq \varnothing$. We further note from Proposition 2 that the contours $\{u(\boldsymbol{x}) = \text{const.}: \boldsymbol{x} \in \overset{o}{\Omega}_m\}$ must be straight lines. Consequently, since the boundary of Ω is not straight, we must have $\overset{o}{\Omega}_m \cap \partial\Omega = \varnothing$, which means that it is only possible for the boundaries of Ω_m and Ω to meet at singular points. We state this in

Proposition 6. Let u be the solution to *Problem P2*. Then, $\partial\Omega \cap \overset{o}{\Omega}_m = \varnothing$.

Proof. Let u be the solution to *Problem P2*. Suppose, without loss of generality, $\partial\Omega_o \cap \overset{o}{\Omega}_m \neq \varnothing$. We let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote two points at which the axis of symmetry of Ω intersects $\partial\Omega_o$ so that $|\boldsymbol{Du}(\boldsymbol{x}_1)| > |\boldsymbol{Du}(\boldsymbol{x}_2)|$. Because u and $|\boldsymbol{Du}|$ are symmetric about the axis of symmetry of Ω, we consider only the top half of Ω with the axis of symmetry of Ω as the base.

Suppose $\tilde{\boldsymbol{x}} \in \partial\Omega_o \cap \partial D(\kappa_1)$. We recall, from Proposition 3, that $|\boldsymbol{Du}|$ is monotone on the top half of $\partial\Omega_o$ and that the maximum and minimum values of $|\boldsymbol{Du}|$ occur on the axis of symmetry of Ω. Also recall, from Lemma 1, that $D(\kappa)$, $\kappa > 0$, is simply connected. Thus, we have either $|\boldsymbol{Du}(\boldsymbol{x}_1)| \geq \kappa_2$ or $\kappa_1 < |\boldsymbol{Du}(\boldsymbol{x}_1)| < \kappa_2$. The latter situation is impossible because otherwise there is a point $\boldsymbol{z} \in \Omega \cap \partial D(\kappa_2)$ that is on the axis of symmetry of Ω such that $\kappa_2 > |\boldsymbol{Du}(\boldsymbol{x})| > \kappa_1$ for all points $\boldsymbol{x}$ in the interval $(\boldsymbol{z}, \boldsymbol{x}_1)$. However, in this case Proposition 5 shows that $\boldsymbol{x}$ is also in $\Omega \cap \partial D(\kappa_1)$, which, because of $(\boldsymbol{DDu})\boldsymbol{n} \cdot \boldsymbol{n} \geq 0$ and $\boldsymbol{Du} \cdot \boldsymbol{n} < 0$ in Theorem 1, implies that $|\boldsymbol{Du}(\boldsymbol{x})| < \kappa_1$ for all points $\boldsymbol{x}$ in the interval $(\boldsymbol{z}, \boldsymbol{x}_1)$. We thus have a contradiction. In the first situation, since $|\boldsymbol{Du}|$ is monotone along the top half of $\partial\Omega_o$, there is a point $\boldsymbol{y} \in \partial\Omega_o \cap \partial D(\kappa_2)$ so that $\boldsymbol{y}$ is in the segment of $\partial\Omega_o$ between $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}_1$. We now show that $\boldsymbol{y}$ must coincide with $\tilde{\boldsymbol{x}}$. Suppose $\boldsymbol{y}$ does not coincide with $\tilde{\boldsymbol{x}}$. Then, the segment of $\partial\Omega_o$ between $\boldsymbol{y}$ and $\tilde{\boldsymbol{x}}$ is in $\overset{o}{\Omega}_m$ and is not a straight line. Because $u(\boldsymbol{x}) = u(\tilde{\boldsymbol{x}}) = u(\boldsymbol{y})$, $\forall \boldsymbol{x} \in \partial\Omega_o$, Proposition 2 then requires the existence of a subdomain of $\overset{o}{\Omega}_m$ adjacent to $\partial\Omega_o \cap \overset{o}{\Omega}_m$ in which $u = u(\tilde{\boldsymbol{x}}) = u(\boldsymbol{y})$, which contradicts the condition that $|\boldsymbol{Du}| \geq \kappa_1$ in $\overset{o}{\Omega}_m$. Thus, $\boldsymbol{y}$ must coincide with $\tilde{\boldsymbol{x}}$ and we conclude that if $\tilde{\boldsymbol{x}} \in \partial\Omega_o \cap \partial D(\kappa_1)$, then $\tilde{\boldsymbol{x}} \in \partial\Omega_o \cap \partial D(\kappa_2)$.

A similar argument will show that if $\tilde{\boldsymbol{x}} \in \partial\Omega_o \cap \partial D(\kappa_2)$, then $\tilde{\boldsymbol{x}} \in \partial\Omega_o \cap \partial D(\kappa_1)$. □

3. Numerical investigations

In order to demonstrate the features of anti-plane shear deformation field that were discussed above, we shall present some numerical results for the relaxed minimization *Problem P2.* Because our interest is on the structure of the domain of phase mixture, we shall focus on the shape of this region and on the contours $u = \text{const.}$ in this domain. These matters, in particular, are numerically delicate because they are concerned with shear strains where the relaxed specific stored energy function for *Problem P2* is no longer strictly convex. We are especially grateful to our colleague Sanjay Mittal who gave us a finite element code which he developed for the Cray-2 with a direct solver to study the *Problem P2.* We revised his code so as to be compatible with the Connection Machine parallel environment and a GMRES iteration scheme, and the numerical results presented here were then obtained using the CM-200.

For the numerical investigation, we suppose that the cross section Ω is defined by the outer radius $R_o = 1$, the inner radius $R_i = 0.25$, and the eccentricity $e = 0.25$. The material constants of the specific stored energy (8) are specified so that $\kappa_1 = 1$, $\kappa_2 = 2$, $\mu^- = 2$, and $\mu^+ = 1$ (see Figures 2 and 3).

For the given geometry of the domain Ω and the specified material constants, it follows from (21) and (22) that the lower and upper limits of the displacement H, between which *Problem P1* has no solution and *Problem P2* has a unique solution with part of Ω supporting shear strains in the nonconvex range (κ_1, κ_2), are $H_m = 0.28512975$ and $H_M = 4.562076$. Recall from Theorem 2, then, that the domain of phase mixture occurs if and only if the prescribed displacement H lies in the range (0.28512975, 4.562076). For the purpose of illustration, we shall consider the cases when $H = 0.9$, $H = 0.31$, and $H = 4.3$. In the first case, when $H = 0.9$, the whole domain of phase mixture Ω_m is contained in Ω. In each of the latter two cases, since $H = 0.31$ and $H = 4.3$ are close to H_m and H_M, respectively, we find that the domain of phase mixture Ω_m intersects the inner and outer boundaries of Ω accordingly. Of course, this intersection can not contain interior points of Ω_m because of Proposition 6, and, in fact, only boundary points are observed numerically.

Case *H* = *0.9*

In Figure 5, the contours of $|\boldsymbol{Du}| = \text{const.}$ in Ω are shown. Since our main concern is the structure of the phase mixture domain Ω_m and the deformation field is symmetric about the symmetry axis of Ω, we plot in Figure 5 only the contours of $|\boldsymbol{Du}| \geq 0.80$ for the top half of Ω. We observe from Figure 5 that the cross section Ω is divided into three regions Ω_1, Ω_2, and Ω_m. In Ω_1 and Ω_2, $|\boldsymbol{Du}|$ is smaller than $\kappa_1 = 1$ and greater than $\kappa_2 = 2$, respectively. Over the thin dark region Ω_m, $|\boldsymbol{Du}|$ varies rapidly from κ_1 on the outside boundary to κ_2 on the inside boundary of Ω_m. Because the region Ω_m is very thin, a set of fine mesh has been introduced in order to

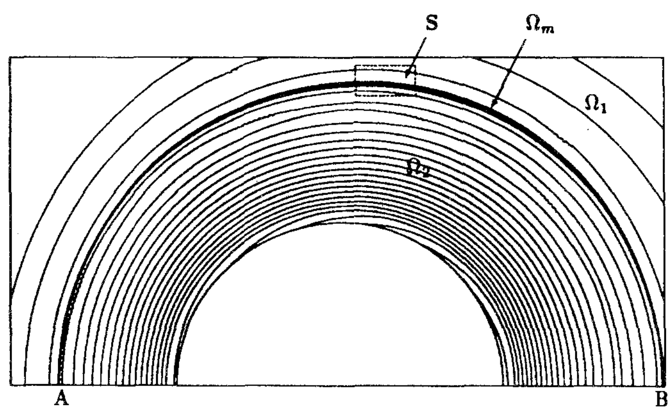


Figure 5
Contours of $|\boldsymbol{Du}| = \text{const.}$ for $H = 0.9$.

capture the phase mixture domain Ω_m. To reduce the computation cost, we employ a set of coarse mesh for the region in Ω where the mixture domain is not expected to occur. From Figure 5 we observe that the distances between the two boundaries of Ω_m at points A and B are very small. Because these two distances are about two or three mesh sizes, and they become smaller when a smaller mesh size is used, we infer from the computation that the two boundaries of Ω_m are tangential to each other at A and B. This is a confirmation of the result of Proposition 5.

The structure of the region Ω_m is further demonstrated in Figure 6. In this figure, we magnify a small region S that is located near the top center of Ω_m as shown in Figure 5 and plot the contours of $|Du| = \text{const.}$ and $u = \text{const.}$ in Figure 6a and Figure 6b, respectively. Comparing these two figures, we observe that in the darker region, where $|Du| = \text{const.}$ varies from κ_1 to κ_2 (Figure 6a), the contours $u = \text{const.}$ are straight lines (Figure 6b). This is in agreement with Proposition 2.

