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On Stochastic Games with Additive Reward and Transition Structure

T. E. S. RAGHAVAN,[1] S. H. TIJS,[2] AND O. J. VRIEZE[3]

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Abstract. In this paper, we introduce a new class of two-person stochastic games with nice properties. For games in this class, the payoffs as well as the transitions in each state consist of a part which depends only on the action of the first player and a part dependent only on the action of the second player.

For the zero-sum games in this class, we prove that the orderfield property holds in the infinite-horizon case and that there exist optimal pure stationary strategies for the discounted as well as the undiscounted payoff criterion. For both criteria also, finite algorithms are given to solve the game. An example shows that, for nonzero sum games in this class, there are not necessarily pure stationary equilibria. But, if such a game possesses a stationary equilibrium point, then there also exists a stationary equilibrium point which uses in each state at most two pure actions for each player.

Key Words. Game theory, stochastic games, pure stationary optimal strategies, additive stochastic games.

1. Introduction

We consider *stochastic games* of the form

$$\Gamma = \langle S, \{A_s; s \in S\}, \{B_s; s \in S\}, r, p \rangle.$$

Here, $S := \{1, 2, \ldots, z\}$ is the *state space* and the finite sets A_s and B_s are the *action spaces*, available to the players I and II, respectively, in state s.

[1] Professor, Department of Mathematics, University of Illinois, Chicago, Illinois.

[2] Professor, Department of Mathematics, Catholic University, Nijmegen, The Netherlands.

[3] Research Associate, Department of Mathematics, Catholic University, Nijmegen, The Netherlands.

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Further, $r=(r_1, r_2)$ is a vector-valued function with domain

$$T := \{(s, i, j);\ s \in S,\ i \in A_s, j \in B_s\}$$

and range $\mathbb{R}^2$, where r_1 and r_2 are the *reward functions* of players I and II, respectively. Finally,

$$p = \{p(t \mid s, i, j);\ t \in S, (s, i, j) \in T\}$$

prescribes the *law of motion*, where $p(t \mid s, i, j)$ denotes the probability that the state moves from s to t when i and j are the actions of the players in state s. Of course, for the *transition probabilities*, we have

$$p(t \mid s, i, j) \geq 0 \quad \text{and} \quad \sum_t p(t \mid s, i, j) = 1,$$

for all $(s, i, j) \in T$.

We say that the game Γ possesses *additive rewards* if, for all $(s, i, j) \in T$,

$$r_1(s, i, j) = r_{11}(s, i) + r_{12}(s, j),$$
$$r_2(s, i, j) = r_{21}(s, i) + r_{22}(s, j),$$

for some functions r_{11}, r_{12}, r_{21}, r_{22} on the appropriate domain. The game Γ is said to be *controlled by one player*, say player II, if

$$p(t \mid s, i, j) = p(t \mid s, i', j),$$

for all $i, i' \in A_s$ and all $s, t \in S$ and $j \in B_s$. We write

$$p(t \mid s, i, j) = p(t \mid s, j),$$

if no confusion is possible. Thus, in such games, the transition probabilities are not influenced by player I. The game Γ is said to be a *switching control game* if the states can be partitioned into two sets S_1 and S_2, such that

$$\text{for } s \in S_1,\ p(t \mid s, i, j) = p(t \mid s, i),$$
$$\text{for } s \in S_2,\ p(t \mid s, i, j) = p(t \mid s, j).$$

That is, the law of motion from states of S_1 is independent of the action of player II, and similarly the law of motion from states of S_2 is independent of the actions of player I.

The game Γ is said to possess *additive transitions* if, for all $(s, i, j) \in T$,

$$p(t \mid s, i, j) = p_1(t \mid s, i) + p_2(t \mid s, j),$$

where p_1 is a function of the state and the action of player I and p_2 is a function of the state and the action of player II.

