Z angew Math Phys 45 (1994)
(ZAMP)

0044-2275/94/010061-20 \$ 1.50 + 0.20

Construction of librational invariant tori in the spin-orbit problem

By Alessandra Celletti, Dipartimento di Matematica Pura e Applicata, Universitá di L'Aquila, Via Vetoio, I-67010 Coppito (L'Aquila), Italia (Fax: 0862/433180)

§1. Introduction

One of the most fascinating subjects of Celestial Mechanics is the study of planetary or satellite systems whose rotational or revolutional periods are close to an exact resonance. More precisely one can distinguish between "orbit-orbit" and "spin-orbit" resonances. In the first case one considers a three body problem which presents an exact commensurability among the mean motions. In the spin-orbit problem one deals with a two body system, e.g. a satellite S and a central planet P, where the satellite S is supposed to be characterized by two main periods: a rotational period T_{rot} about an internal spin-axis and a revolutional period T_{rev} around the central body. We define a spin-orbit resonance of type $p:q$ (for some integers p, q), when the ratio T_{rev}/T_{rot} amounts to p/q.

The classical example of a spin-orbit resonance is provided by the Moon-Earth system. As is well known the Moon always points approximately the same face towards the Earth. This implies that the periods of rotation and revolution of the Moon are approximately equal. Therefore the Moon is observed to be very close to an exact $1:1$ (or "synchronous") spin-orbit resonance. However, due to the torque exerted by the Earth, such commensurability is not precisely exact, but presents *dynamical librations* about the synchronous position.

An astonishing fact is that all the *evolved* satellites of the solar system are trapped in a $1:1$ resonance, namely they always point the same face toward the host planet. The only exception to this rule is provided by the Mercury-Sun system, since radar observations have shown that the ratio between the periods of rotation and revolution of Mercury amount to $\frac{3}{2}\cdot(1\pm 10^{-4})$. In other words, after two revolutions about the Sun, Mercury makes three rotations about the spin-axis.

In this paper we intend to investigate the stability of the synchronous resonance applying perturbation (i.e., Birkhoff-KAM) techniques. More precisely, we introduce a mathematical model which describes an approximation to the real physical situation. The assumptions we make are pretty well satisfied by most satellites of the solar system. Using Hamiltonian formalism our model is described by a nearly-integrable Hamiltonian function of the form

$$H(y, x, t) = \frac{y^2}{2} + \varepsilon V(x, t), \qquad y \in \boldsymbol{R}, \quad (x, t) \in (\boldsymbol{R}/2\pi \boldsymbol{Z})^2 \equiv T^2, \tag{1}$$

where $V(x, t)$ is a trigonometric function and ε is the perturbing parameter, representing the *equatorial oblateness* of the satellite.

Notice that the phase space S associated with (1) is three dimensional: therefore any two dimensional invariant surface divides S in two separate regions, providing a strong stability property in the sense of confinement of the motions.

The phase space around the synchronous resonance has a pendulum-like structure. Therefore one has small librational surfaces whose mean diameter increases as the chaotic separatrix is approached. Such separatrix divides the region of *librational* regime (i.e., oscillations of finite amplitude about the equilibrium position) from the region in which *rotational* surfaces (corresponding to a complete rotation of the pendulum) are found. The confinement of the motion corresponding to the synchronous resonance (or, more widely, of orbits "close" to the exact resonance) can therefore be obtained either proving the existence of two rotational invariant surfaces bounding the motion from above and below or by proving the existence of a librational surface surrounding the motion we want to confine.

The first claim was already investigated in [5, 6]. Here we intend to present a method for the construction of librational trapping surfaces. To this end we make some symplectic changes of variables in order to obtain a Hamiltonian function which is adapted to the description of librational motions. We first center the Hamiltonian to the 1 : 1 periodic orbit and expand in Taylor series around the new origin. We then diagonalize the quadratic terms obtaining a harmonic oscillator plus higher degree and time-dependent terms. We finally transform the Hamiltonian using the action-angle variables (I, φ) of the harmonic oscillator. After these successive changes of variables we obtain a Hamiltonian function of the form

$$H(I, \varphi, t) \equiv \omega I + \varepsilon \bar{h}(I) + \varepsilon R(I, \varphi, t), \qquad I \in \boldsymbol{R}, (\varphi, t) \in T^2, \tag{2}$$

where $\omega \equiv \omega(\varepsilon)$ is the frequency of the harmonic oscillator. We could implement KAM theorem directly on this Hamiltonian but, as was shown in [10, 11], one generally obtains better results applying the theorem after reducing the perturbation to higher orders in ε by a close-to-identity change of variables $(I, \varphi, t) \to (I', \varphi', t')$. Therefore, following [14] we apply the Birkhoff normalization procedure to (2) in order to obtain a new

Hamiltonian of the type

$$H_k(I', \varphi', t) = \omega I' + \varepsilon h_k(I'; \varepsilon) + \varepsilon^{k+1} R_k(I', \varphi', t), \tag{3}$$

for some integer k. Finally we apply the KAM theorem (adapting the set of estimates provided in [7]) to the Hamiltonian (3). Notice that, as already remarked in [11], there is an optimal order of normalization, say $k = k^*$, in the Birkhoff procedure, after which the combination of Birkhoff-KAM estimates provides worst results.

We apply this technique to two specific examples, namely the Moon-Earth and the Rhea-Saturn systems, both observed to move on orbits close to the synchronous resonance. In the Moon-Earth case we are able to establish the existence of invariant surfaces corresponding to a libration of about 8.79°. Unfortunately we are able to state this result for values of the oblateness parameter whose ratio with the real physical value is about 0.19. On the contrary we obtain significant results in the Rhea-Saturn system for values of the parameters which are consistent with the astronomical observations. In particular we are able to confine any motion of librational amplitude less or equal than 1.95° around the synchronous orbit.

This paper is organized as follows. In §2 we introduce the equation of motion describing the mathematical model we want to study. The structure of the phase space and the strategy of confinement of the motions are discussed in §3. The construction of librational tori is presented in §4. Some concluding remarks are given in §5.

§2. The model

In this paragraph we simplify the spin-orbit problem making some assumptions and we introduce the equation of motion governing such reduced model. Suppose that the principal moments of inertia of the satellite S are $A < B < C$. Let P be the central planet around which the satellite moves. Moreover assume that,

i) the orbit of the satellite around P is a fixed Keplerian ellipse (i.e., neglect the secular perturbations of the orbital parameters);
ii) the spin-axis coincides with the direction of the axis of the ellipsoid whose moment of inertia is largest (i.e., it coincides with the shortest physical axis of the satellite);
iii) the spin-axis is perpendicular to the orbit plane (i.e., neglect the mean *obliquity* between the spin-axis and the orbit normal);
iv) the external dissipative torques as well as forces induced by other planets or satellites can be neglected.

