

TECHNICAL NOTE

Games with Vector Payoffs

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Abstract. Two-person games are defined in which the payoffs are vectors. Necessary and sufficient conditions for optimal mixed strategies are developed, and examples are presented.

Key Words. Game theory, vector maximization, Pareto optimality, equilibrium points, minimax theorems.

1. Introduction

An unresolved question in game theory has been whether there exists a theory for vector payoffs similar to standard results. Blackwell established in Ref. 1 an asymptotic analog to the minimax theorem for such payoffs, however, that is used in repeated games. Recent work by Nieuwenhuis (Ref. 2) and Corley (Ref. 3) suggests a generalization to vector payoffs using the following notion of vector maximization (or efficiency or Pareto optimality). Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in D \subset R^n$. If $u_i \leq v_i$, $i = 1, \dots, n$, and $u_j < v_j$, for some j , we write $u < v$ or $v > u$. The point $u \in D$ is said to be a vector maximum of D , denoted $u \in v \max D$, if $u \not< v$, for all $v \in D$. Vector minima and $v \min D$ have similar definitions.

A two-person (noncooperative) bimatrix vector-valued game is defined as follows. Player I has r strategies and Player II s strategies. The payoff with respect to I is represented by an $r \times s$ matrix $A = [a_{ij}]$ of n -tuples $a_{ij} = (a_{ij}^1, \dots, a_{ij}^n) \in R^n$, so A comprises the n real payoff matrices $A_k = [a_{ij}^k]$, $k = 1, \dots, n$. Similarly, the payoff with respect to II is represented by an $r \times s$ matrix $B = [b_{ij}]$ of n -tuples $b_{ij} = (b_{ij}^1, \dots, b_{ij}^n) \in R^n$ determining n real payoff matrices $B_k = [b_{ij}^k]$, $k = 1, \dots, n$. Thus, when I plays his i th strategy and II his j th strategy, the payoff is $(a_{ij}^1, \dots, a_{ij}^n)$ to I and $(b_{ij}^1, \dots, b_{ij}^n)$ to II. Mixed strategies are allowed as usual. Player I assigns a probability x_i

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to choosing strategy i , and II assigns y_j to strategy j . These mixed strategies are given by the column vectors by the column vectors $x = (x_1, \dots, x_r)'$ and $y = (y_1, \dots, y_s)'$ that are members, respectively, of the sets

$$X = \left\{ x: \sum_{i=1}^r x_i = 1, x_i \geq 0, i = 1, \dots, r \right\},$$

$$Y = \left\{ y: \sum_{j=1}^s y_j = 1, y_j \geq 0, j = 1, \dots, s \right\}.$$

The expected payoff of this game is therefore

$$x'Ay = \sum_{i=1}^r \sum_{j=1}^s (x_i y_j a_{ij}^1, \dots, x_i y_j a_{ij}^n) = (x'A_1 y, \dots, x'A_n y), \quad (1)$$

for I, and

$$x'By = \sum_{i=1}^r \sum_{j=1}^s (x_i y_j b_{ij}^1, \dots, x_i y_j b_{ij}^n) = (x'B_1 y, \dots, x'B_n y), \quad (2)$$

for II. The strategy pair (\hat{x}, \hat{y}) is said to be an equilibrium point for this game if

$$\hat{x}'A\hat{y} \preceq x'A\hat{y}, \quad x \in X, \quad (3)$$

and

$$\hat{x}'B\hat{y} \preceq \hat{x}'By, \quad y \in Y; \quad (4)$$

the associated expectations $\hat{x}'A\hat{y}$ and $\hat{x}'B\hat{y}$ are called equilibrium values. When $B = -A$, the result is a zero-sum vector-valued game. In that case (3), (4) can be combined into

$$x'A\hat{y} \succeq \hat{x}'A\hat{y} \succeq \hat{x}'Ay, \quad x \in X, y \in Y, \quad (5)$$

and $\hat{x}'A\hat{y}$ is the equilibrium value. It is obvious that the above definitions reduce to the standard game-theoretic concepts (see Refs. 4 and 5) when $n = 1$.

