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The cyclic *J*-integral as a criterion for fatigue crack growth

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ABSTRACT
The definition of the cyclic J-integral is offered and its physical significance for fatigue crack growth is discussed using the Dugdale model on the assumption that the crack closure, cycle dependent creep deformation, and crack extension under cycling can be neglected. It is shown that the cyclic J-integral for small scale yielding is equivalent to the J-integral for linear elastic crack independent of loading processes, while the value for large scale yielding varies with the loading processes. However, in both cases, the cyclic J-integral remains constant during the reversal of loading under a constant stress range, if the first monotonic loading stage is excluded. In this situation, the cyclic J-integral can be applied as a criterion for fatigue crack growth, since it is evaluated as a generalized force on dislocations to be moved or the energy flow rate to be dissipated to heat by the dislocation movements in an element just attached to the fatigued crack tip during one cycle of loading. It is suggested that the available experimental data of different materials for fatigue crack growth can be generalized to a unified formulation on the basis of the energy criterion. It is also deduced that the threshold ΔJ corresponding to ΔK_{th} should be larger than 4γ where γ is the surface energy of the material. Finally the operational definition of the cyclic J-integral on single load *versus* displacement curves is given for center cracked plate with wide uncracked ligaments in tension.

1. Introduction

In recent work the J-integral [1] has been established as a failure criterion for stable or unstable crack growth [2, 3]. This conception has been extended to apply for the analysis of fatigue crack growth rates [4–6]. The latter application is based on the view that the path independent J-integral is an average measure of the crack tip elastic-plastic field [2]. However, as pointed out by Dowling and Begley [4], the most relevant unanswered question is whether the J-integral concept has meaning relative to the changes occurring in the crack tip stress and strain fields during the loading half of one fatigue cycle.

This paper offers the definition of the cyclic J-integral and discusses its physical significance for fatigue crack growth using the Dugdale model.

2. Definition of the cyclic J-integral

The J-integral is defined for two-dimensional problems [1] as

$$J = \int_{\Gamma} (W \mathrm{d}x_2 - T_m \,\partial u_m/\partial x_1 \,\mathrm{d}s), \tag{1}$$

where x_1 and x_2 are rectangular coordinates normal to the crack front, x_2 being perpendicular to the crack surface; $\mathrm{d}s$ is an increment of arc length along any contour, Γ, beginning along the bottom surface of the crack and ending along the top surface, T_m is the surface traction exerted on the material within the contour; u_m is the displacement. The

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repeated suffix is summed over the values 1, 2 and 3. The quantity W is the strain energy density;

$$W = W(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{kl}\,\mathrm{d}\varepsilon_{kl}, \tag{2}$$

where σ_{kl} and ε_{kl} are the stress and strain tensors, respectively.

The cyclic J-integral for a non-extending crack relevant to loading change from $(\sigma_A)_i$ to $(\sigma_A)_j$ may be evaluated as

$$\Delta J_{j/i} = \int_\Gamma (\Delta W\,\mathrm{d}x_2 - \Delta T_m\,\partial\Delta u_m/\partial x_1\,\mathrm{d}s), \tag{3}$$

where ΔW, ΔT_m and Δu_m are the relative changes between the values corresponding to the two states and given, respectively, as

$$\Delta W = \int_{(\varepsilon_{mn})_i}^{(\varepsilon_{mn})_j} [\sigma_{kl} - (\sigma_{mn})_i]\,\mathrm{d}\varepsilon_{kl},$$

$$\Delta T_m = (T_m)_j - (T_m)_i \tag{4}$$

and

$$\Delta u_m = (u_m)_j - (u_m)_i.$$

In the description of this paper the subscripts i and j are used for the values relevant to the external applied stresses $(\sigma_A)_i$ and $(\sigma_A)_j$, respectively.

The cyclic J-integral in Eqn. (3) is path independent as proven in Appendix 1, if the strain energy is a single-valued function of the strain during the loading change and if the stress and strain fields outside the end-region [3] are specified for the original state i. The use of a total strain theory will be justified for the situation where the strain field outside a non-extending crack receives approximately proportional loading. Everywhere in the region outside the end-region [3], the effective stress will not decrease for the loading stage from state i to j and it will not increase for the unloading stage.

3. The cyclic J-integral for elastic crack

The cyclic J-integral in (3) for elastic crack has the form

$$\Delta J_{j/i} = \int_\Gamma (\Delta\sigma_{mn}\Delta\varepsilon_{mn}/2\,\mathrm{d}x_2 - \Delta T_m\,\partial\Delta u_m/\partial x_1\,\mathrm{d}s), \tag{5}$$

where $\Delta\sigma_{mn} = (\sigma_{mn})_j - (\sigma_{mn})_i$ and $\Delta\varepsilon_{mn} = (\varepsilon_{mn})_j - (\varepsilon_{mn})_i$.

If Γ is chosen as a circle of radius r approaching zero, only the singular terms specified by stress intensity K contribute to the integral. An explicit calculation based on the difference between the singular terms corresponding to the two states j and i leads to (plane stress)

$$\Delta J_{j/i} = \Delta K_{j/i}^2/E, \tag{6}$$

where $\Delta K_{j/i} = K_j - K_i$.