This model provides an accurate description of the spin-orbit interaction in most significant applications of the solar system. For what concerns the Moon-Earth coupling the first hypothesis which should be removed in a more meaningful model is iii), since actually the spin-axis makes an angle of about 6°41′ with the orbit normal. However in a first approximation we retain hypothesis iii), reserving the idea of removing such assumption in future works. Another questionable point is assumption iv), which implies to approximate the system by a conservative model, since we are neglecting dissipative torques. The contribution of the dissipation is relevant for the explanation of the evolutionary history of the satellites and the consequent capture into resonance. In fact, it is generally believed that the satellites rotated fast around their spin axes and that they evolved to the present slower state by means of a dissipative mechanism. Though presently the contribution of the dissipative term may be assumed much smaller than the conservative part, it should not in principle be neglected. However we do not intend to provide explanations about the capture into resonance, but we claim to prove the stability of the periodic orbit, in which the satellite is actually observed, by means of the construction of librational invariant surfaces. To this end one can use, as a first approximation, the conservative model introduced above.

The equation of motion governing the model i)–iv) can be derived from Euler's equations for a rigid body (see, e.g., [12]). After normalizing the mean motion $2\pi/T_{rev}$ to one, the main equation has the form

$$\ddot{x} + \frac{3}{2}\frac{B-A}{C}\left(\frac{a}{r}\right)^3 \sin(2x - 2f) = 0, \tag{4}$$

where a is the major semiaxis of the Keplerian orbit, r and f are respectively the instantaneous orbital radius and the true anomaly, while x is the angle between the longest axis of the ellipsoid (belonging by iii) to the orbit plane) and the periapsis line (see Fig. 1).

Notice that the angle $\psi \equiv x - f$ (Fig. 1) gives the amplitude of librations around the synchronous resonance, being indentically zero for exact 1 : 1 resonance.

Due to the assumption of Keplerian orbit, both functions r and f are 2π-periodic in time and depend on the orbital eccentricity e. Therefore the

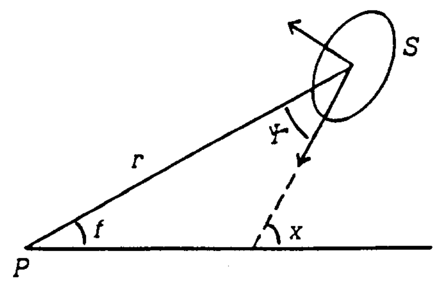

Figure 1
Relative position between the satellite S and the planet P.

term $(a/r)^3 \sin(2x - 2f)$ can be expanded, using standard Keplerian formulae, in Fourier series. After some manipulation (see, e.g., [4]) one obtains in place of (4) the equation

$$\ddot{x} + \varepsilon \sum_{m \neq 0, m = -\infty}^{\infty} \bar{W}\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0, \tag{5}$$

where we refer to $\varepsilon \equiv \frac{3}{2}((B - A)/C)$ as the *equatorial oblateness perturbing parameter* and where the coefficients $\bar{W}(m/2, e)$ decay as powers of the orbital eccentricity e as $\bar{W}(m/2, e) \propto e^{|m-2|}$. For example the first few coefficients are given by the following formulae:

$$\begin{aligned}
\bar{W}(1, e) &= 1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 + O(e^6) \\
\bar{W}(\frac{3}{2}, e) &= \frac{7}{2} e - \frac{123}{16} e^3 + O(e^5) \\
\bar{W}(2, e) &= \frac{17}{2} e^2 - \frac{115}{6} e^4 + O(e^6).
\end{aligned} \tag{6}$$

Since it is more convenient to work with a finite number of terms in the series expansion of (5), we truncate the series according to the following criterion. Due to the assumption iv) we neglect in our model all the dissipative external torques, among which the most important contribution is due to the internal non-rigidity of the satellite. In order to take into account such effect, one should add at the r.h.s. of (5) an additional term T, called "dissipative tidal torque". The explicit expression of T can be given in different forms, according to the assumptions one makes on the internal structure of the satellite (see, e.g., [15]). We do not intend to give the precise formulation of T, but we just mention that in most common situations its average effect is much smaller than the contribution of the gravitational term. Therefore, since according to iv) we neglect such a dissipative force, we will as well neglect in the series expansion of (5) those terms whose size is of the same order of magnitude as the average effect of T. Notice that the coefficients $\bar{W}(m/2, e)$ are themselves infinite series in the eccentricity. Since they decay as powers of e, we truncate the series (see (6)) up to a suitable order in the eccentricity. The order of truncation depends on the physical values of the parameters related to the concrete model one intends to consider. For simplicity of notation we do not explicitely indicate the order of truncation. We just denote by $W(m/2, e)$ the truncated expression of the coefficient $\bar{W}(m/2, e)$. Finally we obtain a finite-sum equation of the type

$$\ddot{x} + \varepsilon \sum_{m \neq 0, m = N_1}^{N_2} W\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0, \tag{7}$$

where the integers N_1, N_2 depend on the physical parameters of the model we want to consider.

§3. The structure of the phase-space and the stability of periodic orbits

Equation (7) is equivalent to Hamilton's equations associated with the Hamiltonian

$$H(y, x, t) \equiv h(y) + \varepsilon V(x, t), \qquad y \in \boldsymbol{R}, \quad (x, t) \in T^2, \tag{8}$$

where

$$h(y) \equiv \frac{y^2}{2}, \qquad V(x, t) = -\frac{1}{2} \sum_{m \neq 0, m = N_1}^{N_2} W\left(\frac{m}{2}, e\right) \cos(2x - mt). \tag{9}$$

The Hamiltonian function (8) describes a nearly-integrable system, since for $\varepsilon = 0$ Hamilton's equations are trivially integrated as $y = y_0$, $x = y_0 t + x_0$. Moreover any plane $\{y_0\} \times T^2$ is invariant under the flow associated with (8)–(9) and motions on such surfaces are either periodic or quasiperiodic, depending on the initial conditions.

Let us define in the standard way the *rotation number* of an invariant surface as $\Omega \equiv \Omega(y_0) \equiv \partial h(y_0)/\partial y$. For non-zero, but sufficiently small values of ε, KAM theory ([17, 1, 20]) ensures the existence of an invariant surface with rotation number Ω, say $\mathcal{T}_\varepsilon(\Omega)$, for the perturbed system provided two assumptions are satisfied, namely

a) non-degeneracy of the unperturbed Hamiltonian, i.e.

$$\frac{\partial^2 h(y)}{\partial y^2} \neq 0 \qquad \forall y \in \boldsymbol{R};$$

b) strong irrationality of the rotation number, i.e. there exists a positive constant C such that

$$\left|\Omega - \frac{p}{q}\right|^{-1} \leq Cq^2, \qquad \forall p, q \in \boldsymbol{Z}, \quad q \neq 0. \tag{10}$$

We shall often refer to (10) as the *diophantine condition* on Ω. Notice that hypothesis a) is trivially covered by (8)–(9), since identically $\partial^2 h(y)/\partial y^2 \equiv 1$.