The game is called a *zero-sum game* if $r_1 + r_2 = 0$ on T. Otherwise, the game is called a *nonzero-sum game*. In the following, we suppose that the players have an infinite horizon. A play proceeds as usual in stochastic games (cf. Ref. 1). We will be concerned with both discounted and undiscounted payoffs. The state and actions on the τth day will be denoted by s_τ, i_τ, and j_τ.

A *stationary strategy* f for player I consists of a z-tuple $f = (f_1, f_2, \ldots, f_z)$, where f_s is a probability distribution on A_s. Intuitively, this means that, when the game is in s, player I, when adopting f as strategy, chooses an action according to f_s. A *behavioral strategy* μ is a sequence $\mu = (\mu_0, \mu_1, \ldots)$ where, on the τth day, $\mu_\tau(s_0, i_0, j_0, s_1, i_1, j_1, \ldots, i_{\tau-1}, j_{\tau-1}, s_\tau)$ is a probability distribution on A_{s_τ}, which depends on the history $h_\tau = (s_0, i_0, j_0, s_1, \ldots, j_{\tau-1}, s_\tau)$ up to the τth day. A stationary strategy f for player I is called *pure* if f_s is degenerate for each $s \in S$; i.e., f_s selects a particular action with probability 1. Let g and ν be similarly defined as stationary and behavioral strategy, respectively, for player II.

Let $V_\beta(\mu, \nu)(s)$ denote the pair of *expected β-discounted rewards* when μ and ν are the strategies of players I and II, s is the starting state, and the discount factor equals $\beta \in [0, 1)$. Thus,

$$V_\beta(\mu, \nu)(s) = \left(E_{\mu\nu s}\left(\sum_{\tau=0}^{\infty} \beta^\tau r_1(s_\tau, i_\tau, j_\tau) \right), E_{\mu\nu s}\left(\sum_{\tau=0}^{\infty} \beta^\tau r_2(s_\tau, i_\tau, j_\tau) \right) \right).$$

Here, $E_{\mu\nu s}$ denotes the expectation with respect to μ, ν and initial state s. The second important evaluation rule in vogue is the so-called *undiscounted payoff* (or *average payoff*), defined by

$$V_1(\mu, \nu)(s) = \left(\liminf_{T\to\infty} E_{\mu\nu s}\left(\frac{1}{T+1} \sum_{\tau=0}^{T} r_1(s_\tau, i_\tau, j_\tau) \right), \right.$$

$$\left. \liminf_{T\to\infty} E_{\mu\nu s}\left(\frac{1}{T+1} \sum_{\tau=0}^{T} r_2(s_\tau, i_\tau, j_\tau) \right) \right).$$

Here again, $s_0 = s$ is the starting state. When $r_1 + r_2 = 0$, we denote by V_β, V_1, etc., the payoff corresponding to player I.

In this paper, special attention is paid to the class of stochastic games with additive rewards and additive transitions (ARAT *games*). Zero-sum ARAT games turn out to have nice optimal strategies, and there are simple

algorithms to solve such games, as we will see in Section 2. In Section 3, some results for nonzero sum ARAT games are derived.

2. Zero-Sum Case

A zero-sum stochastic game is said to possess a *value* if, for each $s \in S$,

$$\inf_{\nu} \sup_{\mu} V_\beta(\mu, \nu)(s) = \sup_{\mu} \inf_{\nu} V_\beta(\mu, \nu)(s) =: V_\beta(s). \tag{1}$$

Here, Eq. (1) corresponds to the undiscounted case if $\beta = 1$. Strategies μ^* and ν^* for players I and II, respectively, are called *optimal strategies* if, for each $s \in S$,

$$\inf_{\nu} V_\beta(\mu^*, \nu)(s) = V_\beta(s), \sup_{\mu} V_\beta(\mu, \nu^*)(s) = V_\beta(s). \tag{2}$$

Shapley introduced stochastic games and showed in his fundamental paper (Ref. 1) that β-discounted stochastic games have a value and that both players possess stationary strategies which are optimal for each starting state. That is, inf and sup can be replaced by min and max in Eq. (1) for $\beta \in [0, 1)$. However, for undiscounted stochastic games, the existence of a value was unknown till recently (cf. Mertens and Neyman, Ref. 2). In general, however, optimal strategies even in the class of behavioral strategies may not exist for this evaluation rule. Thus, without further restrictions on the rewards or the law of motion, one cannot hope for stationary optimal strategies.

