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Minimal Affinizations of Representations of Quantum Groups: the Irregular Case

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Abstract. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $U_q(\mathfrak{g})$ the associated quantum group (q is a nonzero complex number which we assume is transcendental). If V is a finite-dimensional irreducible representation of $U_q(\mathfrak{g})$, an affinization of V is an irreducible representation $\hat{V}$ of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ which contains V with multiplicity one and is such that all other irreducible $U_q(\mathfrak{g})$-components of $\hat{V}$ have highest weight strictly smaller than the highest weight λ of V. There is a natural partial order on the set of $U_q(\mathfrak{g})$-isomorphism classes of affinizations, and we look for the minimal one(s). In earlier papers, we showed that (i) if $\mathfrak{g}$ is of type A, B, C, F or G, the minimal affinization is unique up to $U_q(\mathfrak{g})$-isomorphism; (ii) if $\mathfrak{g}$ is of type D or E and λ is not orthogonal to the triple node of the Dynkin diagram of $\mathfrak{g}$, there are either one or three minimal affinizations (depending on λ). In this paper, we show, in contrast to the regular case, that if $U_q(\mathfrak{g})$ is of type D_4 and λ is orthogonal to the triple node, the number of minimal affinizations has no upper bound independent of λ.

As a by-product of our methods, we disprove a conjecture according to which, if $\mathfrak{g}$ is of type A_n, every affinization is isomorphic to a tensor product of representations of $U_q(\hat{\mathfrak{g}})$ which are irreducible under $U_q(\mathfrak{g})$ (in an earlier paper, we proved this conjecture when $n = 1$).

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Introduction

In [2], we defined the notion of an affinization of a finite-dimensional irreducible representation V of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra and $q \in \mathbb{C}^\times$ is transcendental. An affinization of V is an irreducible representation $\hat{V}$ of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ which, regarded as a representation of $U_q(\mathfrak{g})$, contains V with multiplicity one, and is such that all other irreducible components of $\hat{V}$ have strictly smaller highest weight than that of V. We say that two affinizations are equivalent if they are isomorphic as representations of $U_q(\mathfrak{g})$. We refer the reader to the introduction to [2] for a discussion of the significance of the notion of an affinization.

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In general, a given representation V has finitely many equivalence classes of affinizations (always at least one), and it is natural to ask if there is a canonical affinization. To this end, a natural partial ordering on the set of equivalence classes of affinizations was defined in [2], and it was proposed to study the minimal affinizations. In [2], [4] and [7], it was shown that, if $\mathfrak{g}$ is of type A, B, C, F or G, the minimal affinization is unique up to equivalence. In [6], we considered the case when $\mathfrak{g}$ is of type D or E, and determined the minimal affinizations of those representations V whose highest weight λ is not orthogonal to the simple root corresponding to the 'triple' node of the Dynkin diagram of $\mathfrak{g}$. We found that, under these assumptions, there are three minimal affinizations of V, except when λ is orthogonal to all the simple roots in a 'leg' of the Dynkin diagram, in which case there is only one. In this Letter, we remove this regularity assumption when $\mathfrak{g}$ is of type D_4. We find the surprising result that, if λ is orthogonal to the triple node, the number of minimal affinizations increases with λ (roughly speaking), and has no upper bound independent of λ.

Although the techniques used in the proof of this result are similar to those used in [2], [4], [6] and [7], there is one new feature. In [6] and [7], the crucial step in the classification of the minimal affinizations is to prove that such affinizations remain minimal on restriction to certain type A subdiagrams of the Dynkin diagram of $\mathfrak{g}$ (and to the type B_2 subdiagram in the nonsimply-laced case). The classification is then deduced from that for types A and B_2 proved in [4] and [2], respectively. In the situation considered in this Letter, however, it turns out that there are minimal affinizations of representations V of $U_q(D_4)$ which are not minimal for any of the type A_3 subdiagrams of D_4. This makes it necessary to understand the structure of certain nonminimal affinizations in the A_n case. The main tool used here is the trigonometric R-matrix associated to a pair of representations of $U_q(\hat{A}_n)$. (We thank Gustav Delius for computing the R-matrix we need – see Lemma 3.5.)

As a by-product of this more detailed study of the affinizations of representations of $U_q(A_n)$, we are able to disprove a conjecture made in [4]. In [3], we showed that when $\mathfrak{g}$ is of type A_1, every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ is isomorphic to a tensor product of small representations (i.e. representations of $U_q(\hat{\mathfrak{g}})$ which are irreducible under $U_q(\mathfrak{g})$), and we conjectured that this might extend to type A algebras of higher rank. However, we show in this Letter that, when $\mathfrak{g}$ is of type A_2, the 27-dimensional irreducible representation of $U_q(\mathfrak{g})$ has an affinization of dimension 35. This cannot be a tensor product of small representations, because $U_q(\mathfrak{g})$ has no irreducible representation of dimension 5 or 7.

1. Quantum Affine Algebras and their Representations

In this section, we collect the results about quantum affine algebras which we shall need later.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and symmetric Cartan matrix $A = (a_{ij})_{i,j\in I}$. For any $i \in I$, define $\lambda_i \in \mathbb{Z}^I$ by

$\lambda_i(j) = \delta_{ij}$ for all $j \in I$. Let $P = \sum_{i\in I}\mathbb{Z}.\lambda_i$ and $P^+ = \sum_{i\in I}\mathbb{N}.\lambda_i$. Let R (resp. R^+) be the set of roots (resp. positive roots) of $\mathfrak{g}$. Let α_i ($i \in I$) be the simple roots and let θ be the highest root. Define a nondegenerate symmetric bilinear form (,) on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = a_{ij}$. Let $Q = \oplus_{i\in I}\mathbb{Z}.\alpha_i \subset \mathfrak{h}^*$ be the root lattice, and set $Q^+ = \sum_{i\in I}\mathbb{N}.\alpha_i$. If $\emptyset \neq J \subseteq I$, let $Q_J = \oplus_{i\in J}\mathbb{Z}.\alpha_i \subset \mathfrak{h}^*$ and $Q_J^+ = \sum_{i\in J}\mathbb{N}.\alpha_i$. Define a partial order $\geqslant$ on P by $\lambda \geqslant \mu$ iff $\lambda - \mu \in Q^+$.

In this Letter, we shall be concerned mainly with the case when $\mathfrak{g}$ is of type D_4. We take $I = \{1, 2, 3, 4\}$, with 4 being the 'triple node', so that the Cartan matrix is

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.$$

Let $q \in \mathbb{C}^\times$ be transcendental and, for $r, n \in \mathbb{N}, n \geqslant r$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q,$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

PROPOSITION 1.1. *There is a Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ which is generated as an algebra by elements $x_i^\pm$, $k_i^{\pm1}$ ($i \in I$), with the following defining relations:*

$$k_ik_i^{-1} = k_i^{-1}k_i = 1, \quad k_ik_j = k_jk_i,$$
$$k_ix_j^\pm k_i^{-1} = q^{\pm a_{ij}}x_j^\pm,$$
$$[x_i^+, x_j^-] = \delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}} \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-a_{ij}-r} = 0, \quad i \neq j.$$

The comultiplication Δ, counit ϵ, and antipode S of $U_q(\mathfrak{g})$ are given by

$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+,$$
$$\Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-,$$
$$\Delta(k_i^{\pm1}) = k_i^{\pm1} \otimes k_i^{\pm1},$$
$$\epsilon(x_i^\pm) = 0, \quad \epsilon(k_i^{\pm1}) = 1,$$
$$S(x_i^+) = -x_i^+k_i^{-1}, \quad S(x_i^-) = -k_ix_i^-, \quad S(k_i^{\pm1}) = k_i^{\mp1},$$

for all $i \in I$. □

The Cartan involution ω of $U_q(\mathfrak{g})$ is the unique algebra automorphism of $U_q(\mathfrak{g})$ which takes $x_i^{\pm} \mapsto -x_i^{\mp}$, $k_i^{\pm 1} \mapsto k_i^{\mp 1}$, for all $i \in I$.

Let $\hat{I} = I \amalg \{0\}$ and let $\hat{A} = (a_{ij})_{i,j\in\hat{I}}$ be the extended Cartan matrix of $\mathfrak{g}$, i.e. the generalized Cartan matrix of the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. When $\mathfrak{g}$ is of type D_4,

$$\hat{A} = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix},$$

with the rows and columns numbered $0, 1, 2, 3, 4$.