One of the outcomes of KAM theory is to provide an explicit rigorous algorithm to give a lower bound on the perturbing parameter, say $\varepsilon_r \equiv \varepsilon_r(\Omega)$, ensuring the existence of an invariant surface $\mathcal{T}_\varepsilon(\Omega)$ (for the perturbed system) with rotation number Ω.

If we let ε increase, we reach a critical value, say $\varepsilon_c = \varepsilon_c(\Omega)$, at which the KAM torus breaks down and leaves place to so-called Aubry-Mather sets ([3, 19]), which are closed, invariant and are graphs of Cantor sets. We remark that under some general assumptions one can compute an "experimental" value of the critical break-down threshold $\varepsilon_c(\Omega)$, applying for example the numerical algorithm developed by J. Greene in [16].

The connection between the stability of periodic orbits and the existence of invariant tori is provided by the analysis of the structure of the phase space $\mathscr{S}$ associated with (8):

$$\mathscr{S} \equiv \{(y, x, t)/y = \dot{x} \in \boldsymbol{R}, \quad (x, t) \in T^2\}.$$

Since $\mathscr{S}$ is three dimensional, any KAM surface will separate $\mathscr{S}$ in two invariant regions with the property that any motion starting in one of the two portions of phase space will remain forever in it. In higher dimensional systems (when the number of degrees of freedom is greater than two) the KAM tori do not separate anymore the constant energy surfaces and "Arnold diffusion" is allowed (see [2]). However in our lower dimensional case we can still have the escape of the motions by means of another mechanism: when the perturbing parameter increases up to $\varepsilon_c(\Omega)$, the invariant KAM torus becomes an Aubry-Mather set and the orbits can diffuse through the gaps of the Cantor set.

Let us now look at the shape of the phase space. If we draw a Poincaré section associated with the orbits of (8)–(9) we obtain a pendulum-like structure. Zooming on the synchronous resonance, we find that the periodic orbit, which is just described by a point on the Poincaré map, is surrounded by small curves, giving the amplitude of the librational oscillations around the synchronous position. The librational amplitudes increase as the *chaotic separatrix* is approached. The chaotic separatrix provides a border between the region of librational motion and the region in which invariant rotational curves are found.

Since the analysis of curves inside the librational regime differs significantly according to whether they are close to the periodic orbit or to the separatrix, we shall distinguish between proper librational curves (see Fig. 2b) and close-to-separatrix orbits (Fig. 2c).

According to the phase-space structure presented in this paragraph we conclude that the confinement of the synchronous resonance in the phase space can be obtained in two ways:

a) proving the existence of a librational surface enclosing the periodic orbit;
b) proving the existence of two rotational surfaces bounding the synchronous resonance from above and below.

We remark that the dissipative term will in general destroy such invariant surfaces. Anyhow, if the dissipation is small compared to the conservative term, then the invariant surfaces will in general behave as adiabatic invariants on a dynamical time-scale.

The construction of invariant rotational trapping surfaces has already been investigated in [5, 6], using the KAM algorithm developed in [8].

We briefly report the results for the two satellite-planet systems, which we are going to consider in the following paragraph. Let γ be the golden

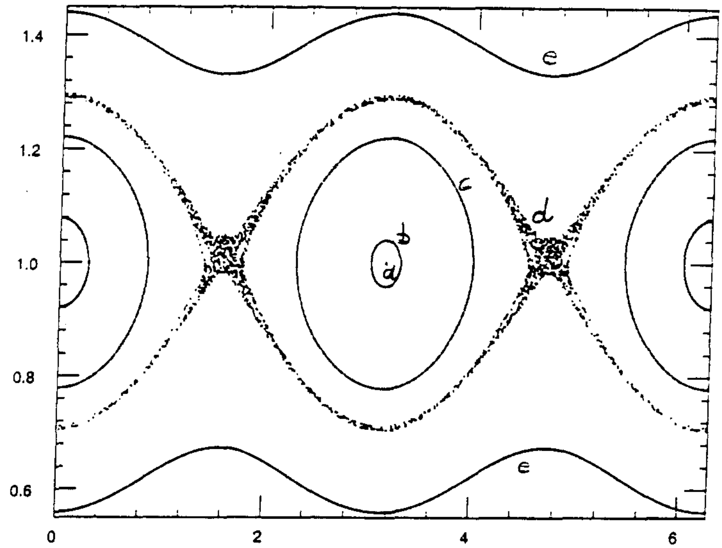

Figure 2
Poincaré map of (8)–(9) (with $N_1 = 1$, $N_2 = 5$) around the synchronous resonance. a) Periodic 1 : 1 orbit, b) librational curve, c) close-to-separatrix librational curve, d) chaotic separatrix, e) rotational invariant curves.

ratio: $\gamma \equiv (\sqrt{5} - 1)/2$. We remark that any irrational number of the type $1 \pm (1/(k + \gamma))$, for any integer $k \geq 2$, satisfies the non-resonance condition (10) with a diophantine constant $C \equiv C_k \equiv k + \gamma$ (see [5]). Then we have the following statements.

Moon-Earth: Consider the system described by the Hamiltonian function (8)–(9) with $N_1 = -1$, $N_2 = 5$ and fix the eccentricity $e = 0.0549$. Let $\varepsilon_{obs} = 3.45 \cdot 10^{-4}$ (i.e., the observed physical value of the equatorial oblateness of the Moon); then for any $\varepsilon \leq \varepsilon_{obs}$, there exist (trapping) rotational invariant surfaces $\mathscr{T}_\varepsilon(\Omega_1)$, $\mathscr{T}_\varepsilon(\Omega_2)$ with $\Omega_1 \equiv 1 - (1/(40 + \gamma)) \simeq 0.9753$, $\Omega_2 \equiv 1 + (1(40 + \gamma)) \simeq 1.0246$.

Rhea-Saturn: Consider the Hamiltonian (8)–(9) with $N_1 = 1$, $N_2 = 5$ and let $e = 0.00098$. Let $\varepsilon_{obs} = 3.45 \cdot 10^{-3}$ (i.e., the observed value); then for any $\varepsilon \leq \varepsilon_{obs}$, there exist (trapping) rotational invariant surfaces $\mathscr{T}_\varepsilon(\Omega_1)$, $\mathscr{T}_\varepsilon(\Omega_2)$ with $\Omega_1 \equiv 1 - (1/(10 + \gamma)) \simeq 0.9058$, $\Omega_2 \equiv 1 + (1/(10 + \gamma)) \simeq 1.0941$.