Cases $H = 0.31$ and $H = 4.5$

Figure 7 shows the contours of $|Du| = \text{const.}$ for $H = 0.31$ and $H = 4.5$, respectively. The very thin dark region defines Ω_m, inside of which $|Du| = \text{const.}$ varies from κ_1 to κ_2. For $H = 0.31$, this phase mixture region is close to the inner cylinder and it occurs on that side of the cylinder which defines the narrow gap for Ω. Notice, in Figure 7a, there is a small crescent domain next to the inner cylinder in which $|Du| > \kappa_2$. For $H = 4.5$, the phase mixture region is close to the outer cylinder and it occurs on that side of the cylinder which is in the wide gap for Ω. Notice, here, in Figure 7b, that there is again a small crescent domain adjacent to the outer cylinder in

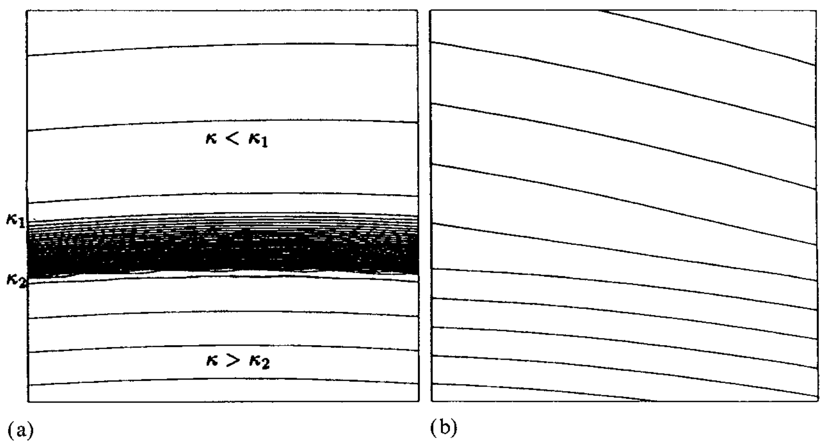


Figure 6
Blow-up of a small region S near the top center of Ω_m in Figure 5 for $H = 0.9$. (a) Contours of $|Du| = \text{const.}$; (b) Contours of $u = \text{const.}$

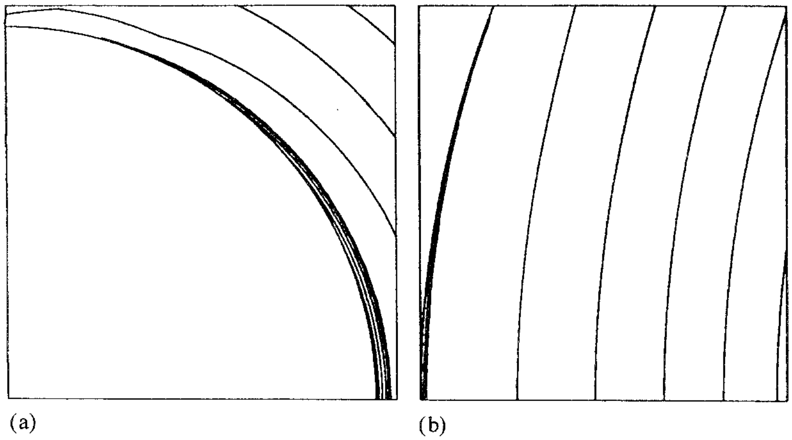


Figure 7
Contours of $|Du| = \text{const.}$ for Ω_m near the boundary of Ω. (a) $H = 0.31$; (b) $H = 4.5$.

which $|\boldsymbol{Du}| < \kappa_1$. As we see from the figures, the domain of phase mixture Ω_m intersects with the inner or outer boundary of Ω, depending on whether $H = 0.31$ or 4.5, respectively. Because we are interested in the geometry of the domain of phase mixture, Figure 7 contains only the contours of $|\boldsymbol{Du}| = \text{const.}$ in a small region near the boundaries of Ω that include the intersection of Ω_m with $\partial\Omega$. The thickness of the thin dark regions in the figure are about two or three mesh sizes. Because the mesh size is not zero, Figure 7 shows that the intersection of the interior of the thin dark regions Ω_m does not vanish. However, the intersection of the interior of the dark regions Ω_m with $\partial\Omega$ becomes thinner as the mesh size is reduced, which we take as an indication that $\partial\Omega \cap \overset{o}{\Omega}_m = \varnothing$. This is the content of our earlier Proposition 6.

In Figure 8, we show the axial force per unit length applied to the inner cylinder as a function of the distance that the cylinder is displaced. At point a in Figure 8 the phase mixture region Ω_m first starts to appear in Ω. Between the origin and a, the minimization *Problem P2* is quadratic and the field theory is linear, so the graph in Figure 8 is linear. Between points a and b the phase mixture region has two points of attachment to the inner boundary. One of them can be seen in Figure 7a where the dark thin region intersects the circular boundary, and the other is symmetric with respect to the horizontal base of this figure. At b Ω_m detaches from the inner boundary $\partial\Omega_i$. As H is increased further, Ω_m consists of a ring in Ω which has zero thickness at the two points where it crosses the axis of symmetry of the cross section of Ω. After point b in Figure 8 the ring Ω_m is inside, and moves toward the outer boundary of Ω. At the same time the axial force per unit length F increases but not as rapidly as it did between the origin and point a, before the phase mixture region appeared. Eventually, for some finite H, the graph in Figure 8 will turn upward and become a straight line with a

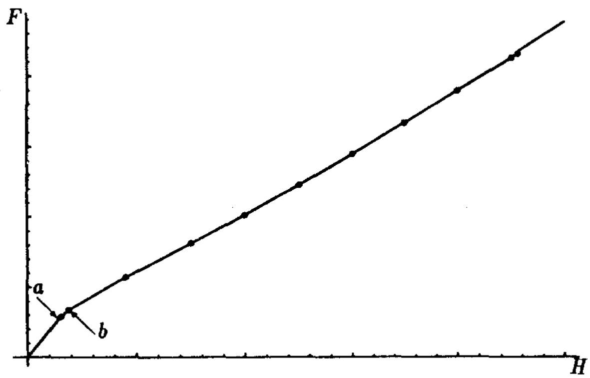


Figure 8
The axial force per unit length F *vs.* the relative axial displacement H.

somewhat smaller slope than that between the origin and point a. When this happens, Ω no longer contains a phase mixture region Ω_m. It is possible that the characteristic slopes of Figure 8 can be significantly altered by choice of the material constants which characterize the specific stored energy function. To some extent these slopes will also depend on the geometry of the cross section Ω. These parametric studies have not been performed.

4. A minimizing sequence

In this section, we shall construct a minimizing sequence for the *Problem P1* for the situation when the eccentricity $e \neq 0$ and the applied displacement $H \in (H_m, H_M)$. While in this case we know from Theorem 2 that *Problem P1* does not have a minimizer in $\mathscr{A}$, this does not preclude the existence of a sequence in $\mathscr{A}$ for which the total potential energy limits to its infimum. Because there is a unique solution to *Problem P2* (cf. Theorem 1), it is natural to base the construction of such a minimizing sequence on this field. In order to do this, we begin with the following

Lemma 3. Let u be the solution to *Problem P2.* Suppose $\overset{o}{\Omega}_m \neq \emptyset$. Then, for any fixed $\boldsymbol{x} \in \overset{o}{\Omega}_m$, there exist unique points $\boldsymbol{y} \in \partial D(\kappa_1) \cap \Omega$ and $\boldsymbol{z} \in \partial D(\kappa_2) \cap \Omega$ such that $u(\boldsymbol{x}) = u(\boldsymbol{y}) = u(\boldsymbol{z})$, and $\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{e}(\boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{z})$, where $\boldsymbol{e}(\,\cdot\,)$ is the unit normal field of (18). Moreover, for any point $\boldsymbol{x}^*(\beta) = \beta\boldsymbol{y} + (1-\beta)\boldsymbol{z}$, $\beta \in (0, 1)$, on the straight line between $\boldsymbol{y}$ and $\boldsymbol{z}$, it happens that $u(\boldsymbol{x}^*(\beta)) = u(\boldsymbol{x})$ and $(\boldsymbol{x}^*(\beta) - \boldsymbol{x}) \cdot \boldsymbol{e}(\boldsymbol{x}) = 0$.

This lemma follows as a consequence of Propositions 2, 4 and 6. First recall that the level set $\{\boldsymbol{x} \in \overset{o}{\Omega}_m : u(\boldsymbol{x}) = \text{const.}\}$ is a straight line by Proposition 2. Because the displacement field is continuous in Ω, the straight line must intersect with one or more of the boundaries $\partial D(\kappa_1)$, $\partial D(\kappa_2)$, or $\partial\Omega$.

However, since Proposition 6 shows that $\overset{o}{\Omega}_m \cap \partial\Omega = \varnothing$, and since Proposition 4 shows that the set $\{\boldsymbol{x} \in \overset{o}{\Omega}_m : u(\boldsymbol{x}) = \text{const.}\}$ can not intersect either $\partial D(\kappa_1)$ or $\partial D(\kappa_2)$ twice, the straight line can only intersect with both $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$.