4. The cyclic J-integral for elastic-plastic crack

If the components of the plastic strain tensor remain in constant proportion to one another at each point of the plastic strain region as in the case of the total strain theory, this permits a

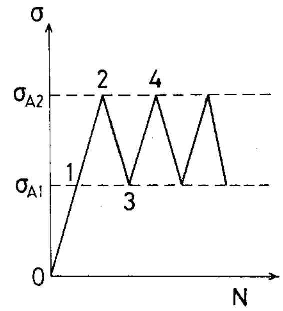

Figure 1. The schematic illustration of load sequence.

general treatment of the response to cyclic loading through the plastic superposition method developed by Rice [7]. On the assumption of the proportional flow, the cyclic J-integral for elastic-plastic crack is calculated using the Dugdale model.

It is supposed that a cracked body undergoes the loading cycles as illustrated in Fig. 1. The resulting deformation procedures occurring at the crack tip are as shown in Fig. 2, where c is the crack length and the region between c and a is the associated plastic zone. At the stage of monotonic loading when the external applied stress is changed from σ_{A1} to σ_{A2}, the plastic zone extends from a_1 to a_2. The change in stresses during this process is evaluated by the subtraction of those for σ_{A2} from those for σ_{A1} (Fig. 2(b)). On unloading from σ_{A2} to σ_{A1}, reverse plastic flow produces a new plastic zone of reversed deformation imbedded in the plastic zone accompanying the original loading (Fig. 2(c)). When flow is proportional, the changes in stresses, strains, and displacements due to load reduction are given by a solution identical to that for original monotonic loading, but with the loading parameter replaced by the load reduction $\Delta\sigma_A = \sigma_{A2} - \sigma_{A1}$ and the yield strain and stress replaced by twice their values for original loading [7] (Fig. 2(d)). The stresses and displacements in the reversed zone, ca_3, are obtained when the changes due to load reduction are subtracted from the distribu-

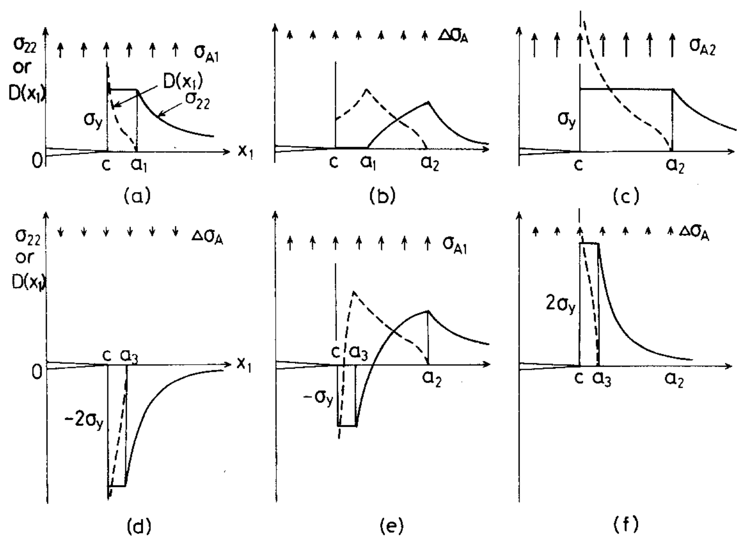

Figure 2. The variations of stress and dislocation distributions at crack tip under load sequence in Fig. 1. The arrow marks indicate the direction of remotely applied stress or stress change. (a) State 1; Loaded state at $\sigma_A = \sigma_{A1}$, (b) State 1 → 2; Loading from σ_{A1} to σ_{A2}, (c) State 2; Loaded state at $\sigma_A = \sigma_{A2}$, (d) State 2 → 3; Unloading from σ_{A2} to σ_{A1}, (e) State 3; Unloaded state at $\sigma_A = \sigma_{A1}$, (f) State 3 → 4; Reloading from σ_{A1} to σ_{A2}.

tions corresponding to the original monotonic loading (Fig. 2(e)). On reloading from σ_{A1} to σ_{A2}, neglecting the possibility of crack closure and crack extension, the plastic superposition is valid up to the point where the reversed plastic zone is equal in size to the original plastic zone accompanying monotonic loading (Fig. 2(f)). Thus, for unloading, reloading and subsequent load cycles, the reversed plastic zone size and cyclic variations in stresses, strains, and displacements depend only on the load fluctuation, $\Delta\sigma_A$ [7].

Following Rice [1], we make the convenient choice of shrinking Γ down to the upper and lower surfaces of the plastic zone for a Dugdale crack, ca_i, in Fig. 2. Since $dx_2 = 0$ in (1) for this choice of Γ, (1) becomes

$$\begin{aligned} J_i &= -\int_\Gamma (T_m)_i\, \partial(u_m)_i/\partial x_1\, ds \\ &= -\int_c^{a_i} (\sigma_{22})_i \partial[v_2(x_1)]_i/\partial x_1\, dx_1 \\ &= \int_c^{a_i} (\sigma_{22})_i [D(x_1))_i\, dx_1, \end{aligned} \tag{7}$$

where σ_{22} is the stress in the plastic zone, $V(x_1)$ is the relative displacement between the upper and lower surfaces of the plastic zone and $D(x_1)\,dx_1$ is the number of dislocations with unit Burgers vector in any distance dx_1. $D(x_1)$ is related to $v(x_1)$ by