§4. Existence of librational invariant surfaces

In order to construct invariant surfaces close to the periodic orbit, we need to construct a Hamiltonian function which suitably describes librational surfaces. The outline of the strategy is the following. We center the

Hamiltonian function on the 1 : 1 periodic orbit and expand in Taylor series around the new equilibrium position. We obtain a perturbed harmonic oscillator which we reduce to diagonal form by a further canonical change of variables. We finally transform to action-angle variables (I, φ) for the harmonic oscillator, obtaining a Hamiltonian function of the form

$$H(I, \varphi, t) = \omega I + \varepsilon \bar{h}(I) + \varepsilon g(I, \varphi, t),$$

for some analytic functions $\bar{h}, g$. We can now remove the perturbation to higher orders in ε; to this end we apply a close-to-identity transformation $(I, \varphi, t) \to (I', \varphi', t)$ (with generating function containing terms up to the order ε^k, for some $k \in \boldsymbol{Z}$) to obtain a new Hamiltonian of the form

$$H_k(I', \varphi', t) = h_k(I'; \varepsilon) + \varepsilon^{k+1} R_k(I', \varphi', t; \varepsilon).$$

We apply the KAM algorithm (developed in [7]) to the last Hamiltonian, in order to obtain the existence of an invariant librational torus surrounding the 1 : 1 resonance.

For simplicity of exposition we subdivide this paragraph in several subsections describing the changes of variables, the Birkhoff-KAM applications and the results.

4.1. Center on the periodic orbit

Let us rewrite the Hamiltonian (8)–(9) putting into evidence the cosine term which corresponds to the synchronous resonance:

$$H(y, x, t) = \frac{y^2}{2} - \varepsilon a \cos(2x - 2t) - \frac{\varepsilon}{2} \sum_{m \neq 0,2} W\left(\frac{m}{2}, e\right) \cos(2x - mt), \tag{11}$$

where for shortness we introduced the coefficient $a \equiv \frac{1}{2} W(1, e)$. Next we perform the linear symplectic change of variables

$$\begin{cases} x' = 2x - 2t \\ y' = \frac{1}{2}(y - 1), \end{cases}$$

which moves the unperturbed position of the synchronous resonance to the point $x' = 0$, $y' = \frac{1}{2}$. We next shift the resonance to the point (0, 0) by using the change of variables

$$\begin{cases} \bar{x} = x' \\ \bar{y} = y' - \frac{1}{2}, \end{cases}$$

such that (11) (up to an inessential additive constant) takes the form

$$H(\bar{y}, \bar{x}, t) = 2\bar{y}^2 - \varepsilon a \cos \bar{x} - \frac{\mu}{2} \sum_{m \neq 0,-2} \tilde{W}\left(\frac{m+2}{2}, e\right) \times [\cos \bar{x} \cos mt + \sin \bar{x} \sin mt], \tag{12}$$

where $\mu \equiv \varepsilon e$ and we have rescaled the coefficients W by a factor e as

$$\tilde{W}\left(\frac{m+2}{2}, e\right) = \frac{1}{e} W\left(\frac{m+2}{2}, e\right).$$

In the pendulum approximation (i.e. setting $\mu = 0$ in (12)) one obtains a conservative system showing librational motions for values of the energy $|E| < \varepsilon a$. Each *unperturbed* librational curve is labeled by a level energy $\bar{\delta}$ defined by the relation

$$2\bar{y}^2 - \varepsilon a \cos \bar{x} = \varepsilon a \bar{\delta} \quad \text{with } |\bar{\delta}| < 1$$

(actually $\bar{\delta}$ will be chosen very close to -1 in order to have small curves around the synchronous resonance). Notice that in the $\mu = 0$ approximation, the angle $\bar{x}$ is exactly the double of the libration angle ψ introduced in Fig. 1.

4.2. *Series expansion around the equilibrium point and diagonalization*

We next expand (12) in Taylor series around the unperturbed equilibrium position $\bar{x} = \bar{y} = 0$, for $|\bar{x}| < 1$. It is easily shown that in the unperturbed situation (i.e. setting $\mu = 0$), the librational curve is described by the equation

$$2\bar{y}^2 + \frac{\varepsilon a}{2} \bar{x}^2 = \tau \quad \text{with } \tau \equiv \varepsilon a(\bar{\delta} + 1). \tag{13}$$

We then diagonalize the time-independent quadratic terms in order to obtain a harmonic oscillator in canonical form. Therefore we make the further symplectic change of variables

$$\begin{cases} p = \alpha \bar{y} \\ q = \beta \bar{x} \end{cases}$$

with

$$\alpha = \frac{\sqrt{2}}{(\varepsilon a)^{1/4}}, \qquad \beta = \frac{(\varepsilon a)^{1/4}}{\sqrt{2}},$$

so that we obtain the Hamiltonian

$$\begin{aligned} H(p, q, t) = \frac{\omega}{2}(p^2 + q^2) - \varepsilon a\left(\frac{q^4}{4!\beta^4} - \frac{q^6}{6!\beta^6} + \cdots\right) \\ - \frac{\mu}{2} \sum_{m \neq 0, -2} \tilde{W}\left(\frac{m+2}{2}, e\right)\left[\cos(mt)\left(1 - \frac{q^2}{2\beta^2} + \frac{q^4}{4!\beta^4} + \cdots\right)\right. \\ \left. + \sin(mt)\left(\frac{q}{\beta} - \frac{q^3}{3!\beta^3} + \frac{q^5}{5!\beta^5} + \cdots\right)\right], \end{aligned} \tag{14}$$

where the frequency ω of the harmonic oscillations is defined as

$$\omega \equiv 2\sqrt{\varepsilon a}.$$

The Taylor expansion converges provided $|q/\beta| < 1$ and the new unperturbed librational curve is described by the equation

$$p^2 + q^2 = \frac{\tau}{\sqrt{\varepsilon a}}$$

with τ as in (13).

