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An Algorithm for Generalized Fractional Programs[1]

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Abstract. An algorithm is suggested that finds the constrained minimum of the maximum of finitely many ratios. The method involves a sequence of linear (convex) subproblems if the ratios are linear (convex-concave). Convergence results as well as rate of convergence results are derived. Special consideration is given to the case of (a) compact feasible regions and (b) linear ratios.

Key Words. Fractional programming, multi-ratio programming, convergence, rate of convergence.

1. Introduction

The purpose of this paper is to describe an algorithm which solves the generalized fractional program

$$\text{(P)} \quad \bar{\theta} = \inf_{x \in S} \left(\max_{1 \leq i \leq p} f_i(x)/g_i(x) \right). \tag{1}$$

We assume that S is a nonempty subset of R^n, the functions f_i and g_i are continuous on S, and the functions g_i are positive on S.

The case $p = 1$ corresponds to tranditional fractional programming and has been actively investigated in the last two decades; see Ref. 1. Generalized fractional programming ($p > 1$) has been studied more recently; see, for example, Refs. 2–4. Fractional programming and its generalization were reviewed in two recent articles: in Ref. 5, basic theoretical results were

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surveyed; in Ref. 6, applications and algorithms were discussed. For a bibliography of fractional programming, see Ref. 7.

The algorithm proposed in this paper is a generalization of a procedure by Dinkelbach (Ref. 8), which was suggested for the case $p=1$; see also Refs. 9–11. An algorithm extending it to the case $p>1$ can already be found in Ref. 2 for the special case of linear functions and constraints, though the method there is not explicitly related to Dinkelbach's algorithm in Ref. 8. A convergence proof was not given in Ref. 2. The method gives rise to finding the root of the equation $F(\theta)=0$, where $F(\theta)$ is the optimal value of the parametric program

$$(\mathrm{P}_\theta) \quad F(\theta)=\inf_{x\in S}\left(\max_{1\leq i\leq p}\,[f_i(x)-\theta g_i(x)]\right). \tag{2}$$

When the algorithm is applied, the optimal solutions and the optimal values of the parametric program (P_θ) are to be determined. The method is especially useful when the structure of (P_θ) is simpler than the one of the initial problem (P). For instance, when f_i are nonnegative and convex on S, g_i are concave on S, and S is a convex set, then (P_θ) is a convex program for every positive value of θ, whereas (P) is only quasi-convex.

In Section 2, we shall analyze the general properties of the function F. The algorithm is then described in Section 3. Special attention is given to the case of a compact feasible region (Section 4) and to the case of linear functions and constraints when S is not necessarily bounded (Section 5). In both sections, we will establish convergence and we will determine the rate of convergence of the algorithm.

2. General Properties of F

For the special case of linear problems (P), the properties of F were already studied in Ref. 3, where they were used to establish duality relations for (P). In the following, we study F in the more general case where f_i and g_i are arbitrary continuous functions and S is an arbitrary set in R^n.

First, notice that $F(\theta)<+\infty$, since S in nonempty. We now show the following proposition.

Proposition 2.1. (a) F is nonincreasing and upper semicontinuous;
(b) $F(\theta)<0$ if and only if $\theta>\bar{\theta}$; hence, $F(\bar{\theta})\geq 0$;
(c) If (P) has an optimal solution, then $F(\bar{\theta})=0$;
(d) If $F(\bar{\theta})=0$, then programs (P) and $(\mathrm{P}_{\bar{\theta}})$ have the same set of optimal solutions (which may be empty).

Proof. (a) The monotonicity of F follows from the positivity of g_i. The function $\max_i[f_i(x)-\theta g_i(x)]$ is continuous jointly in (x, θ). Hence, F is upper semicontinuous in θ.

(b) Suppose that $F(\theta)<0$. Then, there exists $\hat{x}\in S$ such that

$$f_i(\hat{x})-\theta g_i(\hat{x})<0, \qquad i=1,\dots,p.$$

Hence,

$$\theta>\max_i f_i(\hat{x})/g_i(\hat{x})\geq \bar{\theta}.$$

Conversely, if $\theta>\bar{\theta}$, there exists $\tilde{x}\in S$ such that

$$\max_i f_i(\tilde{x})/g_i(\tilde{x})<\theta.$$

Hence,

$$f_i(\tilde{x})-\theta g_i(\tilde{x})<0, \qquad i=1,\dots,p,$$

implying $F(\theta)<0$.