$$D(x_1) = -\partial v(x_1)/\partial x_1. \tag{8}$$

Goodier and Field [8] obtain these values for the Dugdale crack under plane stress as

$$\begin{aligned} v(x_1) = (2/\pi)(\sigma_{ys}a/E)\{&\cos\theta \ln[\sin^2(\beta - \theta)/\sin^2(\beta + \theta)] \\ &+ \cos\beta \ln[(\sin\beta + \sin\theta)^2/(\sin\beta - \sin\theta)^2]\} \end{aligned} \tag{9}$$

and

$$D(x_1) = (2/\pi)(\sigma_{ys}/E)\ln[\sin^2(\beta + \theta)/\sin^2(\beta - \theta)], \tag{10}$$

where σ_{ys} is the yield stress, $\theta = \cos^{-1}(x_1/a)$ and

$$\beta = (\pi/2)(\sigma_A/\sigma_{ys}) = \cos^{-1}(c/a). \tag{11}$$

The cyclic J-integral for the Dugdale model during loading change from $(\sigma_A)_i$ to $(\sigma_A)_j$ is given from (3) referring to (7) as

$$\Delta J_{j/i} = \int_c^{a_{j/i}} [(\sigma_{22})_j - (\sigma_{22})_i]\{[D(x_1)]_j - [D(x_1)]_i\}\, dx_1, \tag{12}$$

where the value for larger plastic zone between a_j and a_i is chosen as $a_{j/i}$.

5. Physical interpretation of the J-integral

The J-integrals and the cyclic J-integrals corresponding to the situations as exhibited in Fig. 2 are calculated, where the stress distributions are schematically illustrated by solid curves and the dislocation distributions by dotted ones. The detailed procedures of calculation are shown in Appendix 2. The J_1 and J_2 values and $\Delta J_{3/2}$ value are explicitly given as

$$\begin{aligned} J_1 &= (8/\pi)(\sigma_y^2 c/E)\ln[\sec(\pi\sigma_{A1}/2\sigma_y)], \\ J_2 &= (8/\pi)(\sigma_y^2 c/E)\ln[\sec(\pi\sigma_{A2}/2\sigma_y)] \end{aligned} \tag{13}$$

and

$$\Delta J_{3/2} = (32/\pi)(\sigma_y^2 c/E)\ln[\sec(\pi \Delta\sigma_A/4\sigma_y)].$$

The results of calculation are shown as a function of stress ratio, $R = \sigma_{A2}/\sigma_{A1}$, in Fig. 3. J-integrals are normalized by the elastic J-integral relevant to loading σ_{A2}, K_2^2/E. The calculation was done for two cases; one is for $\sigma_{A2}/\sigma_y = 0.1$, being representative of small scale yielding and the other is for $\sigma_{A2}/\sigma_y = 0.9$, being representative of large scale yielding.

In the case of small scale yielding, J_3 and $\Delta J_{3/2}$ are coincident with J_1 and $\Delta J_{2/1}$, respectively, within the error of computation. These values are almost equal to those for elastic J-values as

$$J_1 \simeq J_3 \simeq K_1^2/E,$$
$$J_2 \simeq K_2^2/E \tag{14}$$

and

$$\Delta J_{3/2} \simeq \Delta J_{2/1} \simeq \Delta K^2/E,$$

where $\Delta K = K_2 - K_1$. This suggests that the J-integral and the cyclic J-integral for small scale yielding are uniquely determined by applied stress as in the case of linear elastic crack. However, as is demonstrated in Fig. 3(b), for large scale yielding J_3 is not equal to J_1 and $\Delta J_{2/1}$ is not equal to $\Delta J_{3/2}$. This indicates that the J-integral and cyclic J-integral for this case are determined depending on loading processes or histories.

The J-integral is a generalized force acting on a crack tip [9] or the energy flow towards the crack tip per unit of crack extension [3]. It is evident in Fig. 3 that the difference in the J-integrals, the generalized forces or the energy flow rates corresponding to different states, $J_j - J_i$, is not equal to $\Delta J_{j/i}$ defined in (3) and (13) except the special case of $\sigma_{Ai} = 0$ for small scale yielding. For linear elastic crack, J reduces to crack extension force or Irwin's linear elastic energy release rate. Thus, $J_j - J_i$, for elastic crack is interpreted as the difference in the crack extension forces between two different states.