THEOREM 1.2. *Let $U_q(\hat{\mathfrak{g}})$ be the algebra with generators $x_i^{\pm}$, $k_i^{\pm 1}$ ($i \in \hat{I}$) and defining relations those in 1.1, but with the indices i, j allowed to be arbitrary elements of $\hat{I}$. Then, $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra with comultiplication, counit and antipode given by the same formulas as in 1.1 (but with $i \in \hat{I}$).*

Moreover, $U_q(\hat{\mathfrak{g}})$ is isomorphic to the algebra $\mathcal{A}_q$ with generators $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z}\backslash\{0\}$) and $c^{\pm 1/2}$, and the following defining relations:

$$c^{\pm 1/2} \text{ are central},$$
$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad c^{1/2}c^{-1/2} = c^{-1/2}c^{1/2} = 1,$$
$$k_i k_j = k_j k_i, \quad k_i h_{j,r} = h_{j,r} k_i,$$
$$k_i x_{j,r}^{\pm} k_i^{-1} = q^{\pm a_{ij}} x_{j,r}^{\pm},$$
$$[h_{i,r}, h_{j,s}] = \delta_{r,-s} \frac{1}{r} [r a_{ij}]_{q_i} \frac{c^r - c^{-r}}{q_j - q_j^{-1}},$$
$$[h_{i,r}, x_{j,s}^{\pm}] = \pm \frac{1}{r} [r a_{ij}]_q c^{\mp |r|/2} x_{j,r+s}^{\pm},$$
$$x_{i,r+1}^{\pm} x_{j,s}^{\pm} - q^{\pm a_{ij}} x_{j,s}^{\pm} x_{i,r+1}^{\pm} = q^{\pm a_{ij}} x_{i,r}^{\pm} x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm} x_{i,r}^{\pm},$$
$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij} \frac{c^{(r-s)/2}\phi_{i,r+s}^{+} - c^{-(r-s)/2}\phi_{i,r+s}^{-}}{q - q^{-1}},$$
$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q x_{i,r_{\pi(1)}}^{\pm} \cdots x_{i,r_{\pi(k)}}^{\pm} x_{j,s}^{\pm} x_{i,r_{\pi(k+1)}}^{\pm} \cdots x_{i,r_{\pi(m)}}^{\pm} = 0, \quad i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters, and the $\phi^{\pm}_{i,r}$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty} \phi^{\pm}_{i,\pm r} u^{\pm r} = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s} \right).$$

If $\theta = \sum_{i \in I} m_i \alpha_i$, set $k_\theta = \prod_{i \in I} k_i^{m_i}$. Suppose that the root vector $\overline{x}^+_\theta$ of $\mathfrak{g}$ corresponding to θ is expressed in terms of the simple root vectors $\overline{x}^+_i$ $(i \in I)$ of $\mathfrak{g}$ as

$$\overline{x}^+_\theta = \lambda[\overline{x}^+_{i_1}, [\overline{x}^+_{i_2}, \ldots, [\overline{x}^+_{i_k}, \overline{x}^+_j] \cdots]]$$

for some $\lambda \in \mathbb{C}^\times$. Define maps $w_i^{\pm} : \mathrm{U}_q(\hat{\mathfrak{g}}) \to \mathrm{U}_q(\hat{\mathfrak{g}})$ by

$$w_i^{\pm}(a) = x^{\pm}_{i,0} a - k_i^{\pm 1} a k_i^{\mp 1} x^{\pm}_{i,0}.$$

Then, the isomorphism $f : \mathrm{U}_q(\hat{\mathfrak{g}}) \to \mathcal{A}_q$ is defined on generators by

$$\begin{aligned} f(k_0) &= k_\theta^{-1}, \qquad f(k_i) = k_i, \qquad f(x_i^{\pm}) = x^{\pm}_{i,0}, \quad (i \in I), \\ f(x_0^+) &= \mu w^-_{i_1} \cdots w^-_{i_k}(x^-_{j,1}) k_\theta^{-1}, \\ f(x_0^-) &= \lambda k_\theta w^+_{i_1} \cdots w^+_{i_k}(x^+_{j,-1}), \end{aligned}$$

where $\mu \in \mathbb{C}^\times$ is determined by the condition

$$[x_0^+, x_0^-] = \frac{k_0 - k_0^{-1}}{q - q^{-1}}.$$ □

See [1], [5] and [11] for further details.

If $\emptyset \neq J \subseteq I$ defines a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$, let $\mathfrak{g}_J$ be the corresponding simple subalgebra of $\mathfrak{g}$, and let $\mathrm{U}_q(\hat{\mathfrak{g}}_J)$ be the subalgebra of $\mathrm{U}_q(\hat{\mathfrak{g}})$ generated by the $x^{\pm}_{i,r}$, $\phi^{\pm}_{i,r}$ and $c^{\pm 1/2}$ for all $i \in J$, $r \in \mathbb{Z}$. Note that there is a canonical homomorphism from the quantum affine algebra associated to $\mathfrak{g}_J$ onto $\mathrm{U}_q(\hat{\mathfrak{g}}_J)$.

Note that there is a canonical homomorphism $\mathrm{U}_q(\mathfrak{g}) \to \mathrm{U}_q(\hat{\mathfrak{g}})$ such that $x_i^{\pm} \mapsto x_i^{\pm}$, $k_i^{\pm 1} \mapsto k_i^{\pm 1}$ for all $i \in I$. Thus, any representation of $\mathrm{U}_q(\hat{\mathfrak{g}})$ may be regarded as a representation of $\mathrm{U}_q(\mathfrak{g})$ by restriction.

Let $\hat{U}^{\pm}$ (resp. $\hat{U}^0$) be the subalgebra of $\mathrm{U}_q(\hat{\mathfrak{g}})$ generated by the $x^{\pm}_{i,r}$ (resp. by the $\phi^{\pm}_{i,r}$) for all $i \in I$, $r \in \mathbb{Z}$. Similarly, let $U^{\pm}$ (resp. U^0) be the subalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by the $x_i^{\pm}$ (resp. by the $k_i^{\pm 1}$) for all $i \in I$.

PROPOSITION 1.3.

(a) $\mathrm{U}_q(\mathfrak{g}) = U^- . U^0 . U^+$.

(b) $U_q(\hat{\mathfrak{g}}) = \hat{U}^- . \hat{U}^0 . \hat{U}^+$. □

See [5] or [11] for details.

We shall make use of the following automorphisms of $U_q(\hat{\mathfrak{g}})$:

PROPOSITION 1.4. (a) *For all $t \in \mathbb{C}^\times$, there exists a Hopf algebra automorphism τ_t of $U_q(\hat{\mathfrak{g}})$ such that*

$$\tau_t(x_{i,r}^{\pm}) = t^r x_{i,r}^{\pm}, \quad \tau_t(h_{i,r}) = t^r h_{i,r},$$
$$\tau_t(k_i^{\pm 1}) = k_i^{\pm 1}, \quad \tau_t(c^{\pm 1/2}) = c^{\pm 1/2}.$$

(b) *There is a unique algebra involution $\hat{\omega}$ of $U_q(\hat{\mathfrak{g}})$ given on generators by*

$$\hat{\omega}(x_{i,r}^{\pm}) = -x_{i,-r}^{\mp}, \quad \hat{\omega}(h_{i,r}) = -h_{i,r},$$
$$\hat{\omega}(\phi_{i,r}^{\pm}) = \phi_{i,-r}^{\mp}, \quad \hat{\omega}(k_i^{\pm 1}) = k_i^{\mp 1},$$
$$\hat{\omega}(c^{\pm 1/2}) = c^{\mp 1/2}.$$

Moreover, we have

$$(\hat{\omega} \otimes \hat{\omega}) \circ \Delta = \Delta^{op} \circ \hat{\omega},$$

where Δ^{op} is the opposite comultiplication of $U_q(\hat{\mathfrak{g}})$. □

See [2] for the proof. Note that $\hat{\omega}$ is compatible, via the canonical homomorphism $U_q(\mathfrak{g}) \to U_q(\hat{\mathfrak{g}})$, with the Cartan involution ω of $U_q(\mathfrak{g})$.