(c) Let $\bar{x}$ be an optimal solution of (P). Then, $\bar{x}\in S$ and

$$\bar{\theta}=\max_i f_i(\bar{x})/g_i(\bar{x}).$$

Thus,

$$\max_i[f_i(\bar{x})-\bar{\theta}g_i(\bar{x})]=0.$$

Since $F(\bar{\theta})\geq 0$ [see (b)], we have $F(\bar{\theta})=0$.

(d) It was just shown that an optimal solution $\bar{x}$ of (P) is an optimal solution of $(P_{\bar{\theta}})$. Now, assume that $F(\bar{\theta})=0$ and that $\bar{x}$ is an optimal solution of $(P_{\bar{\theta}})$. Then,

$$\max_i[f_i(\bar{x})-\bar{\theta}g_i(\bar{x})]=0.$$

Hence,

$$\max_i f_i(\bar{x})/g_i(\bar{x})=\bar{\theta},$$

implying that $\bar{x}$ is an optimal solution of (P). □

Example 2.1 below shows that F is not necessarily finite on R. Furthermore, it demonstrates that the existence of an optimal solution of $(P_{\bar{\theta}})$ does not imply $F(\bar{\theta})=0$.

Example 2.1. Let $n=1$, $p=1$,

$$f_1(x)=1+x, \qquad g_1(x)=x, \qquad S=\{x|x\geq 1\}.$$

Then, $\bar{\theta}=1$. We have

$$F(\theta)=\inf\{1+(1-\theta)x \mid x\geq 1\}=\begin{cases}2-\theta, & \text{if } \theta\leq 1,\\ -\infty, & \text{if } \theta>1.\end{cases}$$

$(P_{\bar{\theta}})$ has an optimal solution, but $F(\bar{\theta})=1>0$. (P) does not have an optimal solution.

The next example shows that F is not necessarily decreasing on the interval where F is finite. It also demonstrates that $F(\bar{\theta})=0$ does not imply the existence of an optimal solution of (P), even if S is closed.

Example 2.2. Let $n=1$, $p=1$,

$$f_1(x)=\exp(x), \qquad g_1(x)=\exp(2x), \qquad S=R.$$

Then, $\bar{\theta}=0$. We have

$$F(\theta)=\inf\{\exp(x)-\theta\exp(2x) \mid x\in R\}=\begin{cases}0, & \text{if } \theta\leq 0,\\ -\infty, & \text{if } \theta>0.\end{cases}$$

Thus, $F(\bar{\theta})=0$, and (P) and $(P_{\bar{\theta}})$ have no optimal solution.

In case $p=1$, F is concave, since it is the infimum of concave functions $f_1(x)-\theta g_1(x)$ in θ. This was also shown in Ref. 8 in a different way. It follows that F is continuous on the interior of its domain of finiteness. However, the concavity of F is lost when $p>1$, as we see from the next example.

Example 2.3. Let $n=1$, $p=2$,

$$f_1(x)=2x, \qquad g_1(x)=2, \qquad f_2(x)=-x, \qquad g_2(x)=1, \qquad S=[0,1].$$

Then,

$$F(\theta)=\inf\{\max[2x-2\theta,-x-\theta] \mid x\in[0,1]\}$$

$$=\begin{cases}-2\theta, & \text{if } \theta\leq 0,\\ -(4/3)\theta, & \text{if } 0\leq\theta\leq 3,\\ -1-\theta, & \text{if } \theta\geq 3,\end{cases}$$

which is not concave.

In Example 2.3, $F(\theta)$ is continuous at $\bar{\theta}$. In Examples 2.1 and 2.2, $F(\theta)$ is discontinuous at $\bar{\theta}$ with an infinite gap. The next example shows that there may also occur a finite gap at $\bar{\theta}$.

Example 2.4. Let $n=2$, $p=2$,

$$f_1(x)=x_1-2, \qquad g_1(x)=1, \qquad f_2(x)=2x_1, \qquad g_2(x)=x_1+x_2+1,$$

$$S=\{x\in R^2 \mid x_1\geq 0, x_2\geq 0\}.$$

Then, $\bar{\theta}=0$ and $\bar{x}=(0, x_2)$, $x_2\geq 0$ is an optimal solution of (P). We find that

$$F(\theta)=\begin{cases}-\theta, & \text{if } \theta<0,\\ 0, & \text{if } \theta=0,\\ -2-\theta, & \text{if } \theta>0,\end{cases}$$

where the optimal solutions of (P_θ) are

$$\bar{x}(\theta)=\begin{cases}(0,0), & \text{if } \theta<0,\\ (0,x_2),\ x_2\geq 0, & \text{if } \theta=0,\\ (0,x_2),\ x_2\geq 2/\theta, & \text{if } \theta>0.\end{cases}$$

As we see, $F(\theta)$ is discontinuous at $\bar{\theta}$ with a finite gap. Notice that the generalized fractional program is even linear and that S is closed. Also, we observe that (P) does have an optimal solution and still $F(\theta)$ is not continuous at $\bar{\theta}$.