Proof. Let $\boldsymbol{x} \in \overset{o}{\Omega}_m$ be given. Because the level set $\{\boldsymbol{x} \in \overset{o}{\Omega}_m : u(\boldsymbol{x}) = \text{const.}\}$ is a straight line, by Proposition 4 there must exist a $\boldsymbol{y} \in \partial D(\kappa_1) \cap \Omega$ and a $\boldsymbol{z} \in \partial D(\kappa_2) \cap \Omega$ such that $u(\boldsymbol{x}) = u(\boldsymbol{y}) = u(\boldsymbol{z})$ and $\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{e}(\boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{z})$. Moreover, there cannot be another such $\boldsymbol{y}$ and $\boldsymbol{z}$ because neither $\partial D(\kappa_1)$ nor $\partial D(\kappa_2)$ can be intersected more than once. In any other case, the level set will intersect the boundary $\partial\Omega$, which is impossible because $\overset{o}{\Omega}_m \cap \partial\Omega = \varnothing$ by Proposition 6. □

Lemma 3 shows that when $e \neq 0$ and $H \in (H_m, H_M)$, the solution to *Problem P2* has the important property that for any $\boldsymbol{y} \in \partial D(\kappa_1) \cap \Omega$, there exists a unique associated point $\boldsymbol{z} \in \partial D(\kappa_2) \cap \Omega$ such that $u(\boldsymbol{y}) = u(\boldsymbol{z})$ and $\boldsymbol{e}(\boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{z})$. Stated another way, consider the straight line $\mathscr{L}(\boldsymbol{y}) \equiv \{\boldsymbol{x} \in \Omega_m : u(\boldsymbol{x}) = u(\boldsymbol{y})\}$. Then, we see immediately from Lemma 3 that for any $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \partial D(\kappa_1) \cap \Omega$, $\mathscr{L}(\boldsymbol{y}_1) \cap \mathscr{L}(\boldsymbol{y}_2) = \varnothing$ if and only if $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$. We shall use this property in the construction of a minimizing sequence for *Problem P1.*

To aid in this construction we introduce the angular coordinate $\theta \equiv \arctan(x_2/(x_1 - e))$ for points $\boldsymbol{x} = (x_1, x_2) \in \Omega$ as measured from the center of Ω_i, and because of symmetry consider only that half of Ω where $\theta \in [0, \pi]$. We denote by $\boldsymbol{y}(\theta)$ and $\boldsymbol{z}(\theta)$ the two parametric representations of $\partial D(\kappa_1) \cap \Omega$ and $\partial D(\kappa_2) \cap \Omega$, respectively. In both cases, by Theorem 1 and Lemma 2 the range of θ will be $[0, \pi]$ if $\partial D(\kappa_1) \subset \Omega$ and $\partial D(\kappa_1) \cap \partial\Omega = \varnothing$. Otherwise, the range of θ will be subinterval of $[0, \pi]$. We shall concentrate here on the construction of a minimizing sequence when the range of θ is $[0, \pi]$. For the alternative situations, the construction is similar.

Notice that $\theta = 0$ or π if and only if $x_2 = 0$. Thus, the points $\boldsymbol{y}(0)$, $\boldsymbol{z}(0)$, $\boldsymbol{y}(\pi)$, and $\boldsymbol{z}(\pi)$ are on the axis of symmetry, and by Proposition 6 we know that $\boldsymbol{y}(0) = \boldsymbol{z}(0)$ and $\boldsymbol{y}(\pi) = \boldsymbol{z}(\pi)$. We now use the solution u to *Problem P2* to partition the parametric representations $\boldsymbol{y}(\theta)$ and $\boldsymbol{z}(\theta)$, $\theta \in [0, \pi]$, of $\partial D(\kappa_1) \cap \Omega$ and $\partial D(\kappa_2) \cap \Omega$, respectively, into N sections with $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_N$, and $\boldsymbol{z}_0, \ldots, \boldsymbol{z}_N$, as associated end points of each section. In particular, we set (see Figure 9)

$$u_i \equiv u(\boldsymbol{y}_i) = u(\boldsymbol{y}_{i-1}) + \Delta u$$

and

$$u_i \equiv u(\boldsymbol{z}_i) = u(\boldsymbol{z}_{i-1}) + \Delta u$$

for $i = 1, \dots, N$, where

$$\Delta u \equiv \frac{u(y(\pi)) - u(y(0))}{N}. \tag{43}$$

The possibility of this partition is ensured by Lemma 3, from which we also infer that

$$e_i \equiv e(y_i) = e(z_i), \qquad i = 0, 1, \dots, N.$$

Because $e(x)$ is continuous, these normal vectors to the constant u_i contour lines satisfy $e_i - e_{i-1} \to 0$ as $N \to \infty$, a fact we shall use later.

Let y_i and y_{i-1} be two generic end points of the partition on $y(\theta)$, $\theta \in [0, \pi]$, and let Ω^i_m be a subdomain of $\mathring{\Omega}_m$ of the form

$$\Omega^i_m \equiv \{x \in \mathring{\Omega}_m : u(x) \in (u_{i-1}, u_i)\}. \tag{44}$$

For convenience, we shall denote by s^*_i the distance between y_i and z_i, i.e.,

$$s^*_i \equiv |y_i - z_i|, \tag{45}$$

and define

$$t_i \equiv \frac{z_i - y_i}{|z_i - y_i|}. \tag{46}$$

In addition, for any $x \in \Omega^i_m$, let

$$s_i(x) \equiv (x - y_i) \cdot t_i, \tag{47}$$

$$\sigma_i(x) \equiv (y_{i-1} - x) \cdot e_{i-1}, \tag{48}$$

and

$$\tau_i(x) \equiv (x - y_i) \cdot e_i. \tag{49}$$

(All of this notation is exhibited in Figure 9.) Then, for each fixed N, we construct a local displacement field $\tilde{u}_N(x)$ for $x \in \mathring{\Omega}_m$ by

$$\tilde{u}_N(x) = \begin{cases} u_i - \kappa_1 \tau_i(x) & \forall x \in \Omega^i_m \text{ and } \tau_i(x) \in (0, \tau(s_i)), \\ u_{i-1} + \kappa_2 \sigma_i(x) & \forall x \in \Omega^i_m \text{ and } \sigma_i(x) \in (0, \sigma(s_i)), \end{cases} \tag{50}$$

for all $i = 1, \dots, N$, where $\tau(s_i)$ and $\sigma(s_i)$ are two unknowns to be determined so that the field $\tilde{u}_N(x)$ is single valued and continuous in Ω^i_m. We see immediately from (50), (49), and (48) that

$$D\tilde{u}_N(x) = \begin{cases} -\kappa_1 e_i & \forall x \in \Omega^i_m \text{ and } \tau_i(x) \in (0, \tau(s_i)), \\ -\kappa_2 e_{i-1} & \forall x \in \Omega^i_m \text{ and } \sigma_i(x) \in (0, \sigma(s_i)). \end{cases} \tag{51}$$

Now, to determine $\tau(s_i)$ and $\sigma(s_i)$, we note first that any $x \in \Omega^i_m$ may be represented by either

$$x = y_i + s_i t_i + \tau_i e_i,$$

or

$$x = y_{i-1} + s_{i-1} t_{i-1} - \sigma_i e_{i-1}.$$

The continuity of $\tilde{u}_N$ at $\tau_i(x) = \tau(s_i)$ and $\sigma_i(x) = \sigma(s_i)$, then implies that

$$y_i + s_i t_i + \tau(s_i) e_i = y_{i-1} + s_{i-1} t_{i-1} - \sigma(s_i) e_{i-1}, \tag{52}$$

and

$$u_i - \kappa_1 \tau(s_i) = u_{i-1} + \kappa_2 \sigma(s_i). \tag{53}$$

Thus,

$$\tau(s_i) = \frac{\kappa_2[(y_{i-1} - y_i) \cdot e_{i-1} - s_i t_i \cdot e_{i-1}] - (u_i - u_{i-1})}{\kappa_2 e_i \cdot e_{i-1} \kappa_1}, \tag{54}$$

and

$$\sigma(s_i) = \frac{(u_1 - u_{i-1}) e_i \cdot e_{i-1} - \kappa_1[(y_{i-1} - y_i) \cdot e_{i-1} - s_i t_i \cdot e_{i-1}]}{\kappa_2 e_i \cdot e_{i-1} - \kappa_1}. \tag{55}$$

In addition, we find that s_{i-1} is related to s_i by

$$\begin{aligned} s_{i-1} = s_i &\left[t_i \cdot t_{i-1} - t_i \cdot e_{i-1} t_{i-1} \cdot e_i \frac{\kappa_2}{\kappa_2 e_i \cdot e_{i-1} - \kappa_1} \right] \\ &+ (y_i - y_{i-1}) \cdot \left[t_{i-1} - \frac{\kappa_2 e_i \cdot t_{i-1}}{\kappa_2 e_i \cdot e_{i-1} - \kappa_1} e_{i-1} \right] + \frac{(u_{i-1} - u_i) e_i \cdot t_{i-1}}{\kappa_2 e_i \cdot e_{i-1} - \kappa_1}. \end{aligned} \tag{56}$$

From (54) and (55), we emphasize that $\tau(s_i)$ and $\sigma(s_i)$ depend upon $x \in \Omega_m^i$ through s_i. Thus, for a given $x \in \Omega_m^i$, to determine the value of $\tilde{u}_N$, we first compute the value of s_i by using (47). Then, we calculate $\tau(s_i)$ and $\sigma(s_i)$ according to (54) and (55), respectively. Finally, we determine value of $\tilde{u}_N$ from (50), (48) and (49).

The displacement field $\tilde{u}_N$ given in (50) defines a continuous field in $\mathring{\Omega}_m$ for any fixed N. Notice, also, from Proposition 2, that the solution u of *Problem P2* satisfies $u(x) = u_i$ on each of the lines $(x - y_i) \cdot e_i = 0$, $i = 0, 1, 2, \ldots, N$, in the phase mixture region Ω_m. Thus, $\tilde{u}_N(x) = u(x)$ in (50) at both $\tau_i(x) = 0$ and $\sigma_i(x) = 0$. In general, however, $\tilde{u}_N(x) \neq u(x)$ along the curves $y(\theta)$ and $z(\theta)$, $\theta \in [0, \pi]$, except at the $N + 1$ pair of points y_i and z_i, $i = 0, \ldots, N$. Now, we wish to connect the field $u_N(x)$, defined above for all $x \in \Omega_m$, to the field that represents the solution u of *Problem P2* in the remainder of Ω so that the composite field is continuous in all of Ω. We shall do this by introducing thin interpolation regions near the curves $y(\theta)$ and $z(\theta)$, $\theta \in [0, \pi]$. We used a similar construction for our work in [7], but in the present application there are always two transition interpolation regions to consider.