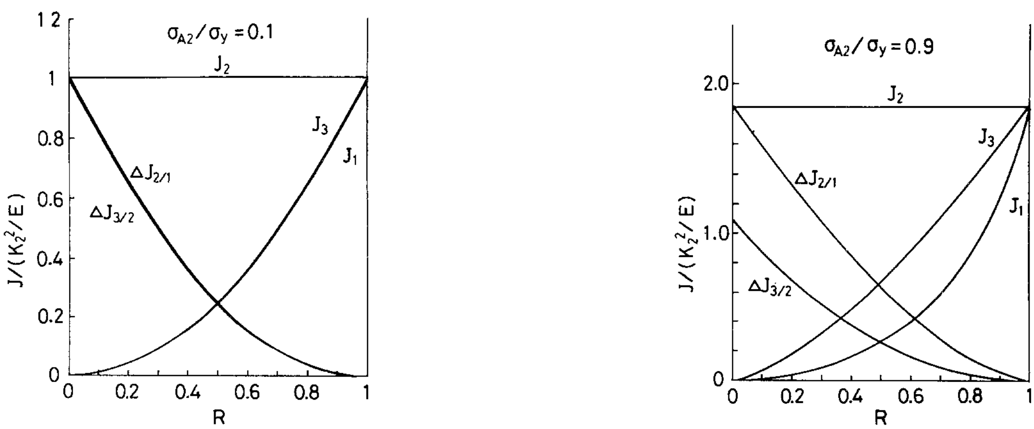

Figure 3. Normalized J-integrals and cyclic J-integrals as a function of stress ratio $R = \sigma_{A2}/\sigma_{A1}$. (a) Small scale yielding ($\sigma_{A2}/\sigma_y = 0.1$), (b) Large scale yielding ($\sigma_{A2}/\sigma_y = 0.9$).

For elastic-plastic materials, the cyclic J-integrals may depend on the loading sequences as shown in Fig. 3(b). Furthermore, even in the case of small scale yielding when the cyclic J-integrals are evaluated to be equal independent of loading sequences, the physical situations occurring at the crack tip are different between the first loading (Fig. 2(b)) and reloading stages (Fig. 2 (f)). However, fortunately the cyclic J-integrals remain constant and equivalent to $\Delta J_{3/2}$ for load cycles unloading from 2 to state 3, reloading from state 3 to 4 and subsequent load cycles, since the reversed plastic zone size and cyclic variations in stresses, strains, and displacements depend only on the load fluctuation [7], when the crack closure, cycle dependent creep deformation, and crack extension under cycling are neglected.

Cyclic plastic straining at the fatigue crack tip motivates the crack growth [7, 10]. Thus, the dislocation movement in the cyclic plastic zone would be the main factor controlling the fatigue crack growth rate. In the case of Dugdale crack, the J-integral defined in (7) is equivalent to the generalized force or the Peach-Koehler force on dislocations in the plastic zone directly ahead of the crack tip [9]. In a similar way, the cyclic J-integral in (12) is defined as the generalized force on dislocations to be moved in the cyclic plastic zone ahead of the crack tip during unloading or reloading. However, the physical situations of the generalized forces between monotonic and cyclic loadings are somewhat different as schematically illustrated in Fig. 4. This shows the relationship between the restraining stress, σ_{22}, and the separation distance, $v(c)$, in the element just attached to the crack tip (at $x_1 = c$). On monotonic loading from state 0 to 2 in Fig. 1, the separation distance varies from 0 to $v(c)_2$ and the plastic work corresponding to area 00′22′ is dissipated to heat as a result of dislocation movements. On unloading from state 2 to 3, the pertinent relationship follows the way, $2 \to 2' \to 2'' \to 3$. In this case, the plastic work corresponding to area 2′2″33′ is dissipated to heat, while the part 22′3′33″ contributes to the reversible potential energy change. The same situation occurs on reloading from state 3 to 4. Hence, the generalized force directly effective in the movement of dislocations in the cyclic zone at the crack tip during unloading from state 2 to 3, $(\Delta J_p)_{3/2}$, is evaluated by

$$(\Delta J_p)_{3/2} = (-\sigma_y)\Delta v(c) \tag{15}$$

and that for reloading from state 3 to 4, $(\Delta J_p)_{4/3}$, by

$$(\Delta J_p)_{4/3} = \sigma_y\{-\Delta v(c)]. \tag{16}$$

Consequently, the generalized force corresponding to the plastic work dissipated to heat in the cyclic plastic zone at the crack tip during one cycle loading is given by

$$(\Delta J_p)_{3/2} + (\Delta J_p)_{4/3} = 2(\Delta J_p)_{3/2} = \Delta J_{3/2}. \tag{17}$$

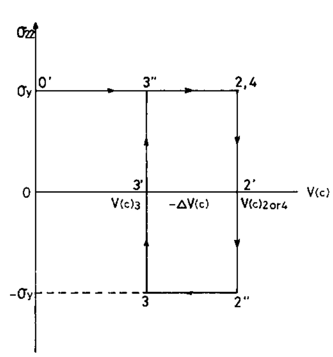

Figure 4. The schematic illustration of relationship between restraining stress *versus* separation distance in an element just attached to a crack tip under monotonic and cyclic loadings.

The right hand relation of this equation is derived in Appendix 2.

According to another explanation of J-integral analogous to that by Broberg [3], (17) may be interpreted as the irreversible part in the energy flow towards the crack-tip during one cycle loading. It is probable that the reversible part contributes to the extension of the crack tip on the loading stage, and the retraction of the crack tip on the unloading stage. Thus, both processes would be compensated by each other and so the reversible part would be totally ineffective for the crack extension during one cycle loading. However, there is a large possibility that the irreversible movements of dislocations at the fatigue crack tip during one cycle loading would result in its extension. This would be related to the irreversible part in the energy flow defined by (17). In other words, (17) is the physical interpretation of the cyclic J-integral for elastic-plastic crack as a fatigue crack growth criterion. Although this interpretation is justified only for the highly simplified elastic-plastic model developed by Rice [7], it is expected one can apply the cyclic J-integral concept for explaining the gross features of elastic-plastic fatique crack growth. That is, the more general expression of (17) is given by

$$\Delta J = \alpha \Delta J_p, \tag{18}$$

where α is a parameter. When the cyclic plastic deformation at the crack tip is sufficiently constrained by internal stress field induced by the monotonic loading as assumed in the Dugdale model, α will be equal to two. This will be strictly fulfilled for small scale yielding condition in all types of specimens and even for large scale yielding condition in a center-cracked plate with sufficiently wide uncracked ligaments in tension. α will be larger than two, when the constraint is not sufficient as in the case of deeply notched plate under large scale yielding. Conversely, α will become smaller than two when the crack closure mechanisms are operated [11].