The next example is similar to Example 2.4, except that (P) does not have an optimal solution. It is obtained by adding $+1$ to the numerators f_1, f_2.

Example 2.5. Let n, p, g_1, g_2, S be as in Example 2.4, but

$$f_1(x)=x_1-1, \qquad f_2(x)=2x_1+1.$$

Again, $\bar{\theta}=0$ but an optimal solution does not exist. We find that

$$F(\theta)=\begin{cases}1-\theta, & \text{if } \theta\leq 0,\\ -1-\theta, & \text{if } \theta>0.\end{cases}$$

The optimal solutions $\bar{x}(\theta)$ are the same as in Example 2.4, since $F(\theta)$ is only translated by $+1$.

We will make use of these examples in various parts of this paper.

We now prove some inequalities that will assist the analysis of the algorithm in Section 3.

For $\theta\in R$, let $M(\theta)$ be the (possibly empty) set of the optimal solutions of (P_θ). Furthermore, let

$$\underline{g}(x)=\min_{1\leq i\leq p} g_i(x), \quad \bar{g}(x)=\max_{1\leq i\leq p} g_i(x), \qquad x\in S.$$

Then, we can show the following proposition.

Proposition 2.2. Let θ be such that $F(\theta)$ is finite and $M(\theta)$ is nonempty. Let $x\in M(\theta)$. Then,

$$F(\mu)\leq F(\theta)+(\theta-\mu)\underline{g}(x), \qquad \text{if } \mu>\theta, \tag{3}$$

$$F(\mu)\leq F(\theta)+(\theta-\mu)\bar{g}(x), \qquad \text{if } \mu<\theta. \tag{4}$$

Proof. Since $x \in M(\theta)$, we have, for all i,

$$F(\theta) \geq f_i(x) - \theta g_i(x);$$

hence,

$$F(\theta) \geq -\mu g_i(x) + f_i(x) - (\theta - \mu) g_i(x),$$

implying that

$$F(\theta) + (\theta - \mu) g_i(x) \geq f_i(x) - \mu g_i(x), \qquad i = 1, \ldots, p.$$

Assume $\mu > \theta$. Then,

$$F(\theta) + (\theta - \mu) \underset{\sim}{g}(x)$$
$$\geq F(\theta) + (\theta - \mu) g_i(x) \geq f_i(x) - \mu g_i(x), \qquad i = 1, \ldots, p.$$

Thus,

$$F(\theta) + (\theta - \mu) \underset{\sim}{g}(x) \geq \max_i [f_i(x) - \mu g_i(x)] \geq F(\mu).$$

In the same way, Ineq. (4) can be derived. □

In case $p = 1$, we have

$$\underset{\sim}{g}(x) = \bar{g}(x) = g_1(x).$$

Then, (3) and (4) imply that $-g_1(x)$ is a subgradient of F at θ for all $x \in M(\theta)$, as noticed already by Ibaraki in Ref. 10.

At the end of this section, we present a sufficient condition for $F(\theta)$ to be strictly monotonic when finite. As Example 2.2 shows, this is not necessarily so in general. In such a situation, we may have $F(\theta) = 0$ for $\theta \neq \bar{\theta}$.

Proposition 2.3. Suppose that there exists $m > 0$ such that $g_i(x) \geq m$, for all $x \in S$ and $i = 1, \ldots, p$. Then, for $\mu > \theta$, we have

$$F(\mu) + (\mu - \theta) m \leq F(\theta). \tag{5}$$

Hence, $F(\theta)$ is decreasing on the interval where it is finite.

Proof. For all $x \in S$ and $i = 1, \ldots, p$,

$$f_i(x) - \theta g_i(x) = f_i(x) - \mu g_i(x) + (\mu - \theta) g_i(x)$$
$$\geq f_i(x) - \mu g_i(x) + (\mu - \theta) m,$$

which implies that

$$\max_i [f_i(x) - \theta g_i(x)] \geq \max_i [f_i(x) - \mu g_i(x)] + (\mu - \theta) m, \qquad \text{for all } x \in S.$$

Hence,

$$\inf_{x\in S}(\max_i[f_i(x)-\theta g_i(x)])\geq \inf_{x\in S}(\max_i[f_i(x)-\mu g_i(x)])+(\mu-\theta)m. \qquad \square$$

Proposition 2.3 together with Proposition 2.1(b) shows that $F(\theta)=0$ implies $\theta=\bar{\theta}$, provided the assumption in Proposition 2.3 holds.