We shall denote by $\Omega_{\varepsilon_o}(\partial D(\kappa_1))$ and $\Omega_{\eta_o}(\partial D(\kappa_2))$ two interpolation regions that neighbor $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$, respectively. These two subdomains of $\Omega\backslash\Omega_m$ are defined as

$$\Omega_{\varepsilon_o}(\partial D(\kappa_1)) \equiv \{\boldsymbol{x} \in \Omega\backslash D(\kappa_1): 0 < d(\boldsymbol{x}, \partial D(\kappa_1)) < \varepsilon_o\}$$

and

$$\Omega_{\eta_o}(\partial D(\kappa_2)) \equiv \{\boldsymbol{x} \in D(\kappa_2)\backslash\Omega_i: 0 < d(\boldsymbol{x}, \partial D(\kappa_2)) < \eta_o\},$$

where ε_o and η_o are positive, and $d(\boldsymbol{x}, \partial D(\kappa_1))$ and $d(\boldsymbol{x}, \partial D(\kappa_2))$ are the distances between $\boldsymbol{x}$ and the boundaries $\partial D(\kappa_1)$ and $\partial D(\kappa_2)$, respectively, i.e.,

$$d(\boldsymbol{x}, \partial D(\kappa_1)) \equiv \min\{|\boldsymbol{x} - \boldsymbol{y}|: \boldsymbol{y} \in \partial D(\kappa_1) \cap \Omega\},$$

and

$$d(\boldsymbol{x}, \partial D(\kappa_2)) \equiv \min\{|\boldsymbol{x} - \boldsymbol{z}|: \boldsymbol{z} \in \partial D(\kappa_2) \cap \Omega\}.$$

The numbers ε_o and η_o define the thickness of the interpolation subdomains $\Omega_{\varepsilon_o}(\partial D(\kappa_1))$ and $\Omega_{\eta_o}(\partial D(\kappa_2))$, respectively. Since, as noted earlier, our present construction assumes $\partial D(\kappa_1) \cap \partial\Omega = \emptyset$, then ε_o and η_o can be chosen sufficiently small and positive such that Ω_{ε_o} and Ω_{η_o} do not intersect the boundaries of Ω.

Consider the auxiliary function

$$h(s; s_o) = -2\left(\frac{s}{s_o}\right)^3 + 3\left(\frac{s}{s_o}\right)^2 \tag{57}$$

for $s \in (0, s_o)$, where s_o is a positive parameter. This function is certainly differentiable for $s \in (0, s_o)$, and it satisfies $h(0; s_o) = 0$ and $h(s_o; s_o) = 1$. We record

$$h'(s; s_o) \equiv \frac{dh(s; s_o)}{ds} = \frac{6}{s_o}\left\{\frac{s}{s_o} - \left(\frac{s}{s_o}\right)^2\right\} \tag{58}$$

for $s \in (0, s_o)$, which shows that for any $s = \alpha s_o$, $\alpha \in (0, 1)$, $|h'(s; s_o)| \propto s_o^{-1}$ as $s_o \to 0$, and $h'(\alpha s_o; s_o) = 0$ at $\alpha = 0$ or 1.

Now, let $f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1))$ and $f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))$ be two functions defined in Ω so that

$$f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1)) = \begin{cases} 0 & \boldsymbol{x} \in D(\kappa_1), \\ h(d(\boldsymbol{x}, \partial D(\kappa_1)); \varepsilon_o) & \boldsymbol{x} \in \Omega_{\varepsilon_o}(\partial D(\kappa_1)), \\ 1 & \boldsymbol{x} \in \Omega\backslash(D(\kappa_1) \cup \Omega_{\varepsilon_o}(\partial D(\kappa_1))), \end{cases} \tag{59}$$

and

$$f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2)) = \begin{cases} 0 & \boldsymbol{x} \in \Omega\backslash D(\kappa_2), \\ h(d(\boldsymbol{x}, \partial D(\kappa_2)); \eta_o) & \boldsymbol{x} \in \Omega_{\eta_o}(\partial D(\kappa_2)), \\ 1 & \boldsymbol{x} \in D(\kappa_2)\backslash(\Omega_i \cup \Omega_{\eta_o}(\partial D(\kappa_2))). \end{cases} \tag{60}$$

Then,

$$\boldsymbol{D}f_1(\boldsymbol{x};\varepsilon_o,\partial D(\kappa_1)) =$$

$$\begin{cases} 0 & \boldsymbol{x}\in D(\kappa_1), \\ h'(d(\boldsymbol{x},\partial D(\kappa_1));\varepsilon_o)\boldsymbol{D}d(\boldsymbol{x},\partial D(\kappa_1)) & \boldsymbol{x}\in\Omega_{\varepsilon_o}(\partial D(\kappa_1)), \\ 0 & \boldsymbol{x}\in\Omega\backslash(D(\kappa_1)\cup\Omega_{\varepsilon_o}(\partial D(\kappa_1))), \end{cases} \tag{61}$$

and

$$\boldsymbol{D}f_2(\boldsymbol{x};\eta_o,\partial D(\kappa_2)) =$$

$$\begin{cases} 0 & \boldsymbol{x}\in \Omega\backslash D(\kappa_2), \\ h'(d(\boldsymbol{x},\partial D(\kappa_2));\eta_o)\boldsymbol{D}d(\boldsymbol{x},\partial D(\kappa_2)) & \boldsymbol{x}\in\Omega_{\eta_o}(\partial D(\kappa_2)), \\ 0 & \boldsymbol{x}\in D(\kappa_2)\backslash(\Omega_i\cup\Omega_{\eta_o}(\partial D(\kappa_2))). \end{cases} \tag{62}$$

Since the curves $|\boldsymbol{D}u| = \text{const.}$ in $\Omega\backslash\Omega_m$ are smooth, due to the nature of the Dirichlet problem that is associated with *Problem P2*, it is clear from (57) to (62) that $f_1(\cdot;\varepsilon_o,\partial D(\kappa_1))$ and $f_2(\cdot;\eta_o,\partial D(\kappa_2))$ are of class C^1.

It is straightforward to show that for any fixed N, ε_o, and η_o, the field

$$\begin{aligned} u_N(\boldsymbol{x}) = \tilde{u}_N(\boldsymbol{x}) &+ f_1(\boldsymbol{x};\varepsilon_o,\partial D(\kappa_1))(u(\boldsymbol{x})-\tilde{u}_N(\boldsymbol{x})) \\ &+ f_2(\boldsymbol{x};\eta_o,\partial D(\kappa_2))(u(\boldsymbol{x})-\tilde{u}_N(\boldsymbol{x})), \end{aligned} \tag{63}$$

where u is the solution to *Problem P2*, is continuous in Ω; the gradient is given by

$$\begin{aligned} \boldsymbol{D}u_N(\boldsymbol{x}) = \boldsymbol{D}\tilde{u}_N(\boldsymbol{x}) &+ (f_1(\boldsymbol{x};\varepsilon_o,\partial D(\kappa_1)) \\ &+ f_2(\boldsymbol{x};\eta_o,\partial D(\kappa_2)))(\boldsymbol{D}u(\boldsymbol{x})-\boldsymbol{D}\tilde{u}_N(\boldsymbol{x})) + (\boldsymbol{D}f_1(\boldsymbol{x};\varepsilon_o,\partial D(\kappa_1)) \\ &+ \boldsymbol{D}f_2(\boldsymbol{x};\eta_o,\partial D(\kappa_2)))(u(\boldsymbol{x})-\tilde{u}_N(\boldsymbol{x})). \end{aligned} \tag{64}$$

We observe from (64) and (59)–(62) that for any $\boldsymbol{x}\notin(\Omega_m\cup\Omega_{\varepsilon_o}(\partial D(\kappa_1))\cup\Omega_{\eta_o}(\partial D(\kappa_2)))$, $\boldsymbol{D}u_N(\boldsymbol{x}) = \boldsymbol{D}u(\boldsymbol{x})$. Also, notice that for any fixed N, ε_o, and η_o, we have $|\boldsymbol{D}u_N(\boldsymbol{x})| = \kappa_1$ or κ_2 when $\boldsymbol{x}\in\mathring{\Omega}_m$, and $|\boldsymbol{D}u_N(\boldsymbol{x})|$ is bounded when $\boldsymbol{x}\in\Omega_{\varepsilon_o}(\partial D(\kappa_1))\cup\Omega_{\eta_o}(\partial D(\kappa_2))$. In addition, the displacement field $u_N(\boldsymbol{x})$ of (63) satisfies the prescribed displacement boundary conditions of (5) because for any $\boldsymbol{x}\in\partial\Omega$, $u_N(\boldsymbol{x}) = u(\boldsymbol{x})$, which is the solution to *Problem P2.*