If stochastic games have to be solved in finite steps, one has to hope for the orderfield property. A zero-sum stochastic game is said to have the *orderfield property* if the coordinates $V_\beta(s)$ of the value of the game and the coordinates of suitable optimal strategies lie in the same ordered subfield of the reals as the data of the stochastic game.

Stern (Ref. 3), in his PhD thesis, first proved the existence of a value in stationary strategies for undiscounted stochastic games controlled by one player. Parthasarathy and Raghavan (Ref. 4) showed that, for this class for both discounted and undiscounted payoffs, the orderfield property holds. Also, they gave a linear programming algorithm for solving these games, when the payoffs are discounted. Vrieze (Ref. 5) and independently Hordijk and Kallenberg (Ref. 6) gave a linear programming algorithm for solving these games with undiscounted payoffs. Filar (Ref. 7) proved the existence of a value in stationary strategies for switching control stochastic games, and he proved that the orderfield property holds also for this class. Vrieze *et al.* (Ref. 8) have given a finite-step algorithm to solve these switching control games.

In looking for stochastic games with the orderfield property and optimal stationary strategies, one needs conditions on the immediate payoffs, or the transition probabilites, or both. We note that one-player control games and switching control games can be considered as subclasses of games with additive transition functions. For such games,

$$p(t \mid s, i, j) = p_1(t \mid s, i) + p_2(t \mid s, j);$$

and, if $p_1 = 0$, then such a game reduces to a player II controlling stochastic game; and if, for each $s \in S$,

$$p_1(t \mid s, i) = 0, \qquad \text{for all } (t, i) \in S \times A_s,$$

or

$$p_2(t \mid s, j) = 0, \qquad \text{for all } (t, j) \in S \times B_s,$$

then the game corresponds to a switching control stochastic game. Thus, a natural question to ask is whether games with additive transition functions admit stationary optimal strategies and whether the orderfield property holds for this more general class.

Shapley's theorem implies the existence of stationary optimal strategies in the discounted case. For the undiscounted case, we have good indications that optimal stationary strategies eixst. However, we have not yet been able to prove this. The following example shows that the orderfield property does not hold.

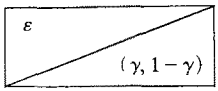
Example 2.1. Let

$$p^1 = q^1 = (1, 0) \quad \text{and} \quad p^2 = q^2 = (0, 1).$$

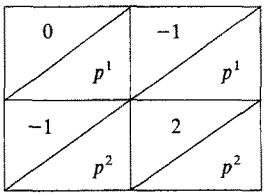
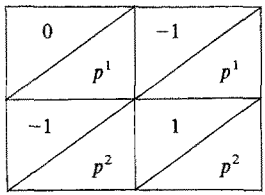
Consider the zero-sum stochastic game with 2 states given by

0	−1		0	−1			
$\frac{1}{2}(p^1+q^1)$	$\frac{1}{2}(p^1+q^2)$	=	$(1, 0)$	$(\frac{1}{2}, \frac{1}{2})$			(state 1),
−1	2		−1	2			
$\frac{1}{2}(p^2+q^2)$	$\frac{1}{2}(p^1+q^1)$		$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$			
0	−1		0	−1			
$\frac{1}{2}(p^1+q^1)$	$\frac{1}{2}(p^1+q^2)$	=	$(1, 0)$	$(\frac{1}{2}, \frac{1}{2})$			(state 2),
−1	1		−1	1			
$\frac{1}{2}(p^2+q^1)$	$\frac{1}{2}(p^2+q^2)$		$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$			

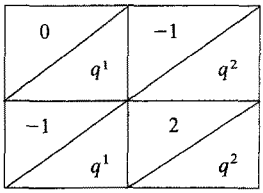
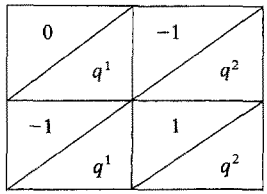
where a box



corresponds to an immediate payoff ε and a jump with probability γ to state 1 and probability $1-\gamma$ to state 2. This game possesses additive transitions. The game can be seen as a kind of mixture of the player I control game

Γ^1:  (state 1),  (state 2),

and the player II control game

Γ^2:  (state 1),  (state 2),

which corresponds to the following game situation.