3. Description and Analysis of the Algorithm

We now introduce the following iterative procedure to solve (P) via (P_θ).

Step 1. Start with some $x^0 \in S$. Let

$$\theta_1 = \max_i f_i(x^0)/g_i(x^0).$$

Let $k=1$.

Step 2. Solve (P_{θ_k}). Let $x^k \in M(\theta_k)$.

Step 3. If $F(\theta_k)=0$, then stop. Then, x^k and x^{k-1} are optimal solutions of (P), and $\theta_k=\bar{\theta}$.

Step 4. If $F(\theta_k)\neq 0$, take

$$\theta_{k+1}=\max_i f_i(x^k)/g_i(x^k).$$

Let $k=k+1$, and go back to Step 2.

From the construction of θ_k, it is obvious that $\theta_k \geq \bar{\theta}$ and

$$\max_i[f_i(x^{k-1})-\theta_k g_i(x^{k-1})]=0, \qquad k\geq 1.$$

Hence, $F(\theta_k)\leq 0$. Then, in Step 3, $F(\theta_k)=0$ implies that $\theta_k=\bar{\theta}$, since otherwise $\theta_k>\bar{\theta}$ would yield $F(\theta_k)<0$, in view of Proposition 2.1(b). Furthermore, we see from Proposition 2.1(d) that x^k is an optimal solution of (P). Also, x^{k-1} solves (P), since

$$\theta_k=\max_i f_i(x^{k-1})/g_i(x^{k-1}),$$

and $\theta_k=\bar{\theta}$ in this case.

In order to apply the algorithm, one needs to determine in addition to an initial feasible solution x^0, optimal solutions x^k of (P_{θ_k}), $k=1,2,\ldots$. We point out that the subproblems (P_{θ_k}) are convex programs if f_i are nonnegative and convex, g_i are concave, and S is a convex set. Instead of (P_{θ_k}), one may solve the equivalent convex program

$$\inf\{\lambda \mid f_i(x)-\theta_k g_i(x)-\lambda\leq 0,\ i=1,\ldots,p,\ x\in S\}. \tag{6}$$

These subproblems are linear programs if f_i, g_i are affine and S is a convex polyhedron (see Ref. 2).

In the following sections, we will consider different situations where the algorithm above converges to an optimal solution of (P).

In preparation for that, we prove Proposition 3.1 below, for which we need the following notation: For $\theta \in R$, $x \in M(\theta)$, let

$$I(x, \theta) = \{i | f_i(x) - \theta g_i(x) = F(\theta)\}.$$

Denote

$$J(x) = \{j | f_j(x)/g_j(x) = \max_{1 \leq i \leq p} f_i(x)/g_i(x)\}, \qquad x \in S.$$

Proposition 3.1. We have:

(a) $$\begin{aligned} -F(\theta_k)/\bar{g}(x^k) &\leq -F(\theta_k)/g_j(x^k) \leq \theta_k - \theta_{k+1} \leq -F(\theta_k)/g_i(x^k) \\ &\leq -F(\theta_k)/\underline{g}(x^k), \qquad \text{for all } j \in J(x^k) \text{ and } i \in I(x^k, \theta_k); \end{aligned} \tag{7}$$

(b) $\theta_k \geq \bar{\theta}$, for all k; and, if $\theta_k > \bar{\theta}$, then $\theta_k > \theta_{k+1} \geq \bar{\theta}$.

Proof. Let $j \in J(x^k)$. Since $x^k \in M(\theta_k)$, we have

$$\begin{aligned} F(\theta_k) &= \max_i [f_i(x^k) - \theta_k g_i(x^k)] \\ &\geq f_j(x^k) - \theta_k g_j(x^k) = g_j(x^k)(\theta_{k+1} - \theta_k). \end{aligned}$$

Let $i \in I(x^k, \theta_k)$. Then,

$$\begin{aligned} F(\theta_k) &= f_i(x^k) - \theta_k g_i(x^k) = g_i(x^k)(f_i(x^k)/g_i(x^k) - \theta_k) \\ &\leq g_i(x^k)(\theta_{k+1} - \theta_k). \end{aligned}$$