In the above considerations we have concentrated on the situation $\partial D(\kappa_1)\cap\partial\Omega = \varnothing$, so let us now consider briefly the alternative case when $\partial D(\kappa_1)\cap\partial\Omega \neq \varnothing$. As an example, suppose $\partial D(\kappa_1)\cap\partial\Omega_o \neq \varnothing$ but $\partial D(\kappa_1)\cap\partial\Omega_i = \varnothing$. We know from Propositions 3 and 6 that the sets $\partial D(\kappa_1)\cap\partial\Omega_o$ and $\partial D(\kappa_2)\cap\partial\Omega_o$ are equal and contain only two points that are located symmetrically with respect to the axis of symmetry of the domain Ω. Let $\tilde{\boldsymbol{x}}$ denote one of the points of this set with $\tilde{\theta}\equiv\arctan(\tilde{x}_2/(\tilde{x}_1-e))\in[0,\pi]$. In this case, the range of the angle θ that describes the boundary $\partial D(\kappa_1) \cap \Omega$ must be $(\tilde{\theta}, 2\pi - \tilde{\theta})$ because of the symmetry of the solution of *Problem P2.* We shall focus on the half-range of $(\tilde{\theta}, \pi]$. As in the previous construction, here we also partition the parametric representations $y(\theta)$ and $z(\theta)$, $\theta \in (\tilde{\theta}, \pi]$, into N sections with $y(\tilde{\theta}) = y_o = \tilde{x}$ and $y(\pi) = y_N$, where y_N lies on the axis of symmetry of the domain Ω. The increment of displacement Δu for this partition is given by

$$\Delta u \equiv \frac{u(y(\pi)) - u(y(\tilde{\theta}))}{N} = \frac{u(y(\pi))}{N}, \tag{65}$$

where u is the solution to *Problem P2.* We now replace (43) by (65) and construct the continuous local displacement field $\tilde{u}_N(x)$, $x \in \Omega_m$, as in (50). Because $\tilde{x} \in \partial D(\kappa_1) \cap \partial\Omega_o$, the distance between the parametric representation $y(\theta)$ and $\partial\Omega_o$ approaches zero as $\theta \to \tilde{\theta}$. In order to guarantee that the interpolation regions Ω_{ε_o} and Ω_{η_o} do not intersect with the boundary $\partial\Omega_o$, we shall choose, for any $y \in \partial D(\kappa_1) \cap \Omega$,

$$\varepsilon_o \equiv \min\{d(y, \partial\Omega), \tilde{\varepsilon}\} = \varepsilon_o(y), \tag{66}$$

and, for any $z \in \partial D(\kappa_2) \cap \Omega$,

$$\eta_o \equiv \min\{d(z, \partial\Omega), \tilde{\varepsilon}\} = \eta_o(z), \tag{67}$$

where $\tilde{\varepsilon}$ is a small but positive number. Keeping the above definitions of ε_o and η_o in mind, we shall again take for the displacement field u_N in Ω the form (63) for any fixed N. Clearly, this field is continuous in Ω, and to see that it satisfies the displacement boundary condition, we note that because of (66) and (67) the interpolation regions Ω_{ε_o} and Ω_{η_o} do not intersect with the boundary $\partial\Omega_o$ and we recall that $u_N(x) = u(x)$ for any $x \in \partial\Omega$ and for every fixed N.

A similar line of reasoning also may be applied to the other situations when $\partial D(\kappa_1) \cap \partial\Omega \neq \emptyset$. Therefore, we consider (63) to be the generic form of an admissible sequence of displacement fields for any $e \neq 0$ and $H \in (H_m, H_M)$. For this sequence, we have the following

Theorem 3. Let $e \neq 0$ and $H \in (H_m, H_M)$. Let u be the solution to *Problem P2* and define u_N through (63). Suppose ε_o and η_o are of order N^{-1} for sufficiently large N as $N \to \infty$. Then,

(i) for large N, $u_N \in W^{1,p}(\Omega)$, $p \geq 1$, and $u_N \in \mathscr{A}$;
(ii) $u_N(x) \rightharpoonup u(x)$ in $W^{1,p}(\Omega)$, $p \geq 1$, as $N \to \infty$; and
(iii) for any fixed sufficiently large N,

$$Du_N(x) = Du(x)$$

for any $x \in \Omega \backslash (\Omega_m \cup \Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2)))$, and

$$Du_N(x) \rightharpoonup -[\lambda(x)\kappa_1 + (1 - \lambda(x))\kappa_2]e(x) \tag{68}$$

in L^p, $p \geq 1$, a.e. in $\overset{o}{\Omega}_m$ as $N \to \infty$, where $\lambda(\boldsymbol{x}) \in (0, 1)$ for any $\boldsymbol{x} \in \overset{o}{\Omega}_m$ and is given by

$$\lambda(\boldsymbol{x}) \equiv \frac{\kappa_2 |\boldsymbol{z} - \boldsymbol{x}|}{\kappa_2 |\boldsymbol{z} - \boldsymbol{x}| + \kappa_1 |\boldsymbol{x} - \boldsymbol{y}|}. \tag{69}$$

Here, $\boldsymbol{y} \in \partial D(\kappa_1) \cap \Omega$ and $\boldsymbol{z} \in \partial D(\kappa_2) \cap \Omega$ are such that $u(\boldsymbol{y}) = u(\boldsymbol{z}) = u(\boldsymbol{x})$ and $(\boldsymbol{y} - \boldsymbol{z}) \cdot \boldsymbol{e}(\boldsymbol{x}) = (\boldsymbol{z} - \boldsymbol{x}) \cdot \boldsymbol{e}(\boldsymbol{x}) = 0$.

Proof. (i) As noted earlier, after (63) and (64), $u_N(\boldsymbol{x})$ and $\boldsymbol{D}u_N(\boldsymbol{x})$ are bounded in Ω, and clearly from (59) and (60), $f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1))$ and $f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))$ are in $W^{1,p}(\Omega), p \geq 1$. Thus, $u_N(\boldsymbol{x}) \in W^{1,p}(\Omega), p \geq 1$. Because $u_N(\boldsymbol{x})$ also satisfies the prescribed displacement boundary condition, as remarked above, it follows that $u_N(\boldsymbol{x}) \in \mathscr{A}$.

(ii) Consider first the subdomain $(\Omega_m \cup \Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2)))$. Let $\boldsymbol{x} = \boldsymbol{x}^* \in \overset{o}{\Omega}_m$, and observe from (50) that for any fixed N there is an integer $i^* \in [1, \ldots, N]$ such that $\boldsymbol{x}^* \in \Omega_m^{i^*}$. Thus, by our earlier construction, we have $u_N(\boldsymbol{x}^*) \to u(\boldsymbol{x}^*)$ as $N \to \infty$. Now, recall that as $N \to \infty$, we have ε_o and $\eta_o \to 0$, $\Omega_{\varepsilon_o}(\partial D(\kappa_1))$ and $\Omega_{\eta_o}(\partial D(\kappa_2)) \to \varnothing$, and $u_N(\boldsymbol{x}) \to u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \partial D(\kappa_1) \cup \partial D(\kappa_2)$. Because $u_N(\boldsymbol{x}) = u(\boldsymbol{x})$ for any $\boldsymbol{x} \in \Omega \backslash (\Omega_m \cup \Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2)))$ and for every fixed N, we see that $u_N(\boldsymbol{x}) \to u(\boldsymbol{x})$ pointwise in Ω. Moreover, because of this, it is straightforward to conclude that

$$\lim_{N \to \infty} \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{D}u_N \, da = \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{D}u \, da,$$

for any $\boldsymbol{\phi} \in C_0^\infty$. Therefore, we have

$$\boldsymbol{D}u_N \rightharpoonup \boldsymbol{D}u$$

in L^p, $p \geq 1$, as $N \to \infty$. Consequently, $u_N \rightharpoonup u$ in $W^{1,p}(\Omega)$, $p \geq 1$, as $N \to \infty$.

(iii) It is clear from the above remarks that, for any $\boldsymbol{x} \in \Omega \backslash (\Omega_m \cup \Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2)))$ and any fixed N,

$$\boldsymbol{D}u_N(\boldsymbol{x}) = \boldsymbol{D}u(\boldsymbol{x}).$$

To see that (68) holds, we need merely to show that

$$\boldsymbol{D}u(\boldsymbol{x}) = -[\lambda(\boldsymbol{x})\kappa_1 + (1 - \lambda(\boldsymbol{x}))\kappa_2]\boldsymbol{e}(\boldsymbol{x}) \tag{70}$$

for any $\boldsymbol{x} \in \overset{o}{\Omega}_m$. Then, the conclusion follows immediately from (ii) of this theorem.

Let $\boldsymbol{x} \in \overset{o}{\Omega}_m$ be given. Then, for any fixed N, there is an integer $i \in [1, \ldots, N]$ so that $\boldsymbol{x} \in \Omega_m^i$. In $\overset{o}{\Omega}_m$ the contour curves are straight lines. Now, consider an integral curve $\boldsymbol{x}(\rho)$, $\rho \in [0, \rho']$, of the normal field $\boldsymbol{e}(\boldsymbol{x}(\rho))$ to these contour curves, that passes through $\boldsymbol{x}$ and is such that

$$u(\boldsymbol{x}(0)) = u_i \quad \text{and} \quad u(\boldsymbol{x}(\rho')) = u_{i-1}.$$

When N is sufficiently large, if we let s_i and $\boldsymbol{t}_i$ be defined as in (47) and (46), then this integral curve is an approximately straight line and we may write

$$\boldsymbol{x}(\rho') \simeq \boldsymbol{x}(0) + L(\boldsymbol{x})\boldsymbol{e}_i = \boldsymbol{y}_{i-1} + s_{i-1}\boldsymbol{t}_{i-1},$$

where

$$\boldsymbol{x}(0) = \boldsymbol{y}_i + s_i\boldsymbol{t}_i.$$

Thus,

$$L(\boldsymbol{x}) = (\boldsymbol{y}_{i-1} - \boldsymbol{y}_i) \cdot \boldsymbol{e}_i + s_{i-1}\boldsymbol{t}_{i-1} \cdot \boldsymbol{e}_i. \tag{71}$$

Because of the regularity properties of the solution to *Problem P2*, when N is sufficiently large, it follows that

$$\boldsymbol{D}u(\boldsymbol{x}(\rho)) \simeq \boldsymbol{D}u(\boldsymbol{x})$$

for $\rho \in (0, \rho')$, and we may write

$$u_{i-1} - u_i = \int_0^{\rho'} \boldsymbol{D}u(\boldsymbol{x}(\rho)) \cdot (d\boldsymbol{x}(\rho)/d\rho)\, d\rho \simeq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{e}_i L(\boldsymbol{x}). \tag{72}$$

On the other hand, from (53), we have

$$u_{i-1} - u_i = -(\kappa_1\tau(s_i) + \kappa_2\sigma(s_i)). \tag{73}$$

Comparing (72) and (73), we obtain

$$\boldsymbol{D}u(\boldsymbol{x}) \simeq -\frac{\kappa_1\tau(s_i) + \kappa_2\sigma(s_i)}{L(\boldsymbol{x})}\boldsymbol{e}_i \tag{74}$$

for sufficiently large N.