4.3. *Action-angle variables for the harmonic oscillator*

Let $(p, q) \to (I, \varphi)$ be a transformation to action-angle variables for the harmonic oscillator:

$$\begin{cases} p = \sqrt{2I}\cos\varphi \\ q = \sqrt{2I}\sin\varphi. \end{cases} \tag{15}$$

If we let $\mu = 0$ the value of $|I|$ represents the radius of the librational circle in the (p, q)-variables. Let $|I_0|$ be the value corresponding to the level energy $\bar{\delta}$:

$$|I_0| = \frac{\sqrt{\varepsilon a}}{2}(\bar{\delta} + 1).$$

Performing the change of variables (15) on (14), after some manipulation one gets the Hamiltonian

$$\begin{aligned} H(I, \varphi, t) = \omega I &- \varepsilon a\left(\frac{I^2}{16\beta^4} - \frac{5I^3}{2\cdot 6!\beta^6} + \cdots\right) \\ &- \varepsilon a\left[-\frac{I^2}{12\beta^4}\cos 2\varphi + \frac{I^2}{48\beta^4}\cos 4\varphi + \frac{I^3}{4\cdot 6!\beta^6}\right. \\ &\left.\cdot(15\cos 2\varphi - 6\cos 4\varphi + \cos 6\varphi) + \cdots\right] \\ &- \frac{\varepsilon e}{2}\sum_{m\neq 0,-2}\tilde{W}\left(\frac{m+2}{2}, e\right)\left\{\cos(mt)\left[1 - \frac{I}{2!\beta^2}(1 - \cos 2\varphi)\right.\right. \\ &+ \frac{I^2}{8\cdot 3!\beta^4}\cdot(3 - 4\cos 2\varphi + \cos 4\varphi) - \frac{I^3}{4\cdot 6!\beta^6} \\ &\left.\cdot(10 - 15\cos 2\varphi + 6\cos 4\varphi - \cos 6\varphi) + \cdots\right] \\ &+ \sin(mt)\left[\frac{\sqrt{2I}}{\beta}\sin\varphi - \frac{\sqrt{2}I^{3/2}}{12\beta^3}(3\sin\varphi - \sin 3\varphi)\right. \\ &\left.\left. + \frac{\sqrt{2}I^{5/2}}{4\cdot 5!\beta^5}(10\sin\varphi - 5\sin 3\varphi + \sin 5\varphi) + \cdots\right]\right\}, \end{aligned} \tag{16}$$

where the dots are short for terms of order greater than three in the action variable.

Remark: In practical applications we choose $\bar{\delta}$ so that fixing $|I_0| = (\sqrt{\varepsilon a}/2)(\bar{\delta}+1)$, the terms of order $(|I_0|/\beta^2)^{7/2}$ are of the same order of magnitude of the dissipative tidal torque, that we have neglected in writing the equation of our mathematical model (see assumption iv), §2). Therefore, for sake of simplicity we decide consistently to neglect those terms in the series expansion of (16) which are of the order of $(|I_0|/\beta^2)^{7/2}$.

We rewrite (16) (or, more precisely, the truncation of (16) according to the remark above) in a compact form as

$$H(I, \varphi, t) = \omega I + \varepsilon \bar{h}(I) + \varepsilon h(I, \varphi) + \varepsilon e f(I, \varphi, t), \tag{17}$$

with an obvious meaning of the symbols. Notice that we have reintroduced εe in place of μ.

Let us define the domain of analyticity of the Hamiltonian (17) as follows. Denote by z_φ, z_t the complexified (angle) variables

$$z_\varphi = e^{i\varphi}, \qquad z_t = e^{it}.$$

Let us denote by

$$D(I_0; \varrho, \xi) \equiv \{(I, z_\varphi, z_t) \in \boldsymbol{C}^3 / |I - I_0| < \varrho,\ e^{-\xi} < |z_\varphi| < e^{\xi},\ e^{-\xi} < |z_t| < e^{\xi}\}$$

for suitable parameters $\varrho > 0$ and $\xi > 0$. Since the change of variables (15) is singular at the origin, we need to fix ϱ in such a way that the annulus $\{I/|I - I_0| < \varrho\}$ does not contain the origin. In practical applications we shall fix

$$\varrho \equiv \frac{4}{5}|I_0| = \frac{2}{5}\sqrt{\varepsilon a}(\bar{\delta}+1).$$

Moreover, the condition under which the Taylor series converges, i.e. $|q/\beta| < 1$, becomes now

$$\left| \frac{\sqrt{2(I_0+\varrho)}}{\beta} \left(\frac{e^{\xi} + e^{-\xi}}{2} \right) \right| < 1.$$

Therefore we will choose the analyticity parameters ϱ (or, equivalently, $\bar{\delta}$) and ξ, so to satisfy the last inequality.

To give an idea of the values of the analyticity parameters in practical applications, we mention that in the Moon-Earth case we fix $\varrho = 1.21 \cdot 10^{-4}$ and $\xi = 0.6$, while in the Rhea-Saturn application we take $\varrho = 3.85 \cdot 10^{-5}$ and $\xi = 0.6$.

4.4. *Birkhoff normalization*

We now proceed to apply the Birkhoff normalization procedure in order to remove the perturbing function $g(I, \varphi, t) \equiv h(I, \varphi) + ef(I, \varphi, t)$ in (17) up to the order ε^{k+1} for some $k \in \boldsymbol{Z}_+$. Following [14], we perform a canonical transformation with generating function close to the identity:

$$I'\varphi + \sum_{j=1}^{k} \varepsilon^j \Phi_j(I', \varphi, t)$$

for some integer k. The induced transformation is given by

$$\begin{cases} I = I' + \sum_{j=1}^{k} \varepsilon^j \dfrac{\partial \Phi_j(I', \varphi, t)}{\partial \varphi} \\ \varphi' = \varphi + \sum_{j=1}^{k} \varepsilon^j \dfrac{\partial \Phi_j(I', \varphi, t)}{\partial I'} \\ t' = t, \end{cases} \tag{18}$$

where the functions Φ_j's have to be chosen so that inserting (18) in (17) one obtains a function which does not depend on the angle variables (φ, t) up to $O(\varepsilon^{k+1})$. Therefore one constructs the function

$$\Psi_k(I', \varphi, t) \equiv \sum_{j=1}^{k} \varepsilon^j \Phi_j(I', \varphi, t)$$

in such a way that the expression

$$\omega\left(I' + \frac{\partial \Psi_k}{\partial \varphi}\right) + \varepsilon \bar{h}\left(I' + \frac{\partial \Psi_k}{\partial \varphi}\right) + \varepsilon h\left(I' + \frac{\partial \Psi_k}{\partial \varphi}, \varphi\right) \\ + \varepsilon e f\left(I' + \frac{\partial \Psi_k}{\partial \varphi}, \varphi, t\right) + \frac{\partial \Psi_k}{\partial t} \tag{19}$$

is independent on (φ, t) up to $O(\varepsilon^{k+1})$. For example, expanding (19) in Taylor series in ε, the first term Φ_1 solves the equation

$$\omega \frac{\partial \Phi_1(I', \varphi, t)}{\partial \varphi} + \frac{\partial \Phi_1(I', \varphi, t)}{\partial t} + h(I', \varphi) + ef(I', \varphi, t) = 0.$$