As seen before, $\theta_k \geq \bar{\theta}$, for all $k = 1, 2, \ldots$. If $\theta_k > \bar{\theta}$, then $F(\theta_k) < 0$, in view of Proposition 2.1 (b). Hence, the first and last inequalities in (7) hold using also the definition of $\underline{g}(x)$ and $\bar{g}(x)$. Furthermore, if $F(\theta_k) < 0$ in (7), then $\theta_{k+1} < \theta_k$. □

From Proposition 3.1, we see that, in case $p = 1$,

$$\theta_{k+1} = \theta_k - F(\theta_k)/(-g_1(x^k)).$$

Above, we concluded from Proposition 2.2 that, in the single-ratio case, $-g_1(x^k)$ is a subgradient of F at θ_k. Thus, the proposed method coincides with Newton's method for $p = 1$; see also Ref. 10. However, for $p > 1$, Newton's method does not coincide with our algorithm. To see this, consider two of the examples above. In Example 2.3, let $x^0 \in (01]$ be a starting point.

Then, the algorithm generates the sequence

$$\theta_k = (1/3^{k-1})x^0, \qquad k = 1, 2, \ldots,$$

converging to $\bar{\theta} = 0$, whereas Newton's method converges in one step. On the other hand, Example 2.5 shows that Newton's method may not even converge when the algorithm above does. To see this, let $x^0 = (1, 1)$ in Example 2.5. Then, $\theta_1 = 1$. The sequence θ_k generated by Newton's method is

$$\theta_k = (-1)^{k+1}, \qquad k = 1, 2, \ldots,$$

which does not converge. On the other hand, the algorithm above yields the sequence

$$x^k = (0, 2/\theta_k),$$

and hence

$$\theta_{k+1} = 1/(1 + 2/\theta_k), \qquad k = 1, 2, \ldots.$$

Then,

$$\theta_k = \left(\sum_{l=0}^{k-1} 2^l\right)^{-1}, \qquad k = 1, 2, \ldots,$$

which does converge, and the limit is $\bar{\theta} = 0$.

In the following, we want to discuss under what conditions the algorithm above converges. In order to apply the method, we need the assumption $M(\theta) \neq \emptyset$, for $\theta \in (\bar{\theta}, \theta_1]$; i.e., we need to assume that $F(\theta) > -\infty$ for such θ and that an optimal solution $\bar{x}(\theta)$ exists. In case of a linear generalized fractional program, this is true provided $F(\theta_1) > -\infty$ for at least one $\theta_1 > \bar{\theta}$, since the equivalent linear program (6) always attains the optimal value $F(\theta)$. Note, however, that, even in the linear case, $F(\theta) = -\infty$, for all $\theta > \bar{\theta}$, is possible as Example 2.1 shows.

In addition to $M(\theta) \neq \emptyset$, for $\theta \in (\bar{\theta}, \theta_1]$, we will assume that (P) has an optimal solution $\bar{x}$ (i.e., $\bar{\theta} > -\infty$) and the optimal value is attained. However, as Example 2.5 shows, the algorithm may still converge if (P) does not have an optimal solution, as long as $\bar{\theta} > -\infty$.

We now prove the following proposition.

Proposition 3.2. If (P) has an optimal solution $\bar{x}$ and $M(\theta_k) \neq \emptyset$, $k = 1, 2, \ldots$, then

$$(\theta_{k+1} - \bar{\theta}) \leq (1 - \underline{g}(\bar{x})/\bar{g}(x^k))(\theta_k - \bar{\theta}), \qquad k = 1, 2, \ldots. \tag{8}$$

Proof. By Proposition 3.1, we have

$$-F(\theta_k)/\bar{g}(x^k) \leq \theta_k - \theta_{k+1};$$

thus,

$$\theta_{k+1} \leq \theta_k + F(\theta_k)/\bar{g}(x^k),$$

yielding

$$\theta_{k+1} - \bar{\theta} \leq \theta_k - \bar{\theta} + F(\theta_k)/\bar{g}(x^k). \tag{9}$$

The existence of an optimal solution $\bar{x}$ of (P) implies that $F(\bar{\theta}) = 0$ and $\bar{x} \in M(\bar{\theta})$, hence $M(\bar{\theta}) \neq \emptyset$ [see Proposition 2.1(b), (c)]. Then, Proposition 2.2 can be applied, where (3) implies that

$$F(\theta_k) \leq F(\bar{\theta}) + (\bar{\theta} - \theta_k)\underline{g}(\bar{x}) = (\bar{\theta} - \theta_k)\underline{g}(\bar{x}).$$

This, together with (9), yields

$$\theta_{k+1} - \bar{\theta} \leq (\theta_k - \bar{\theta})(1 - \underline{g}(\bar{x})/\bar{g}(x^k)). \qquad \square$$

Corollary 3.1. If (P) has an optimal solution $\bar{x}$, if $M(\theta_k) \neq \emptyset$, $k = 1, 2, \ldots$, and if $\sup_k \bar{g}(x^k) < \infty$, then $\{\theta_k\}$ converges to $\bar{\theta}$ and it does so linearly.