Two related notions for zero-sum vector-valued games are generalized minimax and maximin points. A strategy pair (\hat{x}, \hat{y}) for a zero-sum vector-valued game is said to be a minimax point if $\hat{x}'A\hat{y}$ is a member of

$$v \min \left[\bigcup_{y \in Y} v \max \{ x'Ay: x \in X \} \right]; \quad (6)$$

$\hat{x}'Ay$ is the associated minimax value. A maximin point is similarly defined with respect to the set

$$v \max \left[\bigcup_{x \in X} v \min \{ x'Ay: y \in Y \} \right]. \quad (7)$$

These definitions again reduce to the usual ones when $n = 1$, in which case the concepts of an equilibrium, minimax, and maximin point are equivalent due to the minimax theorem (see Ref. 4).

In this note, we present the following results. In Section 2, we establish necessary and sufficient conditions for a strategy pair (x, y) to be an equilibrium point of a bimatrix vector-valued game. The existence of equilibrium points and the fact that a bimatrix vector-valued game is equivalent to a parametric linear complementarity problem are corollaries. In Section 3, examples are given in which some equilibrium points are computed and certain differences between standard games and vector-valued games are illustrated. In particular, the minimax theorem does not have a direct generalization to (6) and (7).

2. Results

We first state an immediate consequence of (3), (4), and the definition of a vector maximum.

Lemma 2.1. Let

$$X_y = \{\hat{x} \in X : \hat{x}'Ay \in v \max\{x'Ay : x \in X\}\},$$

$$Y_x = \{\hat{y} \in Y : x'B\hat{y} \in v \max\{x'By : y \in Y\}\}.$$

Then, (\hat{x}, \hat{y}) is an equilibrium point of a bimatrix vector-valued game if and only if $\hat{x} \in X_{\hat{y}}$ and $\hat{y} \in Y_{\hat{x}}$.

Theorem 2.1. A strategy pair (x, y) is an equilibrium point for a bimatrix vector-valued game if and only if there exist scalars $\hat{\eta}, \theta, \hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\beta}_1, \dots, \hat{\beta}_n, \hat{\lambda}_1, \dots, \hat{\lambda}_r, \hat{\mu}_1, \dots, \hat{\mu}_s$ for which (\hat{x}, \hat{y}) and these scalars satisfy

$$\sum_{k=1}^n \sum_{j=1}^s \alpha_k a_{ij}^k y_j + \lambda_i = \eta, \quad i = 1, \dots, r, \tag{8}$$

$$\sum_{i=1}^r x_i = 1, \tag{9}$$

$$\lambda_i x_i = 0, \quad i = 1, \dots, r, \tag{10}$$

$$\lambda_i x_i \geq 0, \quad i = 1, \dots, r, \tag{11}$$

$$\sum_{k=1}^n \sum_{i=1}^r \beta_k b_{ij} x_i + \mu_j = \theta, \quad j = 1, \dots, s, \tag{12}$$

$$\sum_{j=1}^s y_j = 1, \quad (13)$$

$$\mu_j y_j = 0, \quad j = 1, \dots, s, \quad (14)$$

$$\mu_j y_j \geq 0, \quad j = 1, \dots, s, \quad (15)$$

$$\alpha_k, \beta_k > 0, \quad k = 1, \dots, n. \quad (16)$$

Proof. Since $x'Ay$ is linear in x for fixed y and since $x'By$ is linear in y for fixed x , it follows from Lemma 2.1 and Ref. 6 that (\hat{x}, \hat{y}) is an equilibrium point if and only if there exist scalars $\hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\beta}_1, \dots, \hat{\beta}_n > 0$ [i.e., condition (16)] such that \hat{x} solves the scalar linear program

$$\underset{x}{\text{maximize}} \quad x' \left(\sum_{k=1}^n \hat{\alpha}_k A_k \right) \hat{y}, \quad (17)$$