A representation W of $U_q(\mathfrak{g})$ is said to be of type 1 if it is the direct sum of its weight spaces

$$W_\lambda = \{w \in W \mid k_i.w = q^{\lambda(i)} w\} \quad (\lambda \in P).$$

If $W_\lambda \neq 0$, then λ is a weight of W. A vector $w \in W_\lambda$ is a highest weight vector if $x_i^+.w = 0$ for all $i \in I$, and W is a highest weight representation with highest weight λ if $W = U_q(\mathfrak{g}).w$ for some highest weight vector $w \in W_\lambda$.

It is known (see [5] or [11], for example) that every finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ of type 1 is highest weight. Moreover, assigning to such a representation its highest weight defines a bijection between the set of isomorphism classes of finite-dimensional irreducible type 1 representations of $U_q(\mathfrak{g})$ and P^+; the irreducible type 1 representation of $U_q(\mathfrak{g})$ of highest weight $\lambda \in P^+$ is denoted by $V(\lambda)$. Finally, every finite-dimensional representation W of $U_q(\mathfrak{g})$ is completely reducible: if W is of type 1, then

$$W \simeq \bigoplus_{\lambda \in P^+} V(\lambda)^{\oplus m_\lambda(W)}$$

for some uniquely determined multiplicities $m_\lambda(W) \in \mathbb{N}$.

A representation V of $U_q(\hat{\mathfrak{g}})$ is of type 1 if $c^{1/2}$ acts as the identity on V, and if V is of type 1 as a representation of $U_q(\mathfrak{g})$. A vector $v \in V$ is a highest weight vector if

$$x_{i,r}^+ . v = 0, \quad \phi_{i,r}^\pm . v = \Phi_{i,r}^\pm v, \quad c^{1/2} . v = v,$$

for some complex numbers $\Phi_{i,r}^\pm$. A type 1 representation V is a highest weight representation if $V = U_q(\hat{\mathfrak{g}}).v$, for some highest weight vector v, and the pair of $(I \times \mathbb{Z})$-tuples $(\Phi_{i,r}^\pm)_{i\in I, r\in\mathbb{Z}}$ is its highest weight. Note that $\Phi_{i,r}^+ = 0$ (resp. $\Phi_{i,r}^- = 0$) if $r < 0$ (resp. if $r > 0$), and that $\Phi_{i,0}^+\Phi_{i,0}^- = 1$. (In [5], highest weight representations of $U_q(\hat{\mathfrak{g}})$ are called 'pseudo-highest weight'.)

If $\lambda \in P^+$, let $\mathcal{P}^\lambda$ be the set of all I-tuples $(P_i)_{i\in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1, such that $\deg(P_i) = \lambda(i)$ for all $i \in I$. Set $\mathcal{P} = \cup_{\lambda\in P^+}\mathcal{P}^\lambda$.

THEOREM 1.5. (a) *Every finite-dimensional irreducible representation of* $U_q(\hat{\mathfrak{g}})$ *can be obtained from a type 1 representation by twisting with an automorphism of* $U_q(\hat{\mathfrak{g}})$.

(b) *Every finite-dimensional irreducible representation of* $U_q(\hat{\mathfrak{g}})$ *of type 1 is highest weight.*

(c) *Let* V *be a finite-dimensional irreducible representation of* $U_q(\hat{\mathfrak{g}})$ *of type 1 and highest weight* $(\Phi_{i,r}^\pm)_{i\in I, r\in\mathbb{Z}}$. *Then, there exists* $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}$ *such that*

$$\sum_{r=0}^{\infty} \Phi_{i,r}^+ u^r = q^{\deg(P_i)} \frac{P_i(q^{-2}u)}{P_i(u)} = \sum_{r=0}^{\infty} \Phi_{i,r}^- u^{-r},$$

in the sense that the left- and right-hand terms are the Laurent expansions of the middle term about 0 *and* ∞, *respectively. Assigning to* V *the* I*-tuple* $\mathbf{P}$ *defines a bijection between the set of isomorphism classes of finite-dimensional irreducible representations of* $U_q(\hat{\mathfrak{g}})$ *of type 1 and* $\mathcal{P}$. *We denote by* $V(\mathbf{P})$ *the irreducible representation associated to* $\mathbf{P}$.

(d) *Let* $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ *be as above, and let* $v_\mathbf{P}$ *and* $v_\mathbf{Q}$ *be highest weight vectors of* $V(\mathbf{P})$ *and* $V(\mathbf{Q})$, *respectively. Then, in* $V(\mathbf{P})\otimes V(\mathbf{Q})$,

$$x_{i,r}^+ . (v_\mathbf{P} \otimes v_\mathbf{Q}) = 0, \quad \phi_{i,r}^\pm . (v_\mathbf{P} \otimes v_\mathbf{Q}) = \Psi_{i,r}^\pm (v_\mathbf{P} \otimes v_\mathbf{Q}),$$

where the complex numbers $\Psi_{i,r}^\pm$ *are related to the polynomials* P_iQ_i *as the* $\Phi_{i,r}^\pm$ *are related to the* P_i *in part (c). In particular, if* $\mathbf{P}\otimes\mathbf{Q}$ *denotes the* I*-tuple* $(P_iQ_i)_{i\in I}$, *then* $V(\mathbf{P}\otimes\mathbf{Q})$ *is isomorphic to a quotient of the subrepresentation of* $V(\mathbf{P})\otimes V(\mathbf{Q})$ *generated by* $v_\mathbf{P}\otimes v_\mathbf{Q}$. □

See [5] and [8] for further details. If the highest weight $(\Phi_{i,r}^\pm)_{i\in I, r\in\mathbb{Z}}$ of V is given by an I-tuple $\mathbf{P}$ as in part (c), we shall often abuse notation by saying that V has highest weight $\mathbf{P}$.

If σ is any automorphism of $U_q(\hat{\mathfrak{g}})$, and $\rho : U_q(\hat{\mathfrak{g}}) \to \mathrm{End}(V)$ is a representation of $U_q(\hat{\mathfrak{g}})$, we denote by $\sigma^*(V)$ the representation $\rho \circ \sigma$ of $U_q(\hat{\mathfrak{g}})$. Also, we denote by V^* the vector space dual of V provided with the action of $U_q(\hat{\mathfrak{g}})$ given by

$$(x \,.\, f)(v) = f(\rho(S(x))(v)),$$

where $x \in U_q(\hat{\mathfrak{g}})$, $v \in V$, $f \in V^*$, and S is the antipode of $U_q(\hat{\mathfrak{g}})$.

Let w_0 be the longest element of the Weyl group of $\mathfrak{g}$, and let $i \to \bar{i}$ be the bijection $I \to I$ such that $w_0(\alpha_i) = -\alpha_{\bar{i}}$. It is well known that

$$\omega^*(V(\lambda)) \cong V(-w_0(\lambda)) \quad V(\lambda)^* \cong V(-w_0(\lambda)),$$

for all $\lambda \in P^+$. The following results are proved in [2] and [6]:

PROPOSITION 1.6. *Let* $\lambda \in P^+$, $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}^\lambda$, *and let*

$$P_i(u) = \prod_{r=1}^{\lambda(i)}(1 - a_{r,i}^{-1}u) \quad (a_{r,i} \in \mathbb{C}^\times).$$

(a) *For any* $t \in \mathbb{C}^\times$, $\tau_t^*(V(\mathbf{P})) \cong V(\mathbf{P}^t)$, *where* $\mathbf{P}^t = (P_i^t)_{i\in I}$ *and*

$$P_i^t(u) = \prod_{r=1}^{\lambda(i)}(1 - ta_{r,i}^{-1}u).$$

(b) *Define* $\mathbf{P}^{\hat{\omega}} = (P_i^{\hat{\omega}})_{i\in I} \in \mathcal{P}^{-w_0(\lambda)}$ *by*

$$P_{\bar{i}}^{\hat{\omega}}(u) = \prod_{r=1}^{\lambda(i)}(1 - q^2a_{r,i}u).$$

Then, there exists $t \in \mathbb{C}^\times$, *independent of* $i \in I$, *such that*

$$\hat{\omega}^*(V(\mathbf{P})) \cong \tau_t^*(V(\mathbf{P}^{\hat{\omega}}))$$

as representations of $U_q(\hat{\mathfrak{g}})$.