From (52), we observe that (71) has the equivalent expression

$$L(\boldsymbol{x}) = \tau(s_i) + \sigma(s_i)\boldsymbol{e}_i \cdot \boldsymbol{e}_{i-1}. \tag{75}$$

Let us assume, for the moment, that the following limit is finite, and define

$$\lambda(\boldsymbol{x}) \equiv \lim_{N\to\infty} \frac{\tau(s_i)}{L(\boldsymbol{x})}. \tag{76}$$

Then, because $\boldsymbol{e}_i \cdot \boldsymbol{e}_{i-1} \to 1$ and $\boldsymbol{e}_i \to \boldsymbol{e}(\boldsymbol{x})$ as $N \to \infty$, we find that (74) implies

$$\boldsymbol{D}u(\boldsymbol{x}) \to -[\lambda(\boldsymbol{x})\kappa_1 + (1 - \lambda(\boldsymbol{x}))\kappa_2]\boldsymbol{e}(\boldsymbol{x}). \tag{77}$$

Now, to determine $\lambda(\boldsymbol{x})$, first note from Figure 9 that when N is sufficiently large,

$$\Delta u \simeq \kappa_1 L_i \tag{78}$$

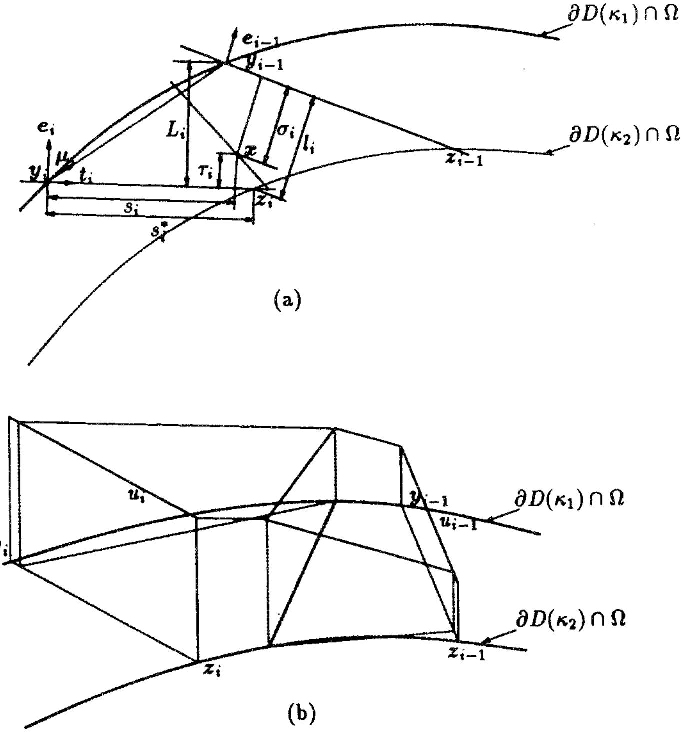


Figure 9
Illustration of the construction of a minimizing sequence in Ω_m.

on $\partial D(\kappa_1) \cap \Omega$, where

$$L_i = |y_{i-1} - y_i| e_i \cdot \mu_i, \tag{79}$$

$$\mu_i \equiv \frac{y_{i-1} - y_i}{|y_{i-1} - y_i|},$$

and

$$\Delta u \simeq \kappa_2 l_i$$

on $\partial D(\kappa_2) \cap \Omega$. Then, for sufficiently large N we have

$$l_i \simeq \frac{\kappa_1}{\kappa_2} L_i. \tag{80}$$

From Figure 9, we also observe that for each N,

$$e_{i-1} \cdot t_i = -e_i \cdot t_{i-1} = \frac{L_i - l_i / e_i \cdot e_{i-1}}{\tilde{s}_i^*}, \tag{81}$$

where

$$\tilde{s}_i^* \equiv |(y_{i-1} - z_i) \cdot t_i|,$$

and $\tilde{s}_i^* \to s_i^*$ as $N \to \infty$. Now with the aid of (79), (80) and (81), we conclude that

$$e_{i-1} \cdot t_i = -e_i \cdot t_{i-1} \simeq \frac{\kappa_2 - \kappa_1/e_i \cdot e_{i-1}}{\kappa_2 \tilde{s}_i^*} |y_{i-1} - y_i| e_i \cdot \mu_i. \tag{82}$$

Upon substituting (82) and (78) into (54) and (55), and the resulting equations into (75) and (76), we find that

$$\lambda(x) = \lim_{N \to \infty} \frac{\kappa_2(s_i^* - s_i)}{\kappa_2(s_i^* - s_i) + \kappa_1 s_i}. \tag{83}$$

Finally, to obtain (69) from (83) we recall (45) and (46), and we note that for any fixed $x \in \Omega_m$ and any fixed N, there is an integer $i \in [1, N]$ such that $x \in \Omega_m^i$, and that as $N \to \infty$,

$$(x - y_i) \cdot e_i \to 0.$$

Because of this, we find that

$$s_i \to |x - y|, \quad \text{and} \quad (s_i^* - s_i) \to |z - x|,$$

where y and z are as claimed in the theorem. □

Finally, we show that the sequence based upon (63) is a minimizing sequence to *Problem P1* in

Theorem 4. Let $e \neq 0, H \in (H_m, H_M)$, and suppose ε_o and η_o are of order N^{-1} as $N \to \infty$. Let u be the solution to *Problem P2* in $W^{1,p}(\Omega)$, $p = 2$, and let u_N be as given in (63). Then,

$$\lim_{N \to \infty} E[u_N] = \inf_{v \in \mathscr{A}} E[v]. \tag{84}$$

Proof. For convenience, we shall define

$$\Omega_s \equiv \{\Omega \backslash (\Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2)))\}.$$

It is clear, from (51), (59)–(64), and the definition of $\overset{o}{\Omega}_m$ in (24), that $\kappa_N \equiv |Du_N(x)|$ is a point of convexity of $\tilde{\omega}(\kappa)$ for any $x \in \Omega_s$ and for any fixed N. Therefore, we have

$$\tilde{\omega}(|Dv|) \geq \tilde{\omega}(\kappa_N) + \tilde{\omega}'(\kappa_N)(|Dv| - \kappa_N) \tag{85}$$

for any $v \in \mathscr{A}$, any $x \in \Omega_s$, and any fixed N. Because $Du_N = -\kappa_N e$ by (18), we observe that

$$\begin{aligned} \tilde{\omega}'(\kappa_N)(|Dv| - \kappa_N) = &\frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} Du_N \cdot (Dv - Du_N) \\ &+ \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} Du_N \cdot (-|Dv| e - Dv). \end{aligned} \tag{86}$$

Since

$$\frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} > 0,$$

and

$$\boldsymbol{D}u_N \cdot (-|\boldsymbol{D}v|e - \boldsymbol{D}v) = \kappa_N |\boldsymbol{D}v| e \cdot \left(e + \frac{\boldsymbol{D}v}{|\boldsymbol{D}v|}\right) \geq 0,$$

we have from (85) and (86) that

$$\tilde{\omega}(|\boldsymbol{D}v|) \geq \tilde{\omega}(\kappa_N) + \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} \boldsymbol{D}u_N \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N),$$

for any $v \in \mathscr{A}$ and, consequently, we obtain

$$\int_{\Omega_s} \tilde{\omega}(|\boldsymbol{D}v|)\, da \geq \int_{\Omega_s} \left[\tilde{\omega}(\kappa_N) + \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} \boldsymbol{D}u_N \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N)\right] da \tag{87}$$

for any $v \in \mathscr{A}$ and any fixed N. Note that for any fixed N, $meas\{\Omega_{\varepsilon_o}(\partial D(\kappa_1)) \cup \Omega_{\eta_o}(\partial D(\kappa_2))\} > 0$ and $\tilde{\omega}(\,\cdot\,) > 0$. Thus, we have

$$\int_{\Omega_s} \tilde{\omega}(|\boldsymbol{D}v|)\, da \leq \int_{\Omega} \tilde{\omega}(|\boldsymbol{D}v|)\, da = E[v].$$

We now show that as $N \to \infty$,

$$\int_{\Omega_s} \left[\tilde{\omega}(\kappa_N) + \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} \boldsymbol{D}u_N \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N)\right] da \to \lim_{N \to \infty} E[u_N],$$

which, with (87), then implies the inequality

$$E[v] \geq \lim_{N \to \infty} E[u_N],$$

for all $v \in \mathscr{A}$, a main element in the proof of (84). Because

$$E[u_N] \equiv \int_{\Omega_s} \tilde{\omega}(\kappa_N)\, da + \int_{\Omega \backslash \Omega_s} \tilde{\omega}(\kappa_N)\, da,$$

it suffices to show that

$$\int_{\Omega_s} \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} \boldsymbol{D}u_N \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N)\, da \to 0 \tag{88}$$

and

$$\int_{\Omega \backslash \Omega_s} \tilde{\omega}(\kappa_N)\, da \to 0 \tag{89}$$

as $N \to \infty$.

Note that

$$meas\{\Omega \backslash \Omega_s\} = O(N^{-1}) \tag{90}$$

for sufficiently large N because ε_o and η_o are of order N^{-1}. Thus, (88) and (89) will hold if the integrands are bounded for all N. It is clear, from (63) and (64), that u_N and $\boldsymbol{D}u_N$ are indeed bounded for any $\boldsymbol{x} \in \Omega_s$ as $N \to \infty$. We shall show now that $\boldsymbol{D}u_N$ is also bounded in $\Omega \backslash \Omega_s$.