Each day, the players observe the current state and then choose one of their possible actions. After the players have chosen their action, an unbiased coin decides whether the transitions are according to the player I control game Γ^1 or to the player II control game Γ^2. For $\beta \in [0, 1)$, the β-discounted value $(V_\beta(1), V_\beta(2))$ is given by the unique solution of

$$v_1 = \operatorname{val}\begin{bmatrix} 0+\beta v_1 & -1+\frac{1}{2}\beta v_1+\frac{1}{2}\beta v_2 \\ -1+\frac{1}{2}\beta v_1+\frac{1}{2}\beta v_2 & 2+\beta v_2 \end{bmatrix},$$

$$v_2 = \operatorname{val}\begin{bmatrix} 0+\beta v_1 & -1+\frac{1}{2}\beta v_1+\frac{1}{2}\beta v_2 \\ -1+\frac{1}{2}\beta v_1+\frac{1}{2}\beta v_2 & 1+\beta v_2 \end{bmatrix}.$$

Both matrix games are completely mixed, resulting in

$$v_1 = \tfrac{1}{4}(\beta v_1(2+\beta v_2) - (-1+\tfrac{1}{2}\beta v_1+\tfrac{1}{2}\beta v_2)^2), \tag{3a}$$

$$v_2 = \tfrac{1}{3}(\beta v_1(1+\beta v_2) - (-1+\tfrac{1}{2}\beta v_1+\tfrac{1}{2}\beta v_2)^2). \tag{3b}$$

Combining (3a) and (3b) yields

$$4v_1 - \beta v_1(2+\beta v_2) = 3v_2 - \beta v_1(1+\beta v_2), \tag{4a}$$

$$v_2 = \tfrac{1}{3}(4-\beta)v_1. \tag{4b}$$

Substitution of (4) into (3a) results in

$$4v_1 = \beta v_1(2+\tfrac{1}{3}\beta(4-\beta)v_1) - (-1+\tfrac{1}{2}\beta v_1+\tfrac{1}{6}\beta(4-\beta)v_1)^2,$$

which leads to

$$V_\beta(1) = v_1 = \frac{12(12-\beta)-12\sqrt{(144-24\beta)}}{-2\beta^2(1-\beta)}$$

$$= \frac{-6}{(1-\beta)(12-\beta+\sqrt{(144-24\beta)})}. \tag{5}$$

Since the value of the undiscounted game with initial state $s=1$ equals $\lim_{\beta\uparrow 1}(1-\beta)V_\beta(1)$ (Ref. 2), it follows by (5) that neither for the discounted case nor for the undiscounted case does the orderfield property hold, by noting that $V_1(1)$ is irrational, while the game parameters all are in the rational field, and by noting that, for the rational discount factor $\beta = 1/2$, $V_\beta(1)$ is also irrational.

This example gives an indication that, in order to obtain a nice solution of the game, one has to look for a further constraint on the game components. Such a constraint is additivity of the rewards. When the rewards and the transitions are both additive, the problem is manageable, as we will show below. Related work on additive games can be found in Parthasarathy and Raghavan (Ref. 9), and Himmelberg *et al.* (Ref. 10). A main result for ARAT zero-sum games is given in the following theorem.

Theorem 2.1. Let the rewards and transitions be additive in a zero-sum stochastic game Γ. Then, there are pure optimal stationary strategies for both players for discounted as well as undiscounted payoffs. Furthermore, the orderfield property holds for both criteria. Also, there are pure stationary strategies for both players which are uniformly optimal for all discount factors sufficiently near to one.