(c) *Define* $\mathbf{P}^* = (P_i^*)_{i\in I} \in \mathcal{P}^{-w_0(\lambda)}$ *by*

$$P_{\bar{i}}^*(u) = \prod_{r=1}^{\lambda(i)}(1 - a_{r,i}^{-1}u).$$

Then, there exists $t^* \in \mathbb{C}^\times$ *such that, as representations of* $U_q(\hat{\mathfrak{g}})$,

$$V(\mathbf{P})^* \cong \tau_{t^*}^*(V(\mathbf{P}^*)).$$

2. Minimal Affinizations

Following [2], we say that a finite-dimensional irreducible representation V of $U_q(\hat{\mathfrak{g}})$ is an affinization of $\lambda \in P^+$ if $V \cong V(\mathbf{P})$ as a representation of $U_q(\hat{\mathfrak{g}})$, for some $\mathbf{P} \in \mathcal{P}^\lambda$. Two affinizations of λ are equivalent if they are isomorphic as representations of $U_q(\mathfrak{g})$; we denote by $[V]$ the equivalence class of V. Let $\mathcal{Q}^\lambda$ be the set of equivalence classes of affinizations of λ.

The following result is proved in [2].

PROPOSITION 2.1. *If $\lambda \in P^+$ and $[V]$, $[W] \in \mathcal{Q}^\lambda$, we write $[V] \preceq [W]$ iff, for all $\mu \in P^+$, either*

(i) $m_\mu(V) \leqslant m_\mu(W)$, *or*
(ii) *there exists $\nu > \mu$ with $m_\nu(V) < m_\nu(W)$.*

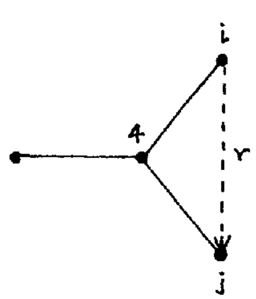
Then, $\preceq$ is a partial order on $\mathcal{Q}^\lambda$. □

An affinization V of λ is minimal if $[V]$ is a minimal element of $\mathcal{Q}^\lambda$ for the partial order $\preceq$, i.e. if $[W] \in \mathcal{Q}^\lambda$ and $[W] \preceq [V]$ implies that $[V] = [W]$. It is proved in [2] that $\mathcal{Q}^\lambda$ is a finite set, so minimal affinizations certainly exist.

The main result of this Letter concerns minimal affinizations of representations of $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is of type D_4. To state it, we introduce the following terminology and notation. First, as in [2], by the q-segment of length $r \in \mathbb{N}$ and centre $a \in \mathbb{C}^\times$, we mean the set of complex numbers $\{aq^{-r+1}, aq^{-r+3}, \ldots, aq^{r-1}\}$. Next, let $\lambda \in P^+$ be such that $\lambda(4) = 0$, and let $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}^\lambda$. Suppose that $i, j \in \{1, 2, 3\}$, $i \neq j$, are such that $\lambda(i) > 0$ and $\lambda(j) > 0$. Then, if the roots of P_i and P_j form q-segments with centres a_i and a_j, and if

$$\frac{a_i}{a_j} = q^{r_i + r_j + 4 - 2r},$$

for some $1 \leqslant r \leqslant \min(r_i, r_j)$, we join nodes i and j of the Dynkin diagram of $\mathfrak{g}$ with a dotted line, as follows:

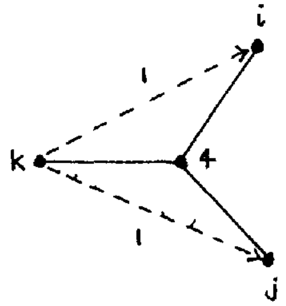


Our main result is the following.

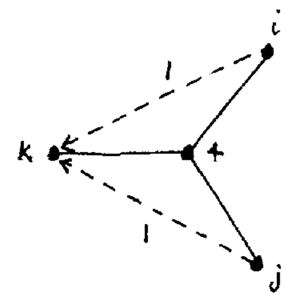
THEOREM 2.2. *Let $\mathfrak{g}$ be of type D_4, let $\lambda \in P^+$ be such that $\lambda(4) = 0$ and $\lambda(i) > 0$ for $i = 1, 2$ and 3, and let $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}^\lambda$.*

If $V(\mathbf{P})$ is a minimal affinization of λ, then, for all $i \in \{1, 2, 3\}$, the roots of P_i form a q-segment with centre a_i (say), where the a_i satisfy one of the following conditions:

(a)$_{i,j}$ *Fix $i, j \in \{1, 2, 3\}$, $i < j$, and let $\{i, j, k\} = \{1, 2, 3\}$. Then, the condition is*

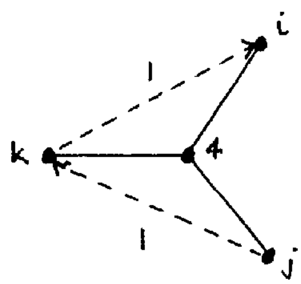


or

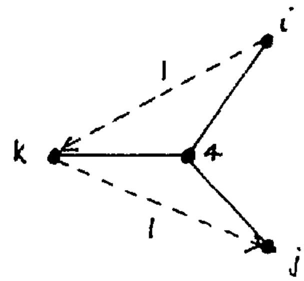


$$i.e.\quad \frac{a_k}{a_i} = q^{\lambda(i)+\lambda(k)+2}, \quad \frac{a_k}{a_j} = q^{\lambda(j)+\lambda(k)+2}, \qquad or \qquad \frac{a_i}{a_k} = q^{\lambda(i)+\lambda(k)+2}, \quad \frac{a_j}{a_k} = q^{\lambda(j)+\lambda(k)+2}.$$

(b)$_{i,j}$ *Fix i, j, k as in* $(\mathrm{a})_{\mathrm{i,j}}$. *Then, the condition is*



or

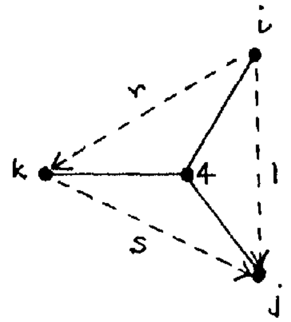


$$i.e.\quad \frac{a_k}{a_i} = q^{\lambda(i)+\lambda(k)+2}, \quad \frac{a_j}{a_k} = q^{\lambda(j)+\lambda(k)+2}, \qquad or \qquad \frac{a_i}{a_k} = q^{\lambda(i)+\lambda(k)+2}, \quad \frac{a_k}{a_j} = q^{\lambda(j)+\lambda(k)+2}.$$

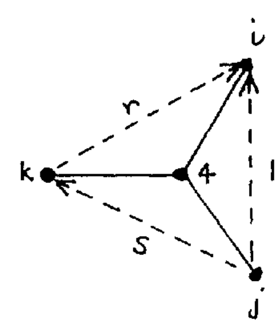
(c)$_{i,j}^{r,s}$ *Fix i, j, k as in* $(\mathrm{a})_{\mathrm{i,j}}$, *and let r, s be integers satisfying*

$$r + s = \lambda(k) + 3, \; 1 \leqslant r \leqslant \min(\lambda(i), \lambda(k)), \; 1 \leqslant s \leqslant \min(\lambda(j), \lambda(k)). \quad (3)$$

Then, the condition is



or



i.e. $\dfrac{a_i}{a_k} = q^{\lambda(i)+\lambda(k)+4-2r}$, $\qquad$ $\dfrac{a_k}{a_i} = q^{\lambda(i)+\lambda(k)+4-2r}$,

$\dfrac{a_k}{a_j} = q^{\lambda(j)+\lambda(k)+4-2s}$, $\qquad$ *or* $\qquad$ $\dfrac{a_j}{a_k} = q^{\lambda(j)+\lambda(k)+4-2s}$,

$\dfrac{a_i}{a_j} = q^{\lambda(i)+\lambda(j)+2}$, $\qquad$ $\dfrac{a_j}{a_i} = q^{\lambda(i)+\lambda(j)+2}$.