$$\text{subject to} \quad \sum_{i=1}^r x_i = 1, \quad (18)$$

$$x_i \geq 0, \quad i = 1, \dots, r, \quad (19)$$

and \hat{y} solves the scalar linear program

$$\underset{y}{\text{maximize}} \quad \hat{x}' \left(\sum_{k=1}^n \hat{\beta}_k B_k \right) y, \quad (20)$$

$$\text{subject to} \quad \sum_{j=1}^s y_j = 1,$$

$$y_j \geq 0, \quad j = 1, \dots, s.$$

Consider problem (17). The linearity in x of both the objective function and the constraints implies that the Kuhn-Tucker conditions for (17) are both necessary and sufficient (see Ref. 7). These conditions are (8)-(12), where η is associated with (18) and the λ_i with (19). Conditions (12)-(16) are similarly obtained from problem (20) to complete the proof. \square

The multipliers η and θ in (8) and (12) represent, respectively, the optimal values of the objective functions in (17) and (20) for fixed α_k, β_k and can be algebraically eliminated. All equilibrium points can be obtained theoretically from (8)-(16) as, say, in Ref. 8 by varying the $\alpha_k, \beta_k > 0$. Obviously a large, even infinite, number of equilibrium points may be obtained (as illustrated in the next section), and thus a player might place secondary criteria on his strategies. If such criteria are formulated as linear constraints (as in goal programming), the proof of Theorem 2.1 can be extended easily to account for these additional constraints. It is for this reason that the Kuhn-Tucker conditions were applied in the above proof

after the scalarizations (17) and (20). Otherwise, the following consequence of these scalarizations and Ref. 9 is perhaps preferable, where e_m denotes an m -dimensional column vector of 1's.

Corollary 2.1. Suppose that

$$a_{ij}^k, b_{ij}^k > 0, \quad i = 1, \dots, r, j = 1, \dots, s, k = 1, \dots, n.$$

Then, a bimatrix vector-valued game is equivalent to the following parametric linear complementarity problem with parameters $\alpha_k, \beta_k > 0, k = 1, \dots, n$,

$$\begin{bmatrix} 0 & \sum_{k=1}^n \beta_k B_k^t \\ \sum_{k=1}^n \alpha_k A_k & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} - \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} e_s \\ e_r \end{bmatrix}, \tag{21}$$

$$\begin{aligned} x_i z_i &= 0, & i &= 1, \dots, r, \\ y_j w_j &= 0, & j &= 1, \dots, s, \\ x_i, z_i &\geq 0, & i &= 1, \dots, r, \\ y_j, w_j &\geq 0, & j &= 1, \dots, s. \end{aligned}$$

A positive constant can be added to the elements of the A_k, B_k without affecting the equilibrium points, so Corollary 2.1 is a general equivalence. Any method for solving linear complementarity problems (as in Ref. 7) may be applied; a solution (x, y) to (21) determines an equilibrium point $(x/x'e_r, y/y'e_s)$. The scalarizations in Theorem 2.1 also establish the existence of equilibrium points for vector-valued games from existence results for scalar games in, for example, Ref. 10.

Corollary 2.2. There exists an equilibrium point for a bimatrix vector-valued game.

3. Examples

Two examples are now presented. For simplicity, both represent the zero-sum case. In Example 3.1, all equilibrium points are obtained utilizing Theorem 2.1, and it is demonstrated that two scalarizations (i.e., both the α_k and β_k) are needed to obtain all equilibrium points. Example 3.2 illustrates some differences between vector-valued and scalar games; for instance, (6) and (7) are not necessarily equal.