Proof. The following is well known (cf. Bewley and Kohlberg, Ref. 11). For a zero-sum stochastic game with finite state and action spaces, there exists a series

$$W(\alpha) = \sum_{k=-\infty}^{K} w_k(\alpha(1-\alpha)^{-1})^{k/K},$$

in fractional powers of $\alpha(1-\alpha)^{-1}$ and vectors $w_k \in \mathbb{R}^Z$ as coefficients, such that, for each $\beta \in (0, 1)$ sufficiently near to one, $W(\beta)$ equals the value V_β of the β-discounted game. Moreover, $W(\alpha)$ satisfies, for each $s \in S$, the so-called limit discount equation

$$W_s(\alpha) = \operatorname{val}_{A_s \times B_s}[G_s(W(\alpha))], \tag{6}$$

where the (i, j)th cell of the matrix game $[G_s(W(\alpha))]$ has content

$$g(s, i, j) = r(s, i, j) + \alpha \sum_t p(t \mid s, i, j) W_t(\alpha). \tag{7}$$

Here, the matrix game $[G_s(W(\alpha))]$ is a game in the field of real Puiseux series (cf. Bewley and Kohlberg, Ref. 11).

Furthermore, it is known that the value of the undiscounted stochastic game equals (cf. Mertens and Neyman, Ref. 2)

$$w_K = \lim_{\alpha \uparrow 1} (1 - \alpha) W(\alpha). \tag{8}$$

In general, optimal actions for the matrix game in (6) are quite complex and belong to the same Puiseux field as to which $W(\alpha)$ belongs. However, for an additive game, $G_s(W(\alpha))$ can be decomposed as follows:

$$G_s(W(\alpha)) = G_{1s}(W(\alpha)) + G_{2s}(W(\alpha)),$$

where

$$g_1(s, i, j) = r_1(s, i) + \alpha \sum_t p_1(t \mid s, i) W_t(\alpha),$$

$$g_2(s, i, j) = r_2(s, j) + \alpha \sum_t p_2(t \mid s, j) W_t(\alpha).$$

So, $G_{1s}(W(\alpha))$ has identical columns and $G_{2s}(W(\alpha))$ has identical rows. But then, when solving $G_s(W(\alpha))$, player I only needs to consider $G_{1s}(W(\alpha))$ and player II only needs to look at $G_{2s}(W(\alpha))$. This observation results in the fact that both players have optimal real pure actions in the limit discount equations. Let f^* be a pure stationary strategy for player I such that f_s^* is an optimal action in $[G_s(W(\alpha))]$ for each $s \in S$, and let g^* be similar for player II. Then, by Theorems 6.1 and 6.2 of Bewley and Kohlberg (Ref. 11), it follows that f^* and g^* are uniformly discount optimal and optimal for the undiscounted case.

That, for each $\beta \in [0, 1)$, both players have optimal pure stationary strategies can be shown in a similar way. Namely, when in (6) we replace α by a fixed $\beta \in [0, 1)$ and $W(\alpha)$ by V_β, we obtain Shapley's equation for the β-discounted game. Again, the matrix game $[G_s(V_\beta)]$ can be decomposed into a part independent of player I and a part independent of player II. Application of Shapley's theorem does the rest. The orderfield property for the discounted case follows from the fact that, for a pair of stationary strategies, the associated discounted payoff is a rational function of β and, for the undiscounted case, then the orderfield property follows from (8); cf. Ref. 4. □

Knowing now that the orderfield property holds for ARAT games, this gives hope that there exists a finite algorithm as was the case for one-player

control games and switching-player control games. We do not know whether, for ARAT games, there exists a one-step solution method, like solving one linear program. But, indeed, for the discounted and also for the undiscounted criterion, we will indicate now a finite-step solution method. For the discounted ARAT game, the method of Hoffman and Karp (Ref. 12) can be used, which proceeds as follows.