Conversely, suppose that the roots of each P_i, $i = 1, 2, 3$, *form a q-segment of centre* a_i. *Then, for each* $i < j$ *in* $\{1, 2, 3\}$,

(d) *at least one of* $(\mathrm{a})_{i,j}$ *and* $(\mathrm{b})_{i,j}$ *defines a minimal affinization, and*

(e) $(\mathrm{c})_{i,j}^{r,s}$ *defines a minimal affinization for all* r, s *satisfying (3).*

Finally, the two diagrams in $(\mathrm{a})_{i,j}$ *(resp.* $(\mathrm{b})_{i,j}$, $(\mathrm{c})_{i,j}^{r,s}$*) define equivalent affinizations; an affinization of type* $(\mathrm{a})_{i,j}$ *(resp.* $(\mathrm{b})_{i,j}$*) is not equivalent to any affinization of any other type, except possibly* $(\mathrm{b})_{i,j}$ *(resp.* $(\mathrm{a})_{i,j}$*); and an affinization of type* $(\mathrm{c})_{i,j}^{r,s}$ *is not equivalent to an affinization of any other type.*

Remarks. (1) We conjecture that the affinization of type $(\mathrm{a})_{i,j}$ is equivalent to that of type $(\mathrm{b})_{i,j}$ (in which case, both types would be minimal).

(2) Each of the diagrams in $(\mathrm{a})_{i,j}$, $(\mathrm{b})_{i,j}$ and $(\mathrm{c})_{i,j}^{r,s}$ actually represent infinitely many nonisomorphic representations of $U_q(\hat{\mathfrak{g}})$. However, it follows from Proposition 1.6 that any two affinizations $V(\mathbf{P})$ satisfying $(\mathrm{a})_{i,j}$ (or any two satisfying $(\mathrm{b})_{i,j}$, or any two satisfying $(\mathrm{c})_{i,j}^{r,s}$) are equivalent.

The next result deals with the case where λ is orthogonal not only to the triple node, but also to one or more of the exterior nodes.

THEOREM 2.3. *Let* $\mathfrak{g}$ *be of type* D_4, *and let* $\lambda \in P^+$ *be such that* $\lambda(4) = 0$.

(a) *Let* $\lambda(i) > 0$, $\lambda(j) > 0$ *and* $\lambda(k) = 0$, *where* $\{i, j, k\} = \{1, 2, 3\}$. *Then, if* $\mathbf{P} \in \mathcal{P}^\lambda$, $V(\mathbf{P})$ *is a minimal affinization of* λ *iff the roots of* P_i *and* P_j *form q-segments with centres* a_i *and* a_j, *where*

$$\frac{a_i}{a_j} = q^{\pm(\lambda(i)+\lambda(j)+2)}.$$

(b) *Let* $\lambda(i) > 0$, $\lambda(j) = 0$ *and* $\lambda(k) = 0$, *where* $\{i, j, k\} = \{1, 2, 3\}$. *Then, if* $\mathbf{P} \in \mathcal{P}^\lambda$, $V(\mathbf{P})$ *is a minimal affinization of* λ *iff the roots of* P_i *form a q-segment.*

The proofs of Theorems 2.2 and 2.3 occupy the next two sections.

3. Affinizations in the Type A Case

The minimal affinizations of representations of $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is of type A were classified in [4] (see Theorem 3.1 of that paper). We recall the result.

THEOREM 3.1. *Let* $\mathfrak{g} = \mathrm{sl}_{n+1}(\mathbb{C})$, *and let* $\lambda \in P^+$. *Then,* $\mathcal{Q}^\lambda$ *has a unique minimal element. Moreover, this element is represented by* $V(\mathbf{P})$, *for* $\mathbf{P} \in \mathcal{P}^\lambda$, *if and only if, for all* $i \in I$ *such that* $\lambda(i) > 0$, *the roots of* P_i *form the q-segment with centre* a_i, *for some* $a_i \in \mathbb{C}^\times$, *and length* $\lambda(i)$, *where either*

(a) *for all* $i < j$, *such that* $\lambda(i) > 0$ *and* $\lambda(j) > 0$,

$$\frac{a_i}{a_j} = q^{\lambda(i)+2(\lambda(i+1)+\cdots+\lambda(j-1))+\lambda(j)+j-i},$$

or

(b) *for all* $i < j$, *such that* $\lambda(i) > 0$ *and* $\lambda(j) > 0$,

$$\frac{a_j}{a_i} = q^{\lambda(i)+2(\lambda(i+1)+\cdots+\lambda(j-1))+\lambda(j)+j-i}.$$

In both cases, $V(\mathbf{P}) \cong V(\lambda)$ *as representations of* $U_q(\mathfrak{g})$. □

To apply this result to the case where $\mathfrak{g}$ is of type D_4, we need the next proposition. If $\lambda \in P^+$, $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}^\lambda$, and J is a nonempty subset of I, let $\mathbf{P}_J$ be the J-tuple $(P_j)_{j\in J}$ and let λ_J be the restriction of $\lambda : I \to \mathbb{Z}$ to J. Also, define $\lambda^J \in P^+$ by

$$\lambda^J(i) = \begin{cases} \lambda(i), & \text{if } i \in J, \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathbf{P}^J \in \mathcal{P}$ by

$$P_i^J = \begin{cases} P_i, & \text{if } i \in J, \\ 1, & \text{otherwise.} \end{cases}$$

Finally, we say that $\emptyset \neq J \subseteq \{1, 2, 3, 4\}$ is *admissible* if $J = \{j, 4\}$ for some $j \in \{1, 2, 3\}$.

PROPOSITION 3.2. *Let* $\mathfrak{g}$ *be of type* D_4, *let* $\lambda \in P^+$, *and let* $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}^\lambda$. *If* $V(\mathbf{P})$ *is a minimal affinization of* λ, *then, for all admissible subsets* J *of* $\{1, 2, 3, 4\}$, $V(\mathbf{P}_J)$ *is a minimal affinization of* λ_J *and* $V(\mathbf{P}^J)$ *is a minimal affinization of* λ^J. □

This is a special case of Proposition 4.2 in [6].

From Theorem 3.1 and Proposition 3.2, we immediately deduce

COROLLARY 3.3. *Let* $\mathfrak{g}$ *be of type* D_4, *let* $\lambda \in P^+$, *and let* $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}^\lambda$. *If* $V(\mathbf{P})$ *is a minimal affinization of* λ, *then for all* $i \in \{1,2,3,4\}$, *either* $P_i = 1$ *or the roots of* P_i *form a q-segment.* □

As we mentioned in the Introduction, we need some information about non-minimal affinizations in the type A case:

PROPOSITION 3.4. *Let* $\mathfrak{g}$ *be of type* A_n, *where* $n \geqslant 2$, *let* $r_1 \geqslant r_n$ *be nonnegative integers, and let* $a_1, a_n \in \mathbb{C}^\times$. *Let* P_1 *(resp.* P_n*) be the polynomial with constant coefficient 1 whose roots form the q-segment with centre* a_1 *(resp.* a_n*) and length* r_1 *(resp.* r_n*). Then, as representations of* $\mathrm{U}_q(\mathfrak{g})$,

$$V(P_1, 1, \ldots, 1, P_n) \cong \bigoplus_{t=0}^{s-1} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n),$$

if $a_1/a_n = q^{\pm(r_1+r_n+n+1-2s)}$ *for some* $1 \leqslant s \leqslant r_n$, *and*

$$V(P_1, 1, \ldots, 1, P_n) \cong V(r_1\lambda_1) \otimes V(r_n\lambda_n)$$

otherwise.

Note that, in view of Proposition 1.6, the assumption $r_1 \geqslant r_n$ involves no loss of generality.

Proof. Note first that, by Theorem 3.1, $V(P_1, 1, \ldots, 1)$ and $V(1, \ldots, 1, P_n)$ are minimal affinizations of $r_1\lambda_1$ and $r_n\lambda_n$, and that they are isomorphic as representations of $\mathrm{U}_q(\mathfrak{g})$ to $V(r_1\lambda_1)$ and $V(r_n\lambda_n)$, respectively. Next, by Theorem 1.5(d), $V(P_1, 1, \ldots, 1, P_n)$ is isomorphic as a representation of $\mathrm{U}_q(\hat{\mathfrak{g}})$ to a subquotient of the tensor product $V(1, \ldots, 1, P_n) \otimes V(P_1, 1, \ldots, 1)$ and, hence, as a representation of $\mathrm{U}_q(\mathfrak{g})$ to a subrepresentation of

$$V(r_n\lambda_n) \otimes V(r_1\lambda_1) \cong \bigoplus_{s=0}^{r_n} V((r_1 - s)\lambda_1 + (r_n - s)\lambda_n). \tag{4}$$

To proceed further, we need the following lemmas.