Example 3.1. Consider a zero-sum game with

$$A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}.$$

Conditions (8)-(16) can be directly solved to yield the set of equilibrium points

$$\begin{aligned} & \{(x, y): 0 \leq x_1 < \frac{1}{3}, \frac{2}{3} < x_1 \leq 1, x_2 = 1 - x_1; 0 \leq y_1 < \frac{1}{3}, \frac{2}{3} < y_1 \leq 1, y_2 = 1 - y_1\} \\ & \cup \{(x, y): \frac{1}{3} < x_1 < \frac{2}{3}, x_2 = 1 - x_1; y_1 = 0, y_2 = 1\} \\ & \cup \{(x, y): x_1 = 1, x_2 = 0; \frac{1}{3} < y_1 < \frac{2}{3}, y_2 = 1 - y_1\}. \end{aligned}$$

In particular,

$$\hat{x} = (0, 1), \quad \hat{y} = (0, 1)$$

determine an equilibrium point. But (\hat{x}, \hat{y}) cannot be obtained with a single scalarization (i.e., $\alpha_k = \beta_k > 0$), since the substitution

$$x_2 = 1 - x_1, \quad y_2 = 1 - y_1$$

would require

$$x_1(2\alpha_1 - \alpha_2) \leq 0 \leq y_1(\alpha_1 - 2\alpha_2),$$

for

$$0 \leq x_1 \leq 1, \quad 0 \leq y_1 \leq 1.$$

This inequality dictates $\alpha_1 = \alpha_2 = 0$, a scalarization that says that any strategy (x, y) is an equilibrium point. We conclude that in the linear case, as in the nonlinear case studied by Nieuwenhuis (Ref. 2), a single scalarization cannot yield all equilibrium points. He did not resolve the issue, however.

Example 3.2. Consider a zero-sum game with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

With the substitution

$$x_2 = 1 - x_1, \quad y_2 = 1 - y_1,$$

we have

$$x^t A y = (x_1 + y_1 - 2x_1 y_1, 1 - y_1 + x_1 y_1). \tag{22}$$

For this game, we show that, contrary to the case for scalar games as discussed in Ref. 4, two equilibrium points $(\hat{x}, \hat{y}), (\bar{x}, \bar{y})$ do not necessarily

determine an equilibrium point (\hat{x}, \hat{y}) . To see this fact, observe that

$$\hat{x} = (1, 0)', \quad \hat{y} = (1, 0)'$$

determine an equilibrium point for (22), as does

$$\bar{x} = (\frac{1}{4}, \frac{3}{4})', \quad \bar{y} = (\frac{3}{4}, \frac{1}{4})'.$$

The strategy pair (\hat{x}, \hat{y}) , however, is not an equilibrium point.

We next illustrate that a strategy pair (x, y) yielding an equilibrium value $x'Ay$ may not itself be an equilibrium point. As before, note that the pair

$$\hat{x} = (1, 0), \quad \hat{y} = (1, 0)$$

is an equilibrium point for (22) giving

$$\hat{x}'A\hat{y} = (0, 1).$$

The pair

$$\bar{x} = (0, 1), \quad \bar{y} = (0, 1)$$

also gives

$$\bar{x}'A\bar{y} = (0, 1).$$

But (\bar{x}, \bar{y}) is not an equilibrium point.

This game also demonstrates that the minimax theorem for scalar games does not generalize to the equality of (6) and (7) for $n \geq 2$. It is an interesting exercise to deduce that, for this game, (6) becomes the set

$$M = \{(\xi, 1 - \xi): 0 \leq \xi \leq 1\}$$

and (7) the (nonclosed) set

$$N = \{2\xi^2 - 2\xi + 1, 2\xi - \xi^2\}: 0 \leq \xi < \frac{1}{2}\}.$$

It is evident that M and N are not equal. One explanation is that the inner vector extremizations in (6) and (7) cannot be formulated as linear constraints, and thus the duality theory of linear programming (either the standard theory or that of Ref. 3) cannot be applied.

4. Remarks

Vector-valued games involving Pareto optimality have been defined here; extensions include n -person games, differential games, and vector criteria other than Pareto optimality. There are also two areas of this note in which further work is needed. First, there may be a way to obtain all

equilibrium points without resorting to the impossible task of solving all possible scalarizations. It is conceivable that one could obtain parametric solutions (x, y) to (21) in terms of the α_k, β_k . Second, it is not clear how minimax and maximin points are related to equilibrium points, except that a joint minimax and maximin point is obviously an equilibrium point. Determining the relationship might lead to a suitable generalization of the minimax theorem for $n \geq 2$.

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