The algorithm may still converge if some of the assumptions in Corollary 3.1 do not hold. This can be seen from Example 2.5. There, $\bar{\theta} = 0$, which is not attained. Starting with $x^0 = (1, 1)$, we obtain the sequence $\{\theta_k\}$ which satisfies

$$(\theta_{k+1} - \bar{\theta}) = [1/(2 + \theta_k)](\theta_k - \bar{\theta});$$

see above. Since

$$\tfrac{1}{3} \leq 1/(2 + \theta_k) \leq \tfrac{1}{2}, \qquad k = 1, 2, \ldots,$$

$\{\theta_k\}$ converges to $\bar{\theta}$, and it does so linearly. Note that, in this example, (P) does not have an optimal solution and

$$\sup_k \bar{g}(x^k) = \sup_k (1 + 2/\theta_k) = \infty.$$

In the following, we study two special cases: the case where S is compact and the linear case.

4. Compact Case

We show the following theorem.

Theorem 4.1. Assume that S is compact. The following results hold.

(a) Programs (P) and (P_θ) always have an optimal solution, $\bar{\theta}$ is finite, and $F(\bar{\theta}) = 0$. Hence, $F(\theta) = 0$ implies $\theta = \bar{\theta}$.

(b) F is finite, continuous, and decreasing on R.

(c) The sequence $\{\theta_k\}$, if not finite, converges linearly to $\bar{\theta}$, and each convergent subsequence of $\{x^k\}$ converges to an optimal soluton of (P).

Proof. Part (a) follows from the compactness of S and the continuity of f_i, g_i; in view of Proposition 2.1, we have $F(\bar{\theta})=0$. In Part (b), continuity of F follows in the same way as upper semicontinuity in Proposition 2.1(a), now using the compactness of S. The strict monotonicity of F is a consequence of Proposition 2.3. Part (c) follows from Corollary 3.1, since S is compact. Let $\{x^{k_l}\}$ be a convergent subsequence of $\{x^k\}$ converging to $\hat{x} \in S$. Since $x^{k_l} \in S$ and S is compact, we have $\hat{x} \in S$. Also,

$$F(\theta_{k_l}) = \max_i [f_i(x^{k_l}) - \theta_{k_l} g_i(x^{k_l})].$$

By continuity,

$$F(\bar{\theta}) = \max_i [f_i(\hat{x}) - \bar{\theta} g_i(\hat{x})].$$

Since $F(\bar{\theta})=0$, it follows that $\hat{x}$ is an optimal solution of (P). □

For $p=1$, it was shown in Ref. 9 that the factor $1-g_1(\bar{x})/g_1(x^k)$ in (8) converges to 0. Hence, the method is superlinear. Unfortunately, this is no longer true if $p>1$ as Example 2.3 demonstrates. Starting with $x^0 \in (0, 1]$, the sequence

$$\theta_k = (1/3^{k-1})x^0$$

is generated as seen before. This sequence converges to 0, but only linearly. Hence, the step from one to two ratios in (P) already destroys superlinear convergence.

Also, (8) indicates that, with increasing p, the algorithm becomes slower and slower, since $\bar{g}$ increases with p. The more ratios are involved, the slower the method will be.

We saw before that, for $p=1$, our method coincides with Newton's algorithm. For $p>1$, Newton's method may be quite different from our method, as Examples 2.3 and 2.5 above illustrate. Our algorithm may be slower than Newton's method in general. However, it has two advantages: (a) a subgradient of F, not readily available, need not be calculated; and (b) even for nonconcave functions F, our method converges.

We finally show under which condition we have, at least locally, concavity of F.

Proposition 4.1. Let

$$I(\theta) = \bigcup_{x \in M(\theta)} I(x, \theta).$$

Assume that, for some θ, the set $I(\hat{\theta})$ is reduced to a singleton. Then, F is concave in a neighborhood of $\hat{\theta}$ and, for $i \in I(\hat{\theta})$ and all $x \in M(\hat{\theta})$, $-g_i(x)$ is a subgradient of $\hat{F}$ at $\hat{\theta}$, where $\hat{F}$ is the restriction of F to the neighborhood.