(i) Choose

$$v_0 = M(1-\beta)^{-1}1_z,$$

with

$$M := \min_{(s,i,j)\in T} r(s,i,j)$$

and

$$1_z = (1, 1, \ldots, 1) \in \mathbb{R}^z.$$

Put $\tau := 0$.

(ii) Determine for player I a pure stationary strategy

$$f^\tau = (f_1^\tau, f_2^\tau, \ldots, f_z^\tau),$$

such that f_s^τ is an optimal action for player I in the matrix game $[G_s(v_\tau)]$ for each $s \in S$; cf. (6) and (7).

(iii) Solve for player II the discounted Markov decision problem which results when player I fixes f^τ. This can be done, for example, by solving one linear programming problem. Let $v_{\tau+1}$ be the optimal value of this problem.

(iv) If $v_{\tau+1} \neq v_\tau$, put $\tau := \tau + 1$ and return to (ii); else, stop.

It is straightforward to show that $v_{\tau+1} \geq v_\tau$ componentwise and when f^τ is not optimal then $v_{\tau+1} \neq v_\tau$. That, in step (ii) of the algorithm, player I possesses optimal pure actions in $[G_s(v_\tau)]$ follows again from the fact that $G_s(v_\tau)$ can be decomposed into a part only depending on player I and a part only depending on player II. Since, in each iteration, player I strictly improves his strategy, and since there are a finite number of pure stationary strategies, it is clear that the algorithm stops after a finite number of iterations.

For the undiscounted additive game, a finite-step algorithm can be developed which resembles the algorithm of Vrieze *et al.* (Ref. 8). Like the algorithm above of Hoffman and Karp, also this algorithm can be described by the term "value-oriented policy iteration." We will not give this algorithm in detail here, but indicate how the algorithm of Vrieze *et al.* (Ref. 8) should be adapted. The notations in their paper are used. Throughout the algorithm,

we have

$$S_1 = S \quad \text{and} \quad S_2 = \varnothing.$$

Further,

$$\sigma^c(\tau) = \{\sigma_k^c(\tau);\, k \in S_1\}, \qquad \tau = 0, 1, 2, \ldots,$$

is a pure stationary strategy now, with the consequence that $\tilde{\Gamma}(\sigma^c(\tau+1))$ is a Markov decision problem. This Markov decision problem can be solved by the same LP1. More changes are not needed. The proof that also this modified algorithm stops after a finite number of iterations proceeds in the same way as in Vrieze *et al.*, using the fact that there are a finite number of pure stationary strategies.

3. Nonzero-Sum Case

For nonzero-sum games, the concept of equilibrium points is relevant. A pair of strategies (μ^*, ν^*) forms an *equilibrium point* if, for all strategies μ and ν,

$$\begin{aligned} V_{\beta 1}(\mu, \nu^*) &\leq V_{\beta 1}(\mu^*, \nu^*), \\ V_{\beta 2}(\mu^*, \nu) &\leq V_{\beta 2}(\mu^*, \nu^*). \end{aligned} \tag{9}$$

Again, in (9), to $\beta = 1$ we associate the undiscounted case.

It is well known, for the discounted case, that there exist equilibrium points of stationary strategy pairs (Refs. 13 and 14). For the undiscounted version, in general, the existence of equilibria is unknown. For different subclasses of stochastic games, this problem is settled by Rogers (Ref. 14), Federgruen (Ref. 15), Parthasarathy and Raghavan (Ref. 4), and Parthasarthy, Tijs, and Vrieze (Ref. 16). In view of the results of the zero-zum case, the question arises whether for nonzero-sum ARAT games, there exist equilibrium points of pure stationary strategy pairs. The following example answers this question in the negative.