LEMMA 3.5. *Let* P_1 *and* P_n *be as in Proposition 3.4. Then, there exists a homomorphism of representations of* $\mathrm{U}_q(\hat{\mathfrak{g}})$

$$I\colon V(1, \ldots, 1, P_n) \otimes V(P_1, 1, \ldots, 1) \to V(P_1, 1, \ldots, 1) \otimes V(1, \ldots, 1, P_n)$$

of the form

$$I = \sum_{s=0}^{r_n} \left(\prod_{r=0}^{s-1} \left(1 - \frac{a_1}{a_n} q^{r_1+r_n+n+1-2r}\right) \prod_{r=s}^{r_n-1} \left(\frac{a_1}{a_n} - q^{r_1+r_n+n+1-2r}\right)\right) Pr_s,$$

where, for $s = 0, 1, \ldots, r_n$,

$$Pr_s\colon V(r_n\lambda_n)\otimes V(r_1\lambda_1) \to V(r_1\lambda_1)\otimes V(r_n\lambda_n)$$

is a homomorphism of representations of $\mathrm{U}_q(\mathfrak{g})$ *(independent of a_1 and a_n) whose image is the unique* $\mathrm{U}_q(\mathfrak{g})$*-subrepresentation of* $V(r_1\lambda_1)\otimes V(r_n\lambda_n)$ *isomorphic to* $V((r_1 - s)\lambda_1 + (r_n - s)\lambda_n)$.

Proof. This is proved by using the techniques of [9]. We omit the details. □

From now on, we denote the unique $\mathrm{U}_q(\mathfrak{g})$-subrepresentation of

$$V(1, \ldots, 1, P_n)\otimes V(P_1, 1, \ldots, 1)$$

isomorphic to $V((r_1-s)\lambda_1+(r_n-s)\lambda_n)$ simply by $V((r_1-s)\lambda_1+(r_n-s)\lambda_n)$.

LEMMA 3.6. *Let P_1 and P_n be as in Proposition 3.4. Then, the tensor product*

$$V(1, \ldots, 1, P_n)\otimes V(P_1, 1, \ldots, 1)$$

is reducible as a representation of $\mathrm{U}_q(\hat{\mathfrak{g}})$ *for only finitely many values of the ratio* a_1/a_n.

Proof. For the duration of this proof, we abuse the notation by writing $V(P_1, P_n)$ for $V(1, \ldots, 1, P_n)\otimes V(P_1, 1, \ldots, 1)$.

Note first that, by Proposition 1.6(a), the reducibility or otherwise of $V(P_1, P_n)$ depends only on the ratio a_1/a_n. Note also that, because each irreducible $\mathrm{U}_q(\mathfrak{g})$-subrepresentation of $V(P_1, P_n)$ occurs with multiplicity one (see (4)), $V(P_1, P_n)$ has only finitely many $\mathrm{U}_q(\mathfrak{g})$-subrepresentations. Thus, it suffices to prove that a given $\mathrm{U}_q(\mathfrak{g})$-subrepresentation M of $V(P_1, P_n)$ is actually an irreducible $\mathrm{U}_q(\hat{\mathfrak{g}})$-subrepresentation for at most finitely many values of a_1/a_n.

Let $\{m_1, \ldots, m_s\}$ be a vector space basis of M, and extend it to a basis $\{m_1, \ldots, m_t\}$ of $V(P_1, P_n)$ (these bases being independent of a_1 and a_n). Then, M is a $\mathrm{U}_q(\hat{\mathfrak{g}})$-subrepresentation of $V(1, \ldots, 1, P_n)\otimes V(P_1, 1, \ldots, 1)$ if and only if, for each $1 \leqslant u \leqslant s$, $x_0^+ . m_u$ and $x_0^- . m_u$ are linear combinations of $m_1, \ldots, m_s$. From the form of the isomorphism f in Theorem 1.2, it is clear *a priori* that

$$x_0^+ . m_u = \sum_{v=1}^{t} c_{uv} m_v,$$

where each coefficient c_{uv} is of the form

$$c_{uv} = c'_{uv} a_1^{-1} + c''_{uv} a_n^{-1},$$

with c'_{uv} and c''_{uv} independent of a_1 and a_n (and similarly for $x_0^- . m_u$). There are two possibilities: either

(a) $c'_{uv} = c''_{uv} = 0$ for all $v > s$, or
(b) the condition $c_{uv} = 0$ for all $v > s$ holds for, at most, one value of a_1/a_n

(and similarly for $x_0^- . m_u$). We claim that possibility (a) cannot hold for both x_0^+ and x_0^-. It follows that possibility (b) must hold either for x_0^+ or for x_0^- (or both). But this means that M is a $U_q(\hat{\mathfrak{g}})$-subrepresentation for, at most, one value of a_1/a_n.

To prove the claim, suppose for a contradiction that (a) does hold for both x_0^+ and x_0^-. Then, M is a $U_q(\hat{\mathfrak{g}})$-subrepresentation of $V(P_1, P_n)$ for all values of a_1/a_n. Taking $a_1/a_n = q^{r_1+r_n+n+1-2s}$ $(0 \leqslant s < r_n)$, we have

$$\ker(I) = \bigoplus_{t=0}^{s} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n),$$

and taking $a_1/a_n = q^{-(r_1+r_n+n+1-2s)}$ $(0 \leqslant s < r_n)$, we have

$$\ker(I) = \bigoplus_{t=s+1}^{r_n} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n).$$

Note that these kernels are $U_q(\hat{\mathfrak{g}})$-subrepresentations of $V(P_1, P_n)$ (for the appropriate value of a_1/a_n). Since M is irreducible under $U_q(\hat{\mathfrak{g}})$, it must either be contained in, or intersect trivially, each of these representations. It is clear that this is possible only if M is irreducible under $U_q(\mathfrak{g})$. But, by Theorem 3.1, M cannot be irreducible under $U_q(\mathfrak{g})$ for all values of a_1/a_n. □

We now return to the proof of Proposition 3.4. Taking $a_1/a_n = q^{-(r_1+r_n+n+1-2s)}$, where $0 \leqslant s \leqslant r_n$, we see that the image of I is a representation of $U_q(\hat{\mathfrak{g}})$ isomorphic to

$$\bigoplus_{t=0}^{s} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n)$$

as a representation of $U_q(\mathfrak{g})$. Moreover, by Theorem 1.5(d), the image of I is an affinization of $V(r_1\lambda_1 + r_n\lambda_n)$. In view of Proposition 1.6(b) and (c), to complete the proof of the first part of Proposition 3.4, it suffices to prove that the image of I is irreducible as a representation of $U_q(\hat{\mathfrak{g}})$. We prove this, and the second part of Proposition 3.4, as follows.

Now let w_s^+ (resp. w_s^-) be a $U_q(\mathfrak{g})$-highest (resp. lowest) weight vector in $V((r_1 - s)\lambda_1 + (r_n - s)\lambda_n)$, and let

$$\pi_s \colon V(P_1, 1, \ldots, 1) \otimes V(1, \ldots, 1, P_n) \to V(P_1, 1, \ldots, 1) \otimes V(1, \ldots, 1, P_n)$$

be the projection onto $V((r_1 - s)\lambda_1 + (r_n - s)\lambda_n)$. As in the proof of Lemma 3.6, it is clear that

$$x_0^+ . w_0^+ = a_1^{-1} u_1 + a_n^{-1} u_n,$$

where u_1 and u_n are vectors in $V(P_1, 1, \ldots, 1)\otimes V(1, \ldots, 1, P_n)$ which do not depend on a_1 or a_n. Moreover, since $x_0^+ . w_0^+$ has weight $(r_1 - 1)\lambda_1 + (r_n - 1)\lambda_n$, and since the weight space

$$\begin{aligned}(V(P_1, 1, \ldots, 1)\otimes V(1, \ldots, 1, P_n))_{(r_1-1)\lambda_1+(r_n-1)\lambda_n} \\ = V(r_1\lambda_1 + r_n\lambda_n)_{(r_1-1)\lambda_1+(r_n-1)\lambda_n} \oplus \mathbb{C}.w_1^+,\end{aligned}$$

it follows that

$$\pi_1(x_0^+ . w_0^+) = (A_1 a_1^{-1} + A_n a_n^{-1})w_1^+,$$

where the complex numbers A_1 and A_n do not depend on a_1 or a_n. By the discussion immediately following the proof of Lemma 3.6, we know that this component vanishes if $a_1/a_n = q^{-(r_1+r_n+n+1)}$. It follows that, for any a_1, a_n,

$$\pi_1(x_0^+ . w_0^+) = B_1^+(a_n^{-1}q^{r_1+r_n+n+1} - a_1^{-1})w_1^+,$$

for some $B_1^+ \in \mathbb{C}$ which is independent of a_1 and a_n. Note also that $B_1^+ \neq 0$, for otherwise $V(r_1\lambda_1+r_n\lambda_n)$ would be a $U_q(\hat{\mathfrak{g}})$-subrepresentation of $V(1, \ldots, 1, P_n)\otimes V(P_1, 1, \ldots, 1)$ for every value of a_1 and a_n, contradicting Lemma 3.6.