6. Energy expression of fatigue crack growth data

Bates and Clark [12] pointed out that the striation s *vs.* ΔK curves for various materials can be normalized using the parameter, $\Delta K/E$. Tanaka *et al.* [13] suggested that in order to complete the normalization, it is more reasonable to represent the crack length, c, by c/b and the parameter $\Delta K/E$ by $\Delta K/E\sqrt{b}$, where b is the Burger vector of the material. They showed that a majority of experimental data on $\mathrm{d}c/\mathrm{d}N$ *vs.* ΔK and s *vs.* ΔK curves for three different metals in the intermediate range of crack growth rate at $R = 0$ are coincident by the general formulation,

$$(\mathrm{d}c/\mathrm{d}N)(1/b) = 10^3(\Delta K/10E\sqrt{b})^m \tag{19}$$

and

$$s/b = 10^3(\Delta K/10E\sqrt{b})^2, \tag{20}$$

respectively. These equations can each be expressed in J-integral form by replacing $\Delta K^2/E$ with ΔJ (1) and $Eb/20$ by γ, the surface energy of material, [14] to obtain

$$(\mathrm{d}c/\mathrm{d}N)(1/b) \simeq 10^3\{\Delta J/10^3(2\gamma)\}^{m/2} \tag{21}$$

and

$$s/b \simeq \Delta J/2\gamma, \tag{22}$$

respectively. It is noted that the final form is related to the Griffith criterion when $s = b$ and $\Delta J = J$. These facts indicate that the fatigue crack growth and the striation formation are intrinsically controlled by the energy criterion, even though they are exhibited as continuum phenomena.

The advantage of the energy criterion also appears in the analysis of data on the threshold range of fatigue crack growth rate. Figure 5(a) shows the threshold values of ΔK, ΔK_{th}, against stress ratio, R, for five different metals, which are collected from the literature [15–29]. Although the results are dispersed considerably even in the same material, particularly for the smaller R-value, they are clearly dependent on material; steels and nickel alloys have the highest ΔK_{th}-values, while aluminum alloys the lowest ones. However, they are in excellent agreement with each other, when they are replotted in Fig. 5(b) using the normalized ΔJ expression, $\Delta J_{th}/2\gamma$, where $\Delta J_{th} = (1 - \nu^2)\Delta K^2/E$ on the assumption of plane strain condition. The Young's moduli, E, and Poisson's ratios, ν, and surface energies, γ, for these materials used in this calculation are listed in Table 1 [14–30]. The normalized ΔJ_{th}-values decrease monotonically as R increases from 0 to 0.9, and then they decrease rapidly to nearly two as R approaches one. This seems to be very reasonable as discussed below.

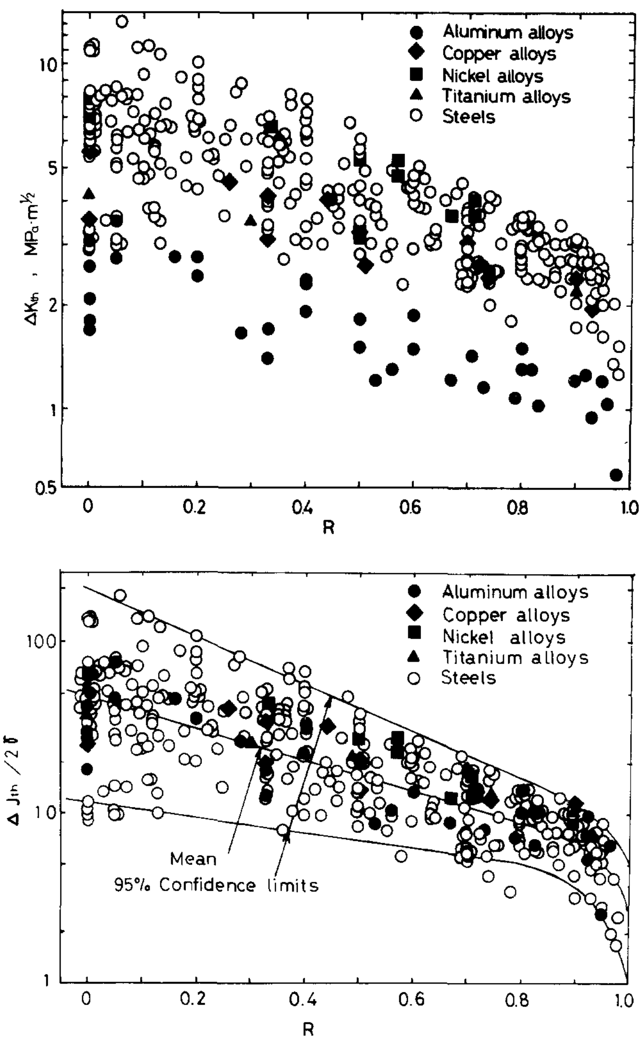

Figure 5. Experimental data on ΔK_{th} against R collected from literature (a) and their normalized ΔJ expression (b).