Expanding $\Phi_1(I', \varphi, t)$ in Fourier series as

$$\Phi_1(I', \varphi, t) \equiv \sum_{n,m \neq 0} \hat{\Phi}^{(1)}_{nm}(I') e^{i(n\varphi + mt)},$$

the coefficients $\hat{\Phi}^{(1)}_{nm}$ are determined by

$$\hat{\Phi}^{(1)}_{nm}(I') \equiv i \frac{\hat{g}_{nm}(I')}{\omega n + m},$$

where $\hat{g}_{nm}(I')$ is the Fourier coefficient of the function

$$g(I', \varphi, t) \equiv h(I', \varphi) + ef(I', \varphi, t).$$

We refer to the terms $1/(\omega n + m)$ as "small divisors"; obviously in order to define properly Φ_1 one has to check that $|\omega n + m| \neq 0$ for any pair of integers n, m belonging to the set of Fourier indexes of the function g. Up to second order the Hamiltonian (in the mixed set of variables (I', φ, t)) is given by

$$\tilde{H}_1(I', \varphi, t) \equiv \tilde{h}_1(I'; \varepsilon) + \varepsilon^2 \tilde{R}_1(I', \varphi, t),$$

where

$$\tilde{h}_1(I'; \varepsilon) \equiv \omega I' + \varepsilon \bar{h}(I') + \varepsilon \langle g \rangle (I')$$

(with $\langle h(I', \varphi, t) \rangle = (1/2\pi)^2 \int_{T^2} h(I', \varphi, t)\, d\varphi\, dt \equiv \langle h \rangle (I')$) and

$$\tilde{R}_1(I', \varphi, t) = \left[\bar{h}'_I(I') \cdot \frac{\partial \Phi_1}{\partial \varphi} + h'_I(I', \varphi) \cdot \frac{\partial \Phi_1}{\partial \varphi} + ef'_I(I', \varphi, t) \cdot \frac{\partial \Phi_1}{\partial \varphi} \right],$$

where $h'_I \equiv \partial h / \partial I'$ and analogous formulae. Inverting the system of equations (18) as

$$I = I' + \sum_{j=1}^{k} \Xi_j(I', \varphi', t) \varepsilon^j$$

$$\varphi = \varphi' + \sum_{j=1}^{k} \Delta_j(I', \varphi', t) \varepsilon^j$$

$$t = t',$$

for some analytic functions Ξ_j, Δ_j, the Hamiltonian function expressed in the new set of variables (I', φ', t) becomes

$$H_1(I', \varphi', t) = h_1(I'; \varepsilon) + \varepsilon^2 R_1(I', \varphi', t),$$

with

$$h_1(I'; \varepsilon) \equiv \omega I' + \varepsilon \bar{h}(I') + \varepsilon \langle g \rangle (I').$$

We do not need the explicit expression of the new perturbing function R_1, but just a bound on its norm which can be obtained as follows. We define the norm of a function $a \equiv a(I', \varphi', t)$ as

$$\| a(I', \varphi', t) \|_{\varrho', \xi'} \equiv \sup_{D(I_0, \varrho', \xi')} | a(I', \varphi', t) |.$$

Then the new perturbing function R_1 can be estimated through the norms of Ξ_1, Δ_1, as well as $\bar{h}, g$, by

$$\| R_1(I', \varphi, t) \|_{\varrho', \xi'} \leq \| \bar{h}'_I(I') \|_{\varrho', \xi'} \cdot \| \Xi_1 \|_{\varrho', \xi'}$$
$$+ \| g'_I(I', \varphi' + \Delta, t) \|_{\varrho', \xi'} \cdot \| \Xi_1 \|_{\varrho', \xi'},$$

where the new parameters ϱ', ξ' have to be chosen in order to define properly the new analyticity domain. More precisely, let $\varrho, \xi > 0$ be the analyticity parameters associated with the initial Hamiltonian function (17). The new parameters $\varrho' < \varrho$ and $\xi' < \xi$ are defined as follows. In the style of [14, 7] we impose some conditions which ensure the correct definition of the analyticity domain in the new set of variables (I', φ', t) as well as the invertibility of the transformation (18). Therefore, the new parameters $\varrho', \xi' \equiv \xi - \delta$ are chosen in order to satisfy the conditions (see, e.g., [7])

$$|I_0 - I_0^{(k)}| + \varrho' + \left\| \frac{\partial \Psi_k}{\partial \varphi} \right\|_{\varrho', \xi} \leq \varrho$$

and

$$\max\left\{ \left\| \frac{\partial^2 \Psi_k}{\partial I' \, \partial \varphi} \right\|_{\varrho', \xi}, \left\| \frac{\partial \Psi_k}{\partial I'} \right\|_{\varrho', \xi} \cdot \delta^{-1} \right\} < 1.$$

We conclude this part remarking that the functions Ξ_j, Δ_j can be estimated in terms of Φ_1 by (see [14])

$$\|\Xi_j\|_{\varrho', \xi'} \leq \left\| \frac{\partial \Phi_j}{\partial \varphi} \right\|_{\varrho', \xi}, \qquad \|\Delta_j\|_{\varrho', \xi'} \leq \left\| \frac{\partial \Phi_j}{\partial I'} \right\|_{\varrho', \xi}.$$

For what concerns the higher order terms of the generating function Ψ_k, they are obtained by the recursive formula

$$\begin{aligned} &\omega \frac{\partial \Phi_j}{\partial \varphi} + \frac{\partial \Phi_j}{\partial t} + \sum_{l=1}^{j-1} \frac{g_I^{(l)}}{l!} \sum_{j_1 + \cdots + j_l = j-1} \frac{\partial \Phi_{j_1}}{\partial \varphi} \cdots \frac{\partial \Phi_{j_l}}{\partial \varphi} \\ &\quad \equiv \omega \frac{\partial \Phi_j}{\partial \varphi} + \frac{\partial \Phi_j}{\partial t} + N_j(I', \varphi, t), \end{aligned}$$

where $j_1, \ldots, j_l \geq 1$ and $g_I^{(l)} \equiv \partial^l g / \partial I^l$. The new Hamiltonian is given by

$$H_k(I', \varphi', t) = h_k(I'; \varepsilon) + \varepsilon^{k+1} R_k(I', \varphi', t; \varepsilon), \tag{20}$$

with

$$h_k(I'; \varepsilon) \equiv \omega I' + \varepsilon \bar{h}(I') + \sum_{j=1}^{k} \langle N_j(I', \varphi, t) \rangle \varepsilon^j,$$

where again $\langle \cdot \rangle$ denotes the average on the angle variables $(\varphi, t) \in T^2$. The perturbation R_k is bounded by

$$\|R_k\|_{\varrho', \xi'} \leq \sum_{l=1}^{k} \frac{\|g_I^{(l)}\|_{\varrho', \xi'}}{l!} \sum_{j_1 + \cdots + j_l = k} \|\Xi_{j_1}\|_{\varrho', \xi'} \cdots \|\Xi_{j_l}\|_{\varrho', \xi'}, \tag{21}$$

where $j_1, \ldots, j_l \geq 1$.