Similar arguments show that, for all $0 < s \leqslant r_n$,

$$\pi_s(x_0^+ . w_{s-1}^+) = B_s^+(a_n^{-1}q^{r_1+r_n+n+3-2s} - a_1^{-1})w_s^+, \tag{5}$$

and that for all $0 \leqslant s < r_n$,

$$\pi_s(x_0^+ . w_{s-1}^-) = B_s^-(a_1^{-1}q^{r_1+r_n+n+3-2s} - a_n^{-1})w_s^-, \tag{6}$$

where the $B_s^\pm \in \mathbb{C}^\times$ are independent of a_1 and a_n. It follows easily from (5) and (6) that

(i) if $a_1/a_n = q^{-(r_1+r_n+n+1-2s)}$, then

$$\operatorname{im}(I) = \bigoplus_{t=0}^{s} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n)$$

is an irreducible $U_q(\hat{\mathfrak{g}})$-subrepresentation of $V(P_1, 1, \ldots, 1)\otimes V(1, \ldots, 1, P_n)$;

(ii) if $a_1/a_n = q^{r_1+r_n+n+1-2s}$, then

$$\operatorname{coker}(I) \cong \bigoplus_{t=0}^{s} V((r_1 - t)\lambda_1 + (r_n - t)\lambda_n)$$

is an irreducible $U_q(\hat{\mathfrak{g}})$-quotient representation of $V(P_1, 1, \ldots, 1)\otimes V(1, \ldots, 1, P_n)$; and

(iii) if $a_1/a_n \neq q^{\pm(r_1+r_n+n+1-2s)}$ for any $0 \leqslant s \leqslant r_n$, then I is surjective and $V(P_1, 1, \ldots, 1)\otimes V(1, \ldots, 1, P_n)$ is an irreducible representation of $U_q(\hat{\mathfrak{g}})$.

These three statements together establish Proposition 3.4. □

EXAMPLE. We consider the case $n = 2$, $\lambda = 2\lambda_1 + 2\lambda_2$, $a_1/a_2 = q^5$. By Proposition 3.4,

$$V(P_1, P_2) \cong V(2\lambda_1 + 2\lambda_2) \oplus V(\lambda_1 + \lambda_2)$$

as representations of $U_q(\mathfrak{g})$. Thus, $V(P_1, P_2)$ is not small (i.e. it is not irreducible under $U_q(\mathfrak{g})$). On the other hand,

$$\dim(V(2\lambda_1 + 2\lambda_2)) = 27, \quad \dim(V(\lambda_1 + \lambda_2)) = 8,$$

so $\dim(V(P_1, P_2)) = 35$; since $U_q(\mathfrak{g})$ has no irreducible representations of dimensions 5 or 7, $V(P_1, P_2)$ is not isomorphic to a tensor product of small representations.

As we mentioned in the Introduction, this example disproves a conjecture made in [4].

4. Proof of the Main Theorem

In this section, we give the proof of Theorem 2.2. We shall not discuss the proof of Theorem 2.3, since it is similar to, but much easier than, that of 2.2.

Suppose first that $V(\mathbf{P})$ is a minimal affinization of $\lambda \in P^+$, where $\lambda(4) = 0$ and $\lambda(i) > 0$ for $i = 1, 2, 3$. Let U_i, for $i = 1, 2, 3$, be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_j^{\pm}$, $k_j^{\pm 1}$ in Proposition 1.1 for which $j \in \{1, 2, 3, 4\}\backslash\{i\}$. For each i, there is an obvious canonical epimorphism $U_q(A_3) \to U_i$, so that, by restricting to U_i, any representation of $U_q(\mathfrak{g})$ may be regarded as a representation of $U_q(A_3)$.

Assume that

$$\frac{a_1}{a_2} \neq q^{\pm(\lambda(1)+\lambda(2)+4-2s)}$$

for all $1 \leqslant s \leqslant \min(\lambda(1), \lambda(2))$. We show that $V(\mathbf{P}_{\{1,3,4\}})$ and $V(\mathbf{P}_{\{2,3,4\}})$ are irreducible as representations of U_2 and U_1, respectively, so that by Theorem 3.1 we are in case $(\mathrm{a})_{1,2}$ or $(\mathrm{b})_{1,2}$. Suppose then that at least one of $V(\mathbf{P}_{\{1,3,4\}})$ or $V(\mathbf{P}_{\{2,3,4\}})$ is reducible, say $V(\mathbf{P}_{\{1,3,4\}})$. Let $\mathbf{Q} \in \mathcal{P}^\lambda$ be such that, for $i = 1, 2, 3$, the roots of Q_i form a q-segment with centre b_i, where

$$\frac{b_1}{b_3} = q^{\lambda(1)+\lambda(3)+2} \quad \text{and} \quad \frac{b_2}{b_3} = q^{\lambda(2)+\lambda(3)+2}. \tag{7}$$

We claim that $[V(\mathbf{Q})] \prec [V(\mathbf{P})]$. This contradiction to the minimality of $[V(\mathbf{P})]$ will prove our assertion.

Let $m_\mu(V(\mathbf{Q})) > 0$, where $\mu = \lambda - \eta$, $\eta = \sum_{j=1}^4 n_j\alpha_j$, $n_j \in \mathbb{N}$. We need the following lemma from [7] (see Lemma 3.2 in that paper).

LEMMA 4.1. *Let* $\emptyset \neq J \subseteq I$ *define a connected subdiagram of the Dynkin diagram of* $\mathfrak{g}$. *Let* $\lambda \in P^+$, $\mathbf{P} \in \mathcal{P}^\lambda$, *and* $\mu \in \lambda - Q_J^+$. *Then, if* V *is any highest weight representation of* $U_q(\hat{\mathfrak{g}})$ *with highest weight* $\mathbf{P}$ *and highest weight vector* v, *we have* $m_\mu(V) = m_{\mu_J}(V_J)$, *where* $V_J = U_q(\hat{\mathfrak{g}}_J).v_{\mathbf{P}}$. *Moreover, if* V *is irreducible, so is* V_J, *and the highest weight of* V_J *is* $\mathbf{P}_J$. □

Since, by Theorem 3.1, $V(\mathbf{Q}_{\{1,3,4\}})$ and $V(\mathbf{Q}_{\{2,3,4\}})$ are irreducible under U_2 and U_1, respectively, it follows from this lemma that n_1 and n_2 are strictly positive. The next lemma, a special case of Lemma 4.3 in [6], shows that $n_4 > 0$ too.

LEMMA 4.2. *Let* $\lambda \in P^+$, $\mathbf{Q} \in \mathcal{P}^\lambda$, $i \in \{1, 2, 3\}$, *and assume that the roots of* Q_i *form a q-segment. Let* $\mu \in P$ *be of the form* $\mu = \lambda - \sum_{j \in I} n_j \alpha_j$, *where* $n_j \geqslant 0$ *for all* j, *and* $n_4 = 0$. *If* $m_\mu(V(\mathbf{Q})) > 0$, *then* $n_i = 0$. □

Next, since $b_1/b_2 = q^{\lambda(1)-\lambda(2)}$ is clearly not of the form $q^{\pm(\lambda(1)+\lambda(2)+4-2s)}$ for any $1 \leqslant s \leqslant \min(\lambda(1), \lambda(2))$, it follows from Proposition 3.4 that

$$V(\mathbf{Q}_{\{1,2,4\}}) \cong V(\mathbf{P}_{\{1,2,4\}}) \cong V(\lambda(1)\lambda_1) \otimes V(\lambda(2)\lambda_2)$$

as representations of U_3. If $n_3 = 0$, Lemma 4.2 implies that $m_\mu(V(\mathbf{P})) = m_\mu(V(\mathbf{Q}))$ (take $J = \{3\}$). On the other hand, if $n_3 > 0$, taking $\eta' = \alpha_1 + \alpha_3 + \alpha_4$, we have $\lambda - \eta' > \lambda - \eta$ and $m_{\lambda-\eta'}(V(\mathbf{P})) > 0$, because $V(\mathbf{P}_{\{1,3,4\}})$ is not irreducible as a representation of U_2 by assumption, while $m_{\lambda-\eta'}(V(\mathbf{Q})) = 0$, because $V(\mathbf{Q}_{\{1,3,4\}})$ is irreducible as a representation of U_2. This completes the proof that $[V(\mathbf{Q})] \prec [V(\mathbf{P})]$.