Proof. Denote by $\hat{i}$ the unique element of $I(\hat{\theta})$. Since $M(\hat{\theta})$ is compact and f_i, g_i are continuous, there exist $\epsilon > 0$ and a neighborhood V of $\hat{\theta}$ such that

$$f_{\hat{i}}(x) - \theta g_{\hat{i}}(x) > f_i(x) - \theta g_i(x),$$
$$\text{for all } i \neq \hat{i},\ \theta \in V,\ x \in M(\hat{\theta}) + B(0, \epsilon),$$

where $B(0, \epsilon)$ denotes the ball of origin 0 and radius ϵ. Now, the compactness of S and the continuity of all the functions involved imply that the correspondence M is upper semicontinuous; i.e., there exists a neighborhood U of $\hat{\theta}$ such that

$$M(\theta) \subset M(\hat{\theta}) + B(0, \epsilon), \qquad \text{for all } \theta \in U$$

(maximum theorem). It follows that, for all $\theta \in U \cap V$,

$$F(\theta) = \inf\{f_{\hat{i}}(x) - \theta g_{\hat{i}}(x) | x \in M(\hat{\theta}) + B(0, \epsilon) \cap S\}.$$

Then, F is concave, since it is the infimum of concave functions in θ. □

Note, that, in Example 2.3 above, F is not locally concave at $\bar{\theta} = 0$. In this case, $I(\bar{\theta})$ is not a singleton. The algorithm converges only linearly.

5. Linear Case

In this section, the feasible region S is allowed to be unbounded. Consider the linear generalized fractional program

$$\text{(P)} \quad \bar{\theta} = \inf\left\{ \max_{1 \leq i \leq p} \frac{a_{i\cdot} x + \alpha_i}{b_{i\cdot} x + \beta_i} \,\middle|\, Cx \leq \gamma, x \geq 0 \right\}, \tag{10}$$

where $a_{i\cdot}$ $[b_{i\cdot}]$ is row i of a $p \times n$ matrix $A\,[B]$. Let

$$A = (a_{\cdot 1}, \ldots, a_{\cdot n}), \qquad B = (b_{\cdot 1}, \ldots, b_{\cdot n}),$$
$$\alpha = (\alpha_1, \ldots, \alpha_p)^T, \qquad \beta = (\beta_1, \ldots, \beta_p)^T.$$

Furthermore, C is a $m \times n$ matrix and $\gamma \in R^m$.

We make the following assumptions (see Refs. 3, 4):

(H1) feasibility assumption: there exists $\hat{x} \geq 0$ such that $C\hat{x} \leq \gamma$;
(H2) positivity assumption: $B > 0$ and $\beta > 0$.

As before, we are only interested in the case where $\bar{\theta}$ is finite.

In Refs. 3-4, the following dual program of (P) was introduced:

$$\text{(D)} \quad \bar{\lambda} = \sup_{\substack{u \geq 0 \\ u \neq 0 \\ w \geq 0}} \left\{ \min\left(\frac{\alpha^T u - \gamma^T w}{\beta^T u}, \min_{1 \leq j \leq n} \frac{a_{.j}^T u + c_{.j}^T w}{b_{.j}^T u} \right) \right\}, \tag{11}$$

and it was shown that $\bar{\lambda} = \bar{\theta}$ if (H1), (H2) hold.

The dual can be written as follows:

$$\bar{\theta} = \sup_{\substack{u \geq 0 \\ e^T u = 1}} \left\{ \sup_{w \geq 0} \left[\min\left(\frac{\alpha^T u - \gamma^T w}{\beta^T u}, \min_{1 \leq j \leq n} \frac{a_{.j}^T u + c_{.j}^T w}{b_{.j}^T u} \right) \right] \right\}. \tag{12}$$

The feasible region of (P) may be unbounded. However, the feasible region of (D) is at least bounded in u. We now want to study whether our algorithm converges when applied to (D), rather than (P). For this, we write (12) as a minimization problem:

$$\bar{\theta} = -\inf_{\substack{u \geq 0 \\ e^T u = 1}} \left\{ \inf_{w \geq 0} \left[\max\left(\frac{-\alpha^T u + \gamma^T w}{\beta^T u}, \max_{1 \leq j \leq n} \frac{-a_{.j}^T u - c_{.j}^T w}{b_{.j}^T u} \right) \right] \right\}. \tag{13}$$

Let $\bar{\mu} = -\bar{\theta}$. We want to determine the F-function of the minimization problem in (13),

$$F_D(\mu) = \inf_{\substack{u \geq 0 \\ e^T u = 1}} \{ \inf_{w \geq 0} [\max(-\alpha^T u + \gamma^T w - \mu \beta^T u, \max_{1 \leq j \leq n} -a_{.j}^T u - c_{.j}^T w - \mu b_{.j}^T u)] \}. \tag{14}$$