4.5. Definition of the rotation number

Define an approximate center $\tilde{I}_0 \equiv (\sqrt{\varepsilon a}/2)(\bar{\delta}+1)$ and let $\varrho \equiv \frac{2}{5}\sqrt{\varepsilon a}(\bar{\delta}+1)$. In this formulae we fix ε equal to the observed physical value, say $\varepsilon = \varepsilon_{obs}$, while $\bar{\delta}$ plays the role of a parameter which will be chosen in order that the KAM algorithm (see next section) converges. For a given $\tilde{I}_0$ we obtain a rotation number

$$\tilde{\Omega} \equiv \omega(\varepsilon) - \varepsilon a\left(\frac{\tilde{I}_0}{8\beta^4} - \frac{15\tilde{I}_0^2}{2\cdot 6!\beta^6}\right), \quad \text{where } \varepsilon = \varepsilon_{obs}.$$

In concrete computations we obtain a numerical value for $\tilde{\Omega}$ which is truncated up to the precision of the computer. In order to have a *diophantine* rotation number we proceed as follows. For a given integer M, let $[a_0, a_1, \ldots, a_M]$ be the continued fraction expansion of $\tilde{\Omega}$. Obviously the number M of the terms in the expansion depends on the number of digits with which $\tilde{\Omega}$ has been computed. It is now necessary to work with the full continued fraction expansion, but one can retain only a few terms $[a_0, a_1, \ldots, a_N]$ $(N < M)$, providing an arbitrary approximation to $\tilde{\Omega}$. Now we obtain a diophantine number adding to $[a_0, a_1, \ldots, a_N]$ an infinite tail of 1's. Therefore let the new rotation number Ω be defined as

$$\Omega \equiv [a_0, a_1, \ldots, a_N, 1, 1, 1, \ldots].$$

The new frequency Ω is a noble number and by number theory results it satisfies the diophantine condition with a constant C which can be evaluated as in [9]. Now we adjust the real center I_0, computing a new value of the action variable I satisfying the relation

$$\omega(\varepsilon) - \varepsilon a\left(\frac{I_0}{8\beta^4} - \frac{15I_0^2}{2\cdot 6!\beta^6}\right) = \Omega, \qquad \varepsilon = \varepsilon_{obs}.$$

Notice that after applying the Birkhoff procedure (at any order k) we have to adjust the new center, say $I = I_0^{(k)}$ as follows. Let $h_k(I'; \varepsilon)$ be the new unperturbed Hamiltonian; having fixed a rotation number Ω we define $I_0^{(k)}$ as the value of the action variable which satisfies the relation

$$\frac{\partial h_k(I_0^{(k)})}{\partial I} \equiv \Omega.$$

We remark that for small values of ε, as in the applications we shall consider later, the correction to the frequency is quite small as the order k of the Birkhoff normalization is increased.

4.6. Application of KAM theorem

Once obtained the new Hamiltonian H_k we can proceed to apply the KAM theorem following closely the algorithm developed in [7]. We just sketch the idea of the proof, referring to [7] for the details.

Rename the initial Hamiltonian (20) as

$$H_0(I, \varphi, t) = h_0(I) + \mu f_0(I, \varphi, t) \qquad (\mu \equiv \varepsilon^k),$$

which is real analytic in the domain $D(I_0; \varrho_0, \xi_0)$ with center I_0. Let Ω be a fixed diophantine rotation number. The idea is to construct a sequence of Hamiltonians $\{H_j\}$ of the form

$$H_j(I', \varphi', t) = h_j(I'; \mu) + \mu^{2^j} f_j(I', \varphi', t; \mu),$$

analytic on smaller domains $D(I_j; \varrho_j, \xi_j)$. The new center I_j relative to H_j is chosen so as to keep the frequency fixed, i.e. $h_j'(I_j) = \Omega$. A new Hamiltonian H_{j+1} can be obtained from H_j applying a (close to identity) canonical transformation $\mathscr{C}_j: (I, \varphi, t) \to (I', \varphi', t)$. The transformation $\mathscr{C}_j$ can be defined under smallness conditions on the perturbing parameter μ. These conditions consist in a set of estimates, controlling the quantities involved in the proof. The invariant torus $\mathscr{T}_\varepsilon(\Omega)$ is finally obtained as the limit of the composition

$$\mathscr{C}_0 \circ \mathscr{C}_1 \circ \cdots \circ \mathscr{C}_j(\{I_{j+1}\} \times T^2).$$

The set of conditions to be imposed for the convergence of the proof are collected in the Appendix A of [7]. We apply that scheme with only one modification in the estimate of the small divisor series. More precisely a crucial point of the proof is the estimate of quantities of the form

$$s_{k_1,k_2}(\eta) \equiv \sum_{(n,m) \in Z^2 \setminus \{0\}} \frac{|n|^{k_1}}{(n\Omega - m)^{k_2}} e^{-\eta(|n| + |m|)}, \quad k_1 \in \{0, 1, 2\}, \quad k_2 \in \{1, 2\} \tag{22}$$

for some $\eta < 1$. One could use the diophantine condition in order to estimate (22). However a substantial improvement is obtained evaluating by computer a certain number of terms in the sum and estimating the remainder using the diophantine condition. The explicit formulae are given by Lemma 9 of [8].

We conclude this part by mentioning that in order to construct the functions Φ_j of the Birkhoff procedure and to control the KAM algorithm, we make use of a computer. However the computer introduces rounding-off and propagation errors. In order to control the numerical errors we apply the so-called "interval arithmetic" technique which was introduced in [18, 13].

4.7. Results

We provide the application of the scheme presented in the previous sections to the Moon-Earth and the Rhea-Saturn systems; both examples are observed to move closely to a synchronous resonance.

We remark here that increasing the order k of the Birkhoff normalization does not imply to obtain better results. More specifically, one of the input data of the KAM theorem is given by the estimate of the perturbing function R_k which is bounded as in (21). The convergence of the KAM algorithm strongly depends on the size of R_k, in the sense that the increase of its norm causes a slower convergence until the algorithm fails to converge. The estimate (21) depends on the order of the Birkhoff normalization; for example, in the Moon-Earth case one finds approximately that $R_{k+1} \simeq 10^3 R_k$, $k \geq 1$. In practical applications, one starts with a Birkhoff normalization at the first order and checks the convergence of the KAM algorithm. If the algorithm converges, the Birkhoff procedure can be pushed to higher orders until the divergence of the KAM scheme (see [11]). For the systems we consider in this paper, the optimal orders are $k = 5$ for the Moon and $k = 4$ for Rhea.