TABLE 1
Material parameters

Alloys	E (GPa)	ν	γ (Jm^{-2})
Aluminum	70	0.34	1.00
Copper	130	0.34	1.65
Nickel	216	0.30	2.00
Titanium	117	0.34	1.72
Steel	206	0.28	2.00

In the threshold level of fatigue crack growth rate, the crack closure mechanism operates strongly at the smaller R level [31]. The crack closure becomes weaker as R increases [11], which explains the decreasing tendency of $\Delta J_{\mathrm{th}}/2\gamma$-value as the increase of R. The ideal state of non-crack-closure will be attained in the extreme case of $R = 1$. In this state, it can be postulated for the minimal condition for the occurrence of fatigue crack growth that

$$\Delta J_p \geq 2\gamma. \tag{23}$$

That is, the generalized force for plastic work relevant to the movement of dislocations at a fatigue crack tip during half cycle loading should overcome the force for the irreversible creation of metal surface. The latter can be the order of 2γ. Therefore, assuming that $\alpha = 2$ in (18), the ΔJ_{th}-value should satisfy that

$$\Delta J_{\mathrm{th}}/2\gamma \geq 2. \tag{24}$$

7. Estimate of ΔJ from single load-displacement records for the case of Dugdale crack

Values of ΔJ for a non-extending Dugdale crack may be determined from load versus displacement curves as was done by Rice *et al.* [32].

Consider a rectangular plate of width $2W$ with a Dugdale crack or a plane stress crack of length $2c$ centrally located, where $2W \gg 2c$. Suppose the applied force per unit thickness, $2P = 2\sigma_A W$, to be centrally applied and write δ for the load point displacement due to the presence of the crack.

An alternate and equivalent definition of J is given by [32]

$$J = \int_0^P (\partial\delta/\partial c)_p \,\mathrm{d}P$$

or

$$(\partial J/\partial P)_c = (\partial\delta/\partial c)_P. \tag{25}$$

Generally the J-integral for a plane stress crack as for a Dugdale one must have the form [33]

$$J = cf(P/W) \tag{26}$$

where $c \ll W$. Making use of (26) Eqn. (25) becomes

$$(\partial\delta/\partial c)_P = (c/W)f'(P/W) \tag{27}$$

or

$$\delta = (c^2/2W)f'(P/W) = (c/2)(\partial J/\partial P)_c. \tag{28}$$

Integrating (28) leads to

$$J = (2/c)\int_0^P \delta\,dP. \tag{29}$$

Hence

$$\Delta J_{j/i} = (2/c)\int_{P_i}^{P_j} (\delta - \delta_i)\,dP. \tag{30}$$

Values of cyclic J-integral for a Dugdale crack are determined from areas under load *versus* displacement lines during cyclic loading as illustrated in Fig. 6. This is constructed using (29) and (30) for the case that $\sigma_{A1}/\sigma_y = 0.9$ and $R = 0.2$. The δ and cyclic δ are obtained substituting (13) into (28) and (29) as

$$\delta = (2\sigma_y c^2/WE)\tan(\pi\sigma_A/2\sigma_y) \tag{31}$$

and

$$\Delta\delta = (4\sigma_y c^2/WE)\tan(\pi\Delta\sigma_A/4\sigma_y). \tag{32}$$

The operational definition of cyclic J-integral employed is essentially coincident with that by Dowling and Begley [4] on a load *versus* deflection curve for a CT specimen with the neglect of crack closure. However, there are some differences in the evaluations of J-integral between the two types of specimens; center-cracked plate with wide uncracked ligaments in tension (present study) and deeply notched plate subject to bending (Dowling and Begley [4]). The cyclic J-integrals are evaluated from the hatched area divided by the crack length for the former and by the ligament length for the latter. The hatched areas differ by twice the area of hysteresis loops between the two specimens. It is curious that although the cyclic J refers to the general force plastic work in the cyclic plastic zone just attached to the crack tip, the area of hyteresis loops, the total plastic work dissipated in the specimen by the presence of crack during one cycle loading, is not included in the derivation of the cyclic J for the case of the Dugdale model. There is not yet any clear explanation for this rather "paradoxical" result.

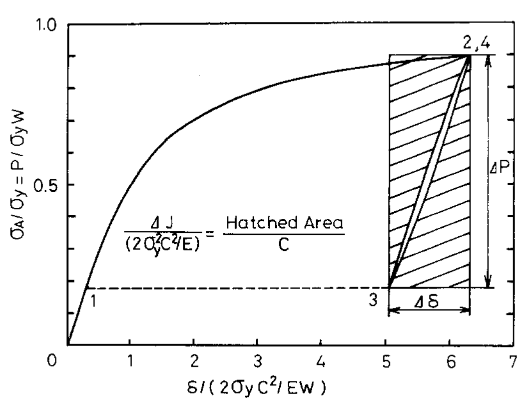

Figure 6. Operational definition of cyclic J-integral for the case that $\sigma_{A2}/\sigma_y = 0.9$ and $R = 0.2$.