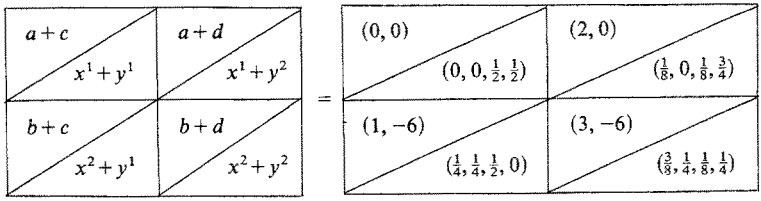
Example 3.1. Let

$$x^1 = (0, 0, 0, \tfrac{1}{2}), \qquad x^2 = (\tfrac{1}{4}, \tfrac{1}{4}, 0, 0),$$

$$y^1 = (0, 0, \tfrac{1}{2}, 0), \qquad y^2 = (\tfrac{1}{8}, 0, \tfrac{1}{8}, \tfrac{1}{4}),$$

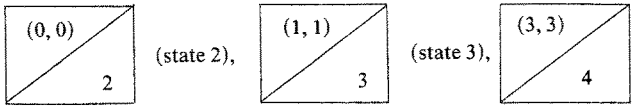
$$a = (0, 0), \qquad b = (1, -6), \qquad c = (0, 0), \qquad d = (2, 0).$$

Consider the ARAT stochastic game with four states, where state 1 is given by



(state 1)

and the absorbing states 2, 3, 4 are given by



(state 2), (state 3), (state 4).

Both players have two pure stationary strategies corresponding to choosing their first and second action, respectively, in state 1. Let us denote these startegies by f^1, f^2, g^1, g^2, respectively. Take $\beta = 1/2$. When we compute $v_{\frac{1}{2}n}(f^k, g^l)$, for $n = 1, 2$, $k = 1, 2$, and $l = 1, 2$, then we obtain

$$\begin{array}{c} \\ f^1 \\ f^2 \end{array} \begin{array}{c} \begin{array}{cc} g^1 & g^2 \end{array} \\ \begin{bmatrix} (2, 2) & (4\frac{2}{3}, 2\frac{8}{15}) \\ (1\frac{5}{5}, -6\frac{2}{7}) & (4\frac{10}{13}, -6\frac{4}{13}) \end{bmatrix} \end{array}. \tag{10}$$

For example $V_{\frac{1}{2}1}(f^1, g^2)$ can be computed as the unique solution v of

$$v = 2 + \tfrac{1}{2}\tfrac{1}{8}v + \tfrac{1}{2}\tfrac{1}{8}(1-\tfrac{1}{2})^{-1} + \tfrac{1}{2}\tfrac{3}{4}(1-\tfrac{1}{2})^{-1}3,$$

resulting in

$$v = 4\tfrac{2}{3}.$$

From (10), it can be seen that there exists no equilibrium point in pure stationary strategies and that, for this example, the unique equilibrium point in stationary strategies is completely mixed.

Also for the undiscounted case, examples of ARAT games can be constructed without an equilibrium point in pure stationary strategies. Like in the general case, also for non-zero-sum ARAT stochastic games there may be several equilibrium points with different payoffs to the players. Furthermore, examples show that, for such games, the ordered field property fails to hold. In the following, for $x \in \mathbb{R}$, We define car(x) as the set

$$\text{car}(x) := \{k; x_k \neq 0\}$$

and, for a finite set T, $|T|$ denotes the number of elements of T.

For discounted nonzero-sum additive stochastic games, we have the next remarkable theorem.

Theorem 3.1. If, for a discounted nonzero-sum ARAT stochastic game, the pair (f^*, g^*) forms an equilibrium point of stationary strategies, then there exists an equilibrium point $(\tilde{f}, \tilde{g})$, such that

$$V_{\beta 1}(f^*, g^*) = V_{\beta 1}(\tilde{f}, \tilde{g}),$$

$$V_{\beta 2}(f^*, g^*) = V_{\beta 2}(\tilde{f}, \tilde{g}),$$

and such that

$$|\mathrm{car}(\tilde{f}_s)| \leq 2,$$

$$|\mathrm{car}(\tilde{g}_s)| \leq 2,$$

for each state $s \in S$.