The initial perturbing function contains 82 Fourier coefficients; after applying the Birkhoff procedure we construct a generating function which contains 1920 Fourier coefficients at the order $k = 4$ and 3475 coefficients at $k = 5$. The computer time needed to construct the generating function amounts to about 50^s of CPU time on a VAX 6000 at the order $k = 4$. The computer execution time multiplies by a factor ~ 12 for the programs in which interval arithmetic has been implemented.

Moon-Earth: Consider the Hamiltonian (8)–(9) with $N_1 = -1$, $N_2 = 5$. Let $e = 0.0549$ and $\varepsilon_{obs} = 3.45 \cdot 10^{-4}$ (i.e., the real physical value). Moreover, let $\bar{\delta} = -0.9766$ and $\xi = 0.6$; then there exists an invariant torus corresponding to a libration of $8^o.79$ for any $\varepsilon \leq \varepsilon_{obs}/5.26$.

Unfortunately in this case we cannot draw conclusions for values of the perturbing parameter which are consistent with the astronomical value. However one obtains significative results in the Rhea-Saturn system as follows.

Rhea-Saturn: Consider the Hamiltonian (8)–(9) with $N_1 = 1$, $N_2 = 5$. Let $e = 0.00098$ and $\varepsilon_{obs} = 3.45 \cdot 10^{-3}$ (i.e., the astronomical value). Moreover, let $\bar{\delta} = -0.9976$ and $\xi = 0.6$; then there exists an invariant torus corresponding to a libration of $1^o.95$ for any $\varepsilon \leq \varepsilon_{obs}$.

§5. Conclusions

Though we are not able to draw definite conclusions about the stability of the Moon, we still believe that one can improve the results using a different KAM algorithm. In particular, the results based on the proof of [7] (which make use of a sequence of canonical transformations as in §4.6) are worse than those obtained applying the method developed in [8] (which was used to prove the existence of rotational invariant tori). However the scheme of [8] cannot be applied straightforwardly to our problem. A new set of estimates in the style of [9] has to be developed.

Anyhow our approach applies for example to the Rhea-Saturn system and gives an insight on the stability of the synchronous resonance associated with our particular mathematical model. It might be interesting to explore the stability of higher order resonances by applying the same method presented here. An analogous approach can also be used in the construction of close-to-separatrix curves, where instead of expanding around the equilibrium position, one might use directly action-angle variables for the pendulum Hamiltonian $h(\bar{y}, \bar{x}) = 2\bar{y}^2 - \varepsilon a \cos \bar{x}$ of formula (12).

Acknowledgements

I am deeply indebted to L. Chierchia, G. Gallavotti and J. Moser for many enlightening suggestions and advices during the accomplishment of this work.

References

[1] Arnold, V. I., *Proof of a Theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian*, Russ. Math. Surveys *18*, 9 (1963).
[2] Arnold, V. I. (ed.), *Encyclopedia of Mathematical Sciences*, Dynamical System III, Springer-Verlag 3 (1988).
[3] Aubry, S. and Le Daeron, P. Y., *The discrete Frenkel-Kontorova model and its extensions I*, Physica *8D*, 381 (1983).
[4] Cayley, A., *Tables of the developments of functions in the theory of elliptic motion*, Mem. Roy. Astron. Soc. *29*, 191 (1859).
[5] Celletti, A., *Analysis of resonances in the spin-orbit problem in Celestial Mechanics: The synchronous resonance (Part I)*, J. Appl. Math. and Phys. (ZAMP) *41*, 174 (1990).
[6] Celletti, A., *Analysis of resonances in the spin-orbit problem in Celestial Mechanics: Higher order resonances and some numerical experiments (Part II)*, J. Appl. Math. and Phys. (ZAMP) *41*, 453 (1990).
[7] Celletti, A. and Chierchia L., *Rigorous estimates for a computer-assisted KAM theory*, J. Math. Phys. *28*, 2078 (1987).
[8] Celletti, A. and Chierchia, L., *Construction of analytic KAM surfaces and effective stability bounds*, Commun. Math. Phys. *118*, 119 (1988).
[9] Celletti, A. and Chierchia, L., *A constructive theory of Lagrangian tori and computer-assisted applications*, to appear in Dynamics Reported.
[10] Celletti, A., Falcolini, C. and Porzio A., *Rigorous numerical stability estimates for the existence of KAM tori in a forced pendulum*, Ann. Inst. Henri Poincaré *47*, 85 (1987).
[11] Celletti, A. and Giorgilli, A., *On the numerical optimization of KAM estimates by classical perturbation theory*, J. Appl. Math. and Phys. (ZAMP) *39*, 743 (1988).
[12] Danby, J. M. A., *Fundamentals of Celestial Mechanics*, Macmillan, New York 1962.
[13] Eckmann, J.-P. and Wittwer, P., *Computer methods and Borel summability applied to Feigenbaum's equation*, Springer Lect. Notes in Phys. 227 (1985).
[14] Gallavotti, G., *The Elements of Mechanics*, Springer-Verlag, New York 1983.
[15] Goldreich, P. and Peale S., *Spin-orbit coupling in the solar system*, Astron. J. *71*, 425 (1966).

[16] Greene, J. M., *A method for determining a stochastic transition*, J. of Math. Phys. *20*, 1183 (1979).
[17] Kolmogorov, A. N., *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian*, Dokl. Akad. Nauk. SSR *98*, 469 (1954).
[18] Lanford, III, O. E., *Computer assisted proofs in analysis*, Physics A *124*, 465 (1984).
[19] Mather, J., *Nonexistence of invariant circles*, Erg. Theory and Dynam. Systems *4*, 301 (1984).
[20] Moser, J., *On invariant curves of area-preserving mappings of an annulus*, Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. II *1*, 1 (1962).

Abstract

We investigate the stability of the synchronous spin-orbit resonance. In particular we construct invariant librational tori trapping periodic orbits in finite regions of phase space. We first introduce a mathematical model describing a simplification of the physical situation. The corresponding Hamiltonian function has the form $H(y, x, t) = (y^2/2) + \varepsilon V(x, t)$, where V is a trigonometric polynomial in x, t and ε is the "perturbing parameter" representing the equatorial oblateness of the satellite.

We perform some symplectic changes of variables in order to reduce the initial Hamiltonian to a form which suitably describes librational tori. We then apply Birkhoff normalization procedure in order to reduce the size of the perturbation. Finally the application of KAM theory allows to prove the existence of librational tori around the synchronous periodic orbit. Two concrete applications are considered: the Moon-Earth and the Rhea-Saturn systems. In the first case one gets the existence of trapping orbits for values of the perturbing oblateness parameter far from the real physical value by a factor ~ 5. In the Rhea-Saturn case we construct the trapping tori for values of the parameters consistent with the astronomical measurements.

(Received: September 15, 1992; revised: January 14, 1993)