8. Concluding remarks

The definition of the cyclic J-integral is offered in (3) and its physical significance for fatigue crack growth is discussed using the Dugdale model. The cyclic J-integral for small scale yielding can be determined independent of loading processes and equivalent to the value for a linear elastic crack as in the case for the "monotonic" J-integral, whereas the cyclic J-integral for large scale yielding depends on the loading processes. However, in both cases, the cyclic deformation occurring in the cyclic plastic zone can be assumed to remain constant during cyclic loading under a constant stress range, if the crack closure, cycle dependent creep deformation and crack extension are neglected. In this situation, the cyclic J-integral is evaluated as a generalized force motivating plastic work dissipated to heat in the cyclic plastic zone during one cycle loading. From this point of view, the cyclic J-integral can be adopted as a criterion for fatigue crack growth. According to this energy conception, the available experimental data of different materials for fatigue crack growth, striation spacing, and the ΔK_{th} values can be generalized respectively to a normalized formulation. It is also deduced that the threshold ΔJ-value ($= (1 - \nu^2)\Delta K_{th}^2/E$) should be larger than 4γ in the extreme case of $R = 1$. Finally it is suggested that the cyclic J-integral for center-cracked plate with wide uncracked ligaments in tension can be determined from single load *versus* displacement record, although in reality the precise measurement of the displacement may be very difficult.

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Appendix 1. Path independence of the cyclic J-integral

The path-independence of the J-integral is shown according to Rice [1] (see p. 210 in his paper).

Consider two curves Γ_1 and Γ_2, suppose Γ_2 to enclose Γ_1, and let ΔJ_1, ΔJ_2 be the associated values of the integral in (3). Then, assuming that the region between Γ_1 and Γ_2 enclosed by the curves and the crack surfaces $A(\Gamma_2, \Gamma_1)$ is simply connected and free of singularities, $\Delta J_2 - \Delta J_1$ is the integral contraclockwise of the integrand in (3) around the boundary of the region, since both terms of the integrand vanish on the flat surfaces. On the assumption that the stress and strain fields in $A(\Gamma_2, \Gamma_1)$ is known for the initial state i, transforming to an area integral and employing Cartesian coordinates

$$\Delta J_2 - \Delta J_1 = \int_{A(\Gamma_2,\Gamma_1)} [\partial \Delta W/\partial x_1 - \Delta\sigma_{kl}\,\partial(\Delta\varepsilon_{kl})/\partial x_1]\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{A1}$$

since

$$\begin{aligned}
\partial \Delta W/\partial x_1 &= \partial/\partial x_1 \int_{(\varepsilon_{mn})_i}^{(\varepsilon_{mn})_j} [\sigma_{kl} - (\sigma_{mn})_i]\,\mathrm{d}\varepsilon_{kl} \\
&= \partial/\partial x_1 \left\{ \int_0^{(\varepsilon_{mn})_j} \sigma_{kl}\,\mathrm{d}\varepsilon_{kl} - \int_0^{(\varepsilon_{mn})_i} \sigma_{kl}\,\mathrm{d}\varepsilon_{kl} - (\sigma_{mn})_i(\varepsilon_{mn})_j \right. \\
&\quad \left. + (\sigma_{mn})_j(\varepsilon_{mn})_i \right\} \\
&= (\sigma_{mn})_j\,\partial(\varepsilon_{mn})_j/\partial x_1 - (\sigma_{mn})_i\,\partial(\varepsilon_{mn})_i/\partial x_1 - (\sigma_{mn})_j\,\partial(\varepsilon_{mn})_i/\partial x_1 \\
&\quad - (\sigma_{mn})_i\,\partial(\varepsilon_{mn})_j/\partial x_1 + 2(\sigma_{mn})_i\,\partial(\varepsilon_{mn})_i/\partial x_1 \\
&= [(\sigma_{mn})_j - (\sigma_{mn})_i][\partial(\varepsilon_{mn})_j/\partial x_1 - \partial(\varepsilon_{mn})_i/\partial x_1] \\
&= \Delta\sigma_{kl}\,\partial\Delta\varepsilon_{kl}/\partial x_1.
\end{aligned}$$

Appendix 2. The cyclic J-integral calculation for elastic plastic crack

The "monotonic" and cyclic J-integrals for loading sequence from states 1 through 4 in Fig. 1 are calculated using (7) through (12).

1. *State 1*

The prescribed stresses are given in Fig. 2(a), where $\sigma_A = \sigma_{A1}$, $\sigma_{ys} = \sigma_y$, $(\sigma_{22})_1 = \sigma_y$. Hence

$$J_1 = \sigma_y[v(c)]_1 = (8/\pi)(\sigma_y^2 c/E)\ln(a_1/c), \tag{A2}$$

where

$$a_1/c = \sec(\pi\sigma_{A1}/2\sigma_y).$$

2. *State 2*

The prescribed stresses are given in Fig. 2(c), where $\sigma_A = \sigma_{A2}$, $\sigma_{ys} = \sigma_y$ and $(\sigma_{22})_2 = \sigma_y$. Hence

$$J_2 = \sigma_y[v(c)]_2 = (8/\pi)(\sigma_y^2 c/E)\ln(a_2/c), \tag{A3}$$

where

$$a_2/c = \sec(\pi_{A2}/2\sigma_y).$$

3. *State 1 → 2*

The prescribed stresses are given in Fig. 2(b).