We have now shown that, if $\mathbf{P}$ does not satisfy condition (a) or condition (b) in 2.2, then, for all $1 \leqslant i < j \leqslant 3$,

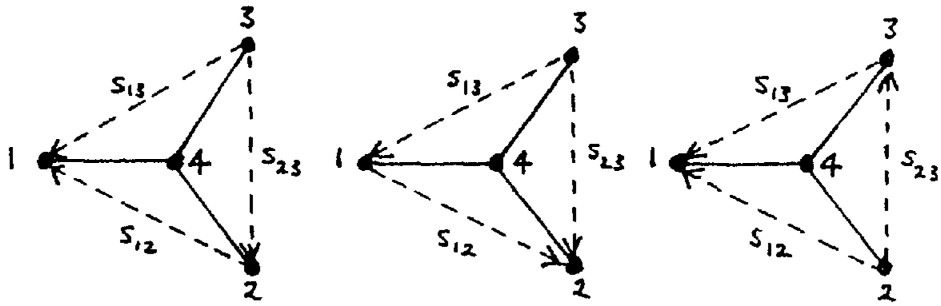
$$\frac{a_i}{a_j} = q^{\pm(\lambda(i)+\lambda(j)+4-2s_{ij})},$$

for some $1 \leqslant s_{ij} \leqslant \min(\lambda(i), \lambda(j))$. It is easy to see that these inequalities and the condition

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_1} = 1$$

are consistent only in the following three cases (together with the other three in which all the arrows are reversed):

I II III



$s_{12}+s_{23}-s_{13}=\lambda(2)+2$ $s_{12}+s_{13}-s_{23}=\lambda(1)+2$ $s_{13}+s_{23}-s_{12}=\lambda(3)+2$

We remind the reader that diagram I, for example, means that

$$\frac{a_2}{a_1}=q^{\lambda(1)+\lambda(2)+4-2s_{12}},\quad \frac{a_3}{a_2}=q^{\lambda(2)+\lambda(3)+4-2s_{23}},$$
$$\frac{a_3}{a_1}=q^{\lambda(1)+\lambda(3)+4-2s_{13}}.$$

We must show that cases I–III can correspond to minimal affinizations only if one of s_{12}, s_{13} and s_{23} is equal to 1.

Consider case I, for example. Note that $s_{12}\geqslant 2$, for otherwise $s_{23}\geqslant\lambda(2)+1$, contradicting $s_{23}\leqslant\min(\lambda(2),\lambda(3))$. Similarly, $s_{23}\geqslant 2$ and $s_{12}>s_{13}$. It suffices to show that $s_{13}=1$.

Assume that $s_{13}>1$. Let $\mathbf{R}\in\mathcal{P}^\lambda$ be such that $R_2=P_2$, $R_3=P_3$ and the roots of R_1 form a q-segment with centre c_1, where $a_3/c_1=q^{\lambda(1)+\lambda(3)+2}$. Note that

$$\frac{a_2}{c_1}=q^{\lambda(1)+\lambda(2)+4-2(s_{12}-s_{13}+1)}.$$

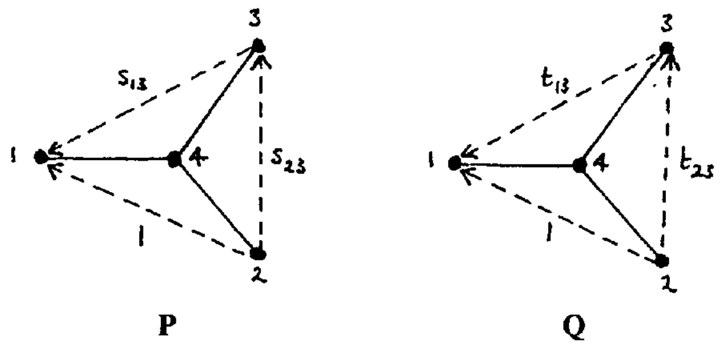
We show that $[V(\mathbf{R})]\prec[V(\mathbf{P})]$, giving the desired contradiction.

Suppose then that $m_\nu(V(\mathbf{R}))>0$, where $\nu=\lambda-\kappa$, $\kappa=\sum_{i=1}^4 n_i\alpha_i$, $n_i\in\mathbb{N}$. If $n_1=0$, then $m_\nu(V(\mathbf{P}))=m_\nu(V(\mathbf{R}))$ by Lemma 4.1, since $\mathbf{P}_{\{2,3,4\}}=\mathbf{R}_{\{2,3,4\}}$. On the other hand, if $n_1>0$, then by Lemma 4.2, $n_4>0$ too. If, in addition, $n_3>0$, let $\nu'=\lambda-\alpha_1-\alpha_3-\alpha_4$. Then, $\nu<\nu'$ and $m_{\nu'}(V(\mathbf{R}))=0$, $m_{\nu'}(V(\mathbf{P}))>0$ by Lemma 4.1 again. The only case left to consider is $n_1>0$, $n_4>0$ and $n_3=0$. But we have $[V(\mathbf{R}_{\{1,2,4\}})]\prec[V(\mathbf{P}_{\{1,2,4\}})]$ by Proposition 3.4, and so another application of Lemma 4.1 deals with this case.

We have now shown that, if $V(\mathbf{P})$ is minimal, one of the conditions (a)–(c) in Theorem 2.2 hold. For the converse, we show first that a minimal affinization $V(\mathbf{P})$ of one of the types (a) or (b) cannot be related under the partial ordering $\preceq$ to any minimal affinization $V(\mathbf{Q})$ of type (c). Suppose, for example, that $\mathbf{P}$ is of type $(\mathrm{a})_{2,3}$, $\mathbf{Q}$ is of type (c), and $[V(\mathbf{P})]\preceq[V(\mathbf{Q})]$. We may assume that $\lambda(2)\leq\lambda(3)$ without loss of generality; note that $s_{23}>1$. If $\mu=\lambda-\lambda(2)(\alpha_2+\alpha_3+\alpha_4)$, then $m_\mu(V(\mathbf{P}))>0$ and $m_\mu(V(\mathbf{Q}))=0$ by Proposition 3.4; and if $\nu>\mu$ and

$m_\nu(V(\mathbf{Q})) > 0$, we must have $\nu = \lambda - r(\alpha_2 + \alpha_3 + \alpha_4)$ for some $r < \lambda(2)$ by Proposition 3.4 again, and then $m_\nu(V(\mathbf{P})) = m_\nu(V(\mathbf{Q}))$. Thus, we cannot have $[V(\mathbf{P})] \preceq [V(\mathbf{Q})]$. Similar arguments apply in the other cases.

To complete the proof of the converse, we have to show, say when **P** and **Q** are of type (c),



that $[V(\mathbf{P})]$ and $[V(\mathbf{Q})]$ are unrelated by $\preceq$ unless $s_{13} = t_{13}$ and $s_{23} = t_{23}$. Note that

$$s_{13} + s_{23} = t_{13} + t_{23} \ (= 4),$$

so if $s_{23} < t_{23}$, then $t_{13} < s_{13}$. Then, considering $\{2, 3, 4\}$ shows that we cannot have $[V(\mathbf{Q})] \preceq [V(\mathbf{P})]$, and considering $\{1, 3, 4\}$ shows that we cannot have $[V(\mathbf{P})] \preceq [V(\mathbf{Q})]$. Finally, the fact that the two diagrams in $(a)_{i,j}$ (resp. those in $(b)_{i,j}$, those in $(c)_{i,j}^{r,s}$) correspond to equivalent affinizations follows from Proposition 1.6. □

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