Proof. Let (f^*, g^*) be a stationary equilibrium point, and let

$$V_1^* = V_{\beta 1}(f^*, g^*),$$

$$V_2^* = V_{\beta 2}(f^*, g^*).$$

This is equivalent to

$$\max_i (r_1(s, i, g_s^*) + \beta \sum_t p(t \mid s, i, g_s^*) V_1^*(t)) = V_1^*(s), \tag{11}$$

$$\max_j (r_2(s, f_s^*, j) + \beta \sum_t p(t \mid s, f_s^*, j) V_2^*(t)) = V_2^*(s), \tag{12}$$

where the maximum in (11) is reached at least for each $i \in \mathrm{car}(f_s^*)$ and in (12) the maximum is attained at least for each $j \in \mathrm{car}(g_s^*)$. By the additivity of the game, (11) and (12) are equivalent to

$$\max_i \left(r_{11}(s, i) + \beta \sum_t p_1(t \mid s, i) V_1^*(t) \right) + r_{12}(s, g_s^*) + \beta \sum_t p_2(t \mid s, g_s^*) V_1^*(t) = V_1^*(s), \tag{13}$$

$$r_{21}(s, f_s^*) + \beta \sum_t p_1(t \mid s, f_s^*) V_2^*(t) + \max_j \left(r_{22}(s, j) + \beta \sum_t p_2(t \mid s, j) V_2^*(t) \right) = V_2^*(s). \tag{14}$$

Put

$$W_1 = r_{12}(s, g_s^*) + \beta \sum_t p_2(t \mid s, g_s^*) V_1^*(t).$$

Since

$$g_s \mapsto r_{12}(s, g_s) + \beta \sum_t p_2(t \mid s, g_s) V_1^*(t)$$

is a linear function of the weights $g_s(j)$ on the pure actions, there exists a $\tilde{g}_s$, with

$$\text{car}(\tilde{g}_s) \subset \text{car}(g_s^*) \quad \text{and} \quad |\text{car}(\tilde{g}_s)| \leq 2,$$

such that

$$W_1 = r_{12}(s, \tilde{g}_s) + \beta \sum_t p_2(t \mid s, \tilde{g}_s) V_1^*(t).$$

Hence, replacing g_s^* by $\tilde{g}_s$ does not disturb Eqs. (13) and (11); and, since

$$\text{car}(\tilde{g}_s) \subset \text{car}(g_s^*),$$

the maxima in (14) and (12) are reached for each $j \in \text{car}(\tilde{g}_s)$. This procedure can be carried out for each state $s \in S$ and also for player I by considering

$$W_2 = r_{21}(s, f_s^*) + \beta \sum_t p(t \mid s, f_s^*) V_2^*(t).$$

This leads to

$$\max_i (r_1(s, i, \tilde{g}_s) + \beta \sum_t p(t \mid s, i, \tilde{g}_s) V_1^*(t)) = V_1^*(s), \tag{15}$$

$$\max_j (r_2(s, \tilde{f}_s, j) + \beta \sum_t p(t \mid s, \tilde{f}_s, j) V_2^*(t)) = V_2^*(s), \tag{16}$$

for each $s \in S$, where in (15) the maximum is attained at least for each $i \in \text{car}(\tilde{f}_s)$ and in (16) the maximum is reached at least for each $j \in \text{car}(\tilde{g}_s)$. Hence, $(\tilde{f}, \tilde{g})$ forms an equilibrium point and

$$V_{\beta 1}(\tilde{f}, \tilde{g}) = V_1^* = V_{\beta 1}(f^*, g^*),$$

$$V_{\beta 2}(\tilde{f}, \tilde{g}) = V_2^* = V_{\beta 2}(f^*, g^*). \qquad \square$$

We conclude with some remarks.

(i) Example 3.1 above shows that Theorem 3.1 cannot be sharpened.

(ii) An analogous statement like Theorem 3.1 can be given for the undiscounted case. The proof uses Markov chain theory and Markov decision theory. We do not give this proof, because it would take too much space.

(iii) We do not know whether, for the undiscounted ARAT case, equilibria of stationary strategies always exist, though we have good reasons to believe that this indeed is the case.

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