$$\begin{aligned}\Delta J_{2/1} &= \int_c^{a_2} [(\sigma_{22})_2 - (\sigma_{22})_1]\{[D(x_1)]_2 - [D(x_1)]_1\}\,dx_1 \\ &= \int_{a_1}^{a_2} [\sigma_y - (\sigma_{22})_1][D(x_1)]_2\,dx_1.\end{aligned} \tag{A4}$$

$[D(x_1)]_2$ is evaluated by substituting $\sigma_{ys} = \sigma_y$, $\theta = \cos^{-1}(x_1/a_2)$ and $\beta = \cos^{-1}(c/a_2)$ into (10). $(\sigma_{22})_1$ is the stress outside the plastic zone produced by the application of load, σ_{A1}. The latter stress can be solved analytically, but has a very complicated form as given by Hahn and Rosenfield [34]. Equation (A4) is calculated numerically using (10) and the analytical equation given by Hahn and Rosenfield.

4. *State 2 → 3*

The prescribed stresses are given in Fig. 2(d). Since the changes in the distributions of dislocation, $\Delta D(x_1)$, and displacement, $\Delta v(x_1)$, accompanying the state change from 2 to 3 are given substituting $\sigma_A = -\Delta\sigma_A$, $\sigma_{ys} = -2\sigma_y$, and $(\sigma_{22})_2 = -2\sigma_y$ into (9) and (10).

$$\begin{aligned}\Delta J_{3/2} &= \int_c^{a_2} [(\sigma_{22})_3 - (\sigma_{22})_2\{[D(x_1)]_3 - [D(x_1)]_2\}\,dx_1 \\ &= \int_c^{a_3} (-2\sigma_y)\Delta D(x_1)\,dx_1 = (-2\sigma_y)\Delta v(c) \\ &= (8/\pi)(4\sigma_y^2 c/E)\ln(a_3/c),\end{aligned} \tag{A5}$$

where

$$a_3/c = \sec(\Delta\sigma_A/4\sigma_y).$$

5. *State 3*

The prescribed stresses are given in Fig. 2(e). Since $D(x_1)_3 = [D(x_1)]_2 + \Delta D(x_1)$ and $(\sigma_{22})_3 = (\sigma_{22})_2 + \Delta\sigma_{22}$,

$$J_3 = \int_c^{a_2} (\sigma_{22})_3[D(x_1)]_3\,dx_1$$

$$= \int_{c}^{a_3} (\sigma_y - 2\sigma_y)\{[D(x_1)]_2 + \Delta D(x_1)\}\,\mathrm{d}x_1$$
$$+ \int_{a_3}^{a_2} (\sigma_y + \sigma_{22})[D(x_1)]_2\,\mathrm{d}x_1. \tag{A6}$$

$\Delta\sigma_{22}$ in the right hand of (A6) is the stress field outside the cyclic plastic zone produced by the application of load fluctuation, $-\Delta\sigma_A$, and calculated under the situation as illustrated in Fig. 2(d) using the Hahn and Rosenfield solution. The integral was done in a similar way to that in the calculation of (A4).

6. *State 3 → 4*

The calculation was identical to that in (A5), but the sign of stresses was conversed. Thus

$$\Delta J_{4/3} = (2\sigma_y)[-\Delta v(c)] = \Delta J_{3/2}. \tag{A7}$$

RÉSUMÉ
On présente la définition de l'intégrale J cyclique et sa signification physique dans le cas d'une croissance de fissure de fatigue est discutée en utilisant le modèle de Dugdale et l'hypothèse que la fermeture de la fissure, la déformation de fluage liée au cycle, et l'extension de la fissure au cours du cycle peuvent être négligées. On montre que l'intégrale J cyclique est, dans le cas d'écoulement plastique à faible échelle, équivalente à l'intégrale J utilisée pour les fissures élastiques linéaires indépendamment des processus de mise en charge, tandis que la valeur de déformation plastique macroscopique varie avec le processus de mise en charge. Cependant, dans les 2 cas, l'intégrale J cyclique demeure constante au cours du renversement de charge appliquée à tension constante si l'on exclut le premier stade de mise en charge monotone. Dans cette situation, l'intégrale J cyclique peut être appliquée comme critère de la croissance d'une fissure de fatigue puisqu'elle représente une forme généralisée de force, un effort sur les dislocations à mouvoir ou à l'écoulement d'énergie dissipée en chaleur par les mouvements de dislocation dans un élément immédiatement solidaire de l'extrémité de la fissure de fatigue, au cours d'un cycle de mise en charge. On suggère que les données expérimentales disponibles sur différents matériaux pour la croissance d'une fissure de fatigue puissent être généralisées à une formulation unifiée, sur base d'un critère d'énergie. On déduit également que le seuil ΔJ correspondant à ΔK_{th} devrait être plus grand que 4γ, où γ est l'énergie de surface du matériau. Finalement, la définition opérationnelle de l'intégrale J cyclique sous simple charge en fonction des courbes de déplacement est fournie dans le cas d'une plaque comportant une fissure centrale et de larges ligaments non fissurés, soumises à traction.