Let $\boldsymbol{y}(\theta)$ and $\boldsymbol{z}(\theta)$ be parametric representations of $\partial D(\kappa_1) \cap \Omega$ and $\partial D(\kappa_2) \cap \Omega$, respectively, with $\theta \equiv \arctan(x_2/(x_1 - e))$. We shall denote by $(a, b) \subset [0, 2\pi)$ the range of θ. When $\partial D(\kappa_1) \cap \partial\Omega = \emptyset$, $(a, b) = [0, 2\pi)$. Otherwise, (a, b) is an open subset of $[0, 2\pi)$.

First, for a given $\boldsymbol{x} \in \Omega_{\varepsilon_o}(\partial D(\kappa_1))$, there exist a corresponding $\theta \in (a, b)$ and $\alpha \in (0, 1)$ so that

$$\boldsymbol{x} = \boldsymbol{x}(\alpha, \theta) = \boldsymbol{y}(\theta) + \alpha\varepsilon_o \boldsymbol{v}(\boldsymbol{y}(\theta)), \tag{91}$$

where $\boldsymbol{v}(\boldsymbol{y})$ is the unit outer normal to $D(\kappa_1)$ at $\boldsymbol{y}$. This is true because, by definition, the distance, $d(\boldsymbol{x}, \partial D(\kappa_1))$, from $\boldsymbol{x}$ to $\partial D(\kappa_1)$ is smaller than ε_0, and because the boundary $\partial D(\kappa_1) \cap \Omega$ is continuously differentiable. When ε_o is sufficiently small, the differentiability of the solution u to *Problem P2* shows that

$$u(\boldsymbol{x}(\alpha, \theta)) = u(\boldsymbol{y}(\theta)) + \alpha\varepsilon_o \boldsymbol{D}u(\boldsymbol{y}(\theta)) \cdot \boldsymbol{v}(\boldsymbol{y}(\theta)) + O(\varepsilon_o^2)$$

for any $\theta \in (a, b)$ and any $\alpha \in (0, 1)$. Thus, since $\varepsilon_o = O(N^{-1})$, we obtain

$$|u(\boldsymbol{x}(\alpha, \theta)) - u(\boldsymbol{y}(\theta))| = O(N^{-1})$$

for any $\theta \in (a, b)$ and any $\alpha \in (0, 1)$ as $N \to \infty$. On the other hand, by the construction of (50), we have

$$|\tilde{u}_N(\boldsymbol{x}(\alpha, \theta)) - u(\boldsymbol{y}(\theta))| = O(\Delta u) = O(N^{-1})$$

for any $\theta \in (a, b)$ and $\alpha \in (0, 1)$. Consequently, we obtain

$$\begin{aligned} |u(\boldsymbol{x}(\alpha, \theta)) - \tilde{u}_N(\boldsymbol{x}(\alpha, \theta))| &\leq |u(\boldsymbol{x}(\alpha, \theta)) - u(\boldsymbol{y}(\theta))| + |\tilde{u}_N(\boldsymbol{x}(\alpha, \theta)) - u(\boldsymbol{y}(\theta))| \\ &= O(N^{-1}) \end{aligned} \tag{92}$$

for any $\theta \in (a, b)$ and any $\alpha \in (0, 1)$.

Upon employing a similar argument to the subdomain $\Omega_{\eta_o}(\partial D(\kappa_2))$, we show again that, for a given $\boldsymbol{x} \in \Omega_{\eta_o}(\partial D(\kappa_2))$, there exist a corresponding $\theta \in (a, b)$ and $\beta \in (0, 1)$ such that

$$\boldsymbol{x} = \boldsymbol{x}(\beta, \theta) = \boldsymbol{z}(\theta) - \beta\eta_o \boldsymbol{v}(\boldsymbol{z}(\theta)),$$

where $\boldsymbol{v}(\boldsymbol{z})$ is the unit outer normal to $D(\kappa_2)$ at $\boldsymbol{z}$, and

$$|u(\boldsymbol{x}(\beta, \theta)) - \tilde{u}_N(\boldsymbol{x}(\beta, \theta))| = O(N^{-1}). \tag{93}$$

From (64), we readily have

$$|\boldsymbol{D}u_N(\boldsymbol{x})| < |f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1)) + f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))| |\boldsymbol{D}u(\boldsymbol{x})|$$
$$+ |1 - f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1)) - f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))| |\boldsymbol{D}\tilde{u}_N(\boldsymbol{x})|$$
$$+ |\boldsymbol{D}f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1)) + \boldsymbol{D}f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))| |u(\boldsymbol{x}) - \tilde{u}_N(\boldsymbol{x})|$$

for any $\boldsymbol{x} \in \Omega \backslash \Omega_s$, and it is clear from (58), (61), and (62) that

$$|\boldsymbol{D}f_1(\boldsymbol{x}; \varepsilon_o, \partial D(\kappa_1)) + \boldsymbol{D}f_2(\boldsymbol{x}; \eta_o, \partial D(\kappa_2))| = O(N) \tag{94}$$

as $N \to \infty$. Thus, because $|\boldsymbol{D}u|$, $|\boldsymbol{D}\tilde{u}_N|$, f_1, and f_2 are bounded, and (92), (93) and (94) imply

$$|\boldsymbol{D}f_1 + \boldsymbol{D}f_2| |u - \tilde{u}_N| < \infty$$

as $N \to \infty$, we conclude that

$$|\boldsymbol{D}u_N| < \infty \tag{95}$$

as $N \to \infty$. Consequently, $\tilde{\omega}(\kappa_N)$ and $\tilde{\omega}'(\kappa_N)$ are bounded as $N \to \infty$, and because of (90) we see that (89) holds.

To show (88), we first recall that $\kappa_N(\boldsymbol{x})$ is a point of convexity of $\tilde{\omega}(\,\cdot\,)$ for every $\boldsymbol{x} \in \Omega_s$ and any fixed N, with $\kappa_N(\boldsymbol{x}) = |\boldsymbol{D}u(\boldsymbol{x})|$ in $\Omega_s \backslash \Omega_m$, and $\kappa_N(\boldsymbol{x})$ equal either κ_1 or κ_2 if $\boldsymbol{x} \in \Omega_m$. Hence, $\tilde{\omega}'(\kappa_N(\boldsymbol{x})) = \tilde{\omega}^{*\prime}(|\boldsymbol{D}u(\boldsymbol{x})|)$ for every $\boldsymbol{x} \in \Omega_s$ and for any fixed N. We further observe that

$$\lim_{N \to \infty} \frac{\boldsymbol{D}u_N(\boldsymbol{x})}{\kappa_N(\boldsymbol{x})} = -\boldsymbol{e}(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \Omega$, where $\boldsymbol{e}(\,\cdot\,)$ is the unit normal field of (18). Since $\Omega_s \to \Omega$ and $\tilde{\omega}'(\kappa_N)$ is bounded as $N \to \infty$, the order of integration and limit may be interchanged to obtain

$$\lim_{N \to \infty} \int_{\Omega_s} \frac{\tilde{\omega}'(\kappa_N)}{\kappa_N} \boldsymbol{D}u_N \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N)\, da$$
$$= -\int_{\Omega} \lim_{N \to \infty} \tilde{\omega}^{*\prime}(|\boldsymbol{D}u|)\boldsymbol{e} \cdot (\boldsymbol{D}v - \boldsymbol{D}u_N)\, da.$$

After further interchanging the order of integration and limit, we see that the right hand side of the above equation becomes

$$\lim_{N \to \infty} \left\{ \int_{\Omega} \boldsymbol{D} \cdot (\tilde{\omega}^{*\prime}(|\boldsymbol{D}u|)\boldsymbol{e})(v - u_N)\, da - \int_{\Omega} \boldsymbol{D} \cdot (\tilde{\omega}^{*\prime}(|\boldsymbol{D}u|)\boldsymbol{e}(v - u_N))\, da \right\}.$$

Because $u_N \to u$ in $L^p, p = 2$, pointwise, upon employing the Divergence Theorem and the fact that both u and v are in the admissible class

$\mathscr{A}$, we have

$$\lim_{N\to\infty}\int_{\Omega_s}\frac{\tilde{\omega}'(\kappa_N)}{\kappa_N}\boldsymbol{D}u_N\cdot(\boldsymbol{D}v-\boldsymbol{D}u_N)\,da=\int_{\Omega}\boldsymbol{D}\cdot(\tilde{\omega}^{*\prime}(|\boldsymbol{D}u|)\boldsymbol{e})(v-u)\,da. \tag{96}$$

The right hand side of (96) vanishes because, as noted earlier after (16),

$$\boldsymbol{D}\cdot(\tilde{\omega}^{*\prime}(|\boldsymbol{D}u|)\boldsymbol{e})=0$$

in Ω is a necessary condition for u to be the solution to *Problem P2.* With this, we have shown that (88) holds.

Moreover, if we let $\mu(\kappa_1)\equiv meas\{\boldsymbol{x}\in\Omega_m:|\boldsymbol{D}u_N|=\kappa_1\}$ and $\mu(\kappa_2)\equiv meas\{\boldsymbol{x}\in\Omega_m:|\boldsymbol{D}u_N|=\kappa_2\}$, then it is clear that

$$\lim_{N\to\infty}\int_{\Omega_m}\tilde{\omega}(|\boldsymbol{D}u_N|)\,d\boldsymbol{x}=\tilde{\omega}(\kappa_1)\mu(\kappa_1)+\tilde{\omega}(\kappa_2)\mu(\kappa_2). \tag{97}$$

From Theorem 3 it readily follows that the right hand side of (97) may be rewritten so as to obtain

$$\begin{aligned}\lim_{N\to\infty}\int_{\Omega_m}\tilde{\omega}(|\boldsymbol{D}u_N|)\,da&=\int_{\Omega_m}(\tilde{\omega}(\kappa_1)\lambda(\boldsymbol{x})-\tilde{\omega}(\kappa_1)(1-\lambda(\boldsymbol{x})))\,da\\&=\int_{\Omega_m}\tilde{\omega}^*(\lambda(\boldsymbol{x})\kappa_1+(1-\lambda(\boldsymbol{x}))\kappa_2)\,da.\end{aligned} \tag{98}$$

We further observe that, with the aid of (89),

$$\lim_{N\to\infty}\int_{\Omega}\tilde{\omega}(|\boldsymbol{D}u_N|)\,da=\lim_{N\to\infty}\int_{\Omega_s\backslash\Omega_m}\tilde{\omega}(|\boldsymbol{D}u_N|)\,da+\lim_{N\to\infty}\int_{\Omega_m}\tilde{\omega}(|\boldsymbol{D}u_N|)\,da, \tag{99}$$

and that, for any fixed N and any points $\boldsymbol{x}\in\Omega_s\backslash\Omega_m$,

$$\tilde{\omega}(|\boldsymbol{D}u_N(\boldsymbol{x})|)=\tilde{\omega}^*(|\boldsymbol{D}u_N(\boldsymbol{x})|).$$

Consequently, we conclude from Theorem 3, (98), and (99) that

$$\lim_{N\to\infty}\int_{\Omega}\tilde{\omega}(|\boldsymbol{D}u_N|)\,da=\int_{\Omega}\tilde{\omega}^*(|\boldsymbol{D}u|)\,da=E^*[u]. \tag{100}$$

Because u is the solution to *Problem P2,*

$$E^*[u]=\min_{v\in\mathscr{A}}E^*[v].$$

The proof of (84) is, then, completed once we see that

$$\min_{v\in\mathscr{A}}E^*[v]=\inf_{v\in\mathscr{A}}E[v]. \tag{101}$$

To show this, we call upon Corollary 3.8, Chapter X of [3]. The function $\tilde{\omega}(\cdot)$, here, satisfies the regularity condition of this corollary and

$$\tfrac{1}{2}\mu^{+}\kappa^2 \leq \tilde{\omega}(\kappa) \leq \tfrac{1}{2}\mu^{-}\kappa^2.$$

Therefore, according to this corollary, (101) holds where the admissible class of functions $\mathscr{A}$ is contained in the Sobolev space $W^{1,p}(\Omega)$ with $p = 2$. □

5. Some remarks

We have shown that when the prescribed displacement H is in the interval (H_m, H_M) defined by (21) and (22), *Problem P1* does not possess a minimizer in the admissible class $\mathscr{A}$, but rather the problem has an associated minimizing sequence. The minimizing sequence converges weakly to a deformation of Ω which, in rough terms, consists of two regions of smooth anti-plane shear separated by a domain Ω_m of finely mixed shears. In (50) and (63) we constructed such a sequence of deformations that is piecewise affine and continuous in Ω_m consisting of shear strain layers $\kappa_1/\kappa_2/\kappa_1/\kappa_2, \ldots$, and we showed how to connect each member of this sequence continuously to neighboring regions of smooth deformation by two thin interpolation layers. The sequence (63) is admissible and drives the total stored energy to its infimum as the distribution of shear layers gets finer and finer (cf., Theorem 4). While the thickness of these shear layers vanishes in the limit, there is associated with each point of Ω_m a residual relative density of strain κ_1 and strain κ_2 which characterizes the "average shear strain" at that point and how it is achieved by a convex combination (mixture) of κ_1 and κ_2 (cf., Theorem 3).

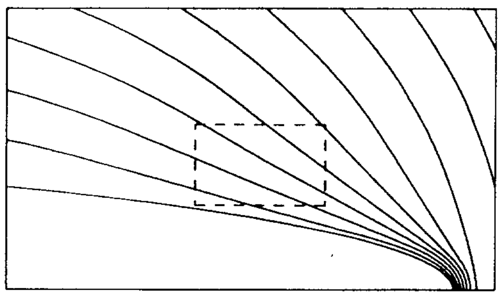
In the domain of phase mixture Ω_m the shear strain, i.e., the weak limit of $|\boldsymbol{D}u_N|$ as $N \to \infty$, varies on an infinitely fine scale. Because it is common to interpret the weak limit of $\tilde{\omega}'(|\boldsymbol{D}u_N|)$ as shear stress in Ω_m, and because $\tilde{\omega}'(|\boldsymbol{D}u_N|)$ has the constant value $\tilde{\omega}'(\kappa_1) = \tilde{\omega}'(\kappa_2)$ for all points in Ω_m, we find that the shear stress is constant in Ω_m. This is reminiscent of general phenomena associated with plasticity.

According to the numerical results presented in Figures 5 and 7 of Section 3, the phase mixture region Ω_m is relatively narrow and long. This being a thin localized region in Ω over which the shear strain suffers a large change from κ_1 on one side to κ_2 on the other (cf., (68)), suggests the idea of a *shear band*, also reminiscent of studies in plasticity.

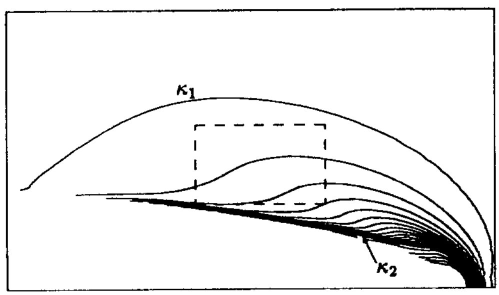
We have considered only a special class of materials (trilinear) in our work so that computation could be performed. We also constructed a minimizing sequence to the nonconvex *Problem P1* based upon the minimizer to the relaxed (convexified) *Problem P2*. We emphasize that when Ω

is a convex ring Bauman and Phillips [2] have shown that there exists a unique minimizer to *Problem P2* for a more general class of materials than those considered here. Also, they have shown that for a certain range of prescribed axial displacements there is a subdomain $\Omega_m \subset \Omega$ in which the minimizer of *Problem P2* takes on shear strain values in the interval (κ_1, κ_2)—that interval which spans the nonconvex portion of the stored energy function. Since, it can be shown that the $u =$ const. contours for a minimizer are again straight lines in Ω_m independent of the material, the construction of a minimizing sequence for *Problem P1* can be carried out the same as we did here for the class of trilinear materials.

In Figure 10, we show a preliminary calculation for the relaxed anti-plane shear *Problem P2* when the outer fixed cylinder is again circular, but the inner concentric cylinder is a relatively thin ellipse of major diameter equal to one-tenth the diameter of the outer circle and ratio of minor to major diameter equal to one-tenth. The material constants of the specific stored energy function (8) are given so that $\kappa_1 = 1$, $\kappa_2 = 10$, $\mu^- = 2$, and $\mu^+ = 0.2$ (see Figures 2 and 3), and the prescribed axial displacement of the inner cylinder is $H = 0.15$, which is sufficiently large to cause a region Ω_m to



(a) $u =$ const.



(b) $|Du| =$ const.

Figure 10
Contours of $u =$ const. and $|Du| =$ const. near the tip of the inner boundary of the domain Ω that is bounded by a circle and an ellipse. The ratio of the major diameter to the minor diameter of the ellipse is 10, the displacement $H = 0.15$, and $\kappa_1 = 1$ and $\kappa_2 = 10$.

appear, in which the shear strain varies between κ_1 and κ_2. While the computations are not accurate and need to be refined with additional mesh in the neighborhood of the tip of the ellipse, within the dashed boxes of Figure 10 we believe that the computations are trustworthy. Note that here $|\boldsymbol{D}u| \in (\kappa_1, \kappa_2)$ and the $u = \text{const.}$ contours are straight lines, as we expect. In front of the tip of the ellipse we expect a small region of shear strains greater than κ_2 to occur, and this much is suggested from Figure 10b since the $\kappa_2 = \text{const.}$ curve in this figure detaches from the boundary and does indeed encircle the tip. But we also expect, from theoretical considerations such as Proposition 5, that the $\kappa_1 = \text{const.}$ curve in Figure 10b would meet the $\kappa_2 = \text{const.}$ curve in front of the tip in Ω, on the axis of symmetry. This we do not see in Figure 10, and so a mesh refinement in this vicinity will be essential in order to clearly identify the region Ω_m. Finally, the theoretical considerations of Proposition 6 suggest that in Figure 10b the $\kappa_1 = \text{const.}$ and $\kappa_2 = \text{const.}$ curves should only intersect the boundary of the ellipse at coincident points. It is difficult to see this unequivocally in the figure, but there is a clear tendency for all of the $|\boldsymbol{D}u| = \text{const.}$ curves to become tangent to the inner elliptical boundary, and this is at least consistent with the possibility of coalescence at a common boundary point.

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Summary

We study the detailed structure of the deformed configuration of an elastic tube whose cross section is a convex ring that is subjected to a prescribed relative axial displacement of its lateral boundaries. The material is assumed to have a non-convex stored energy function. Special attention is paid to the situation when there is no minimizer of the associated anti-plane shear minimization problem, but, nevertheless, the energy functional has an infimum. The non-existence of a minimizer to this problem for a certain interval of prescribed relative axial displacement of the lateral boundaries implies that among all "admissible" deformations there is none with this boundary data for which the values of the stored energy function correspond to its convex points almost everywhere in the body. Because of this, we find that to reach the infimum the tube divides into three subdomains: one of high strain, one of low strain, and one of intermediate "mixed" strain. In the intermediate "mixed" strain subdomain, the field values of the stored energy correspond to convex combinations of convex, but not strictly convex, points of the stored energy function. The main variational problem then gives rise to a free boundary problem in which the subdomain where the strict convexity of the stored energy function breaks down must be determined as part of the solution. The characterization of this intermediate phase mixture region is one of the goals of this work.

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