Hence,

$$F_D(\mu) = \inf_{\substack{u \geq 0 \\ e^T u = 1}} h(u; \mu), \tag{15}$$

where

$$h(u; \mu) = \inf_{w \geq 0} \max[-\alpha^T u + \gamma^T w - \mu \beta^T u, \max_{1 \leq j \leq n} (-a_{.j}^T u - c_{.j}^T w - \mu b_{.j}^T u)]. \tag{16}$$

Then,

$$\begin{aligned} h(u; \mu) &= \inf_{\substack{w \geq 0 \\ t \in R}} \{ t \mid t \geq -\alpha^T u + \gamma^T w - \mu \beta^T u, t \\ &\geq -a_{.j}^T u - c_{.j}^T w - \mu b_{.j}^T u, j = 1, \ldots, n \} \\ &= \inf_{w \geq 0, t} \{ o^T w + t \mid -\gamma^T w + t \geq (-\alpha - \mu \beta)^T u, c_{.j}^T w + t \\ &\geq (-a_{.j}^T - \mu b_{.j}^T) u, j = 1, \ldots, n \}. \end{aligned} \tag{17}$$

From (16), we see that $h(\mu;\mu)<\infty$, for all u, μ. Also, $h(u;\mu)>-\infty$, for all u, μ since assumption (H1) implies that there does not exist $\hat{w}\geq 0$ such that $C^T\hat{w}\geq 0$ and $\gamma^T\hat{w}<0$. Hence, $h(u;\mu)$ is finite for all u, μ. On the other hand, we see from (17) that $h(u;\mu)$ is the optimal value of a linear program in the parameters u, μ. Hence, $h(u;\mu)$ is continuous in u, μ. Then, (15) implies the continuity of $F_D(\mu)$. Such reasoning, together with Proposition 2.3, shows the following proposition.

Proposition 5.1. Suppose that (H1), (H2) hold and $\bar{\theta}$ is finite. Then, the function $F_D(\mu)$ is finite, continuous, and decreasing for all $\mu\in R$.

From Proposition 2.1(b), we conclude that the following corollary holds.

Corollary 5.1. If (H1), (H2) hold and $\bar{\theta}$ is finite, then $\bar{\mu}=-\bar{\theta}$ is the unique zero of F_D; i.e., $F_D(\bar{\mu})=0$.

In view of Proposition 2.1(d), an optimal solution $\bar{u}$, $\bar{w}$ of $F_D(\bar{\mu})=0$ is also an optimal solution of (D). Furthermore, as already said, $F_D(\mu)$ is attained for all μ at some $\bar{u}(\mu)$, $\bar{w}(\mu)$, where $\bar{w}(\mu)$ solves the linear program (17). Hence, the assumptions of Proposition 3.2 are satisfied if our algorithm is applied to $F_D(\mu)$. Thus, (8) holds. Moreover, the additional condition in Corollary 3.1, requiring that

$$\sup_{\substack{u\geq 0\\ e^Tu=1\\ w\geq 0}} \max(\beta^T u, \max_{1\leq j\leq n} b_{.j}^T u)<\infty,$$

is satisfied. Therefore, we have shown the following theorem.

Theorem 5.1. The sequence $\{\mu_k\}$, if not finite, converges linearly to $\bar{\mu}=-\bar{\theta}$.

Note that this is true even if the primal feasible region S is unbounded. If the primal optimal value $\bar{\theta}$ is attained in S, such a primal optimal solution can be calculated from a dual optimal solution $(\bar{u}, \bar{w})$ with the help of the complementary slackness conditions established in Ref. 3.

We demonstrate the algorithm by the following example where (D), but not (P), can be solved by the algorithm.

Example 5.1. Consider the following problem:

$$\text{(P)}\quad \inf\{1/(x+1) \mid -x\leq -1, x\geq 0\}=0.$$

Then,

$$F(\theta)=\begin{cases} 1-\theta, & \text{if } \theta\leq 0,\\ -\infty, & \text{if } \theta>0.\end{cases}$$

Thus, our algorithm cannot be applied to (P), since $F(\theta) = -\infty$, for all $\theta > \bar{\theta}$.

On the other hand, (P) does satisfy all the assumptions of Corollary 5.1; therefore, the method converges if applied to the dual (D). To see this, we determine the dual program

$$\text{(D)} \quad \sup_{\substack{u=1 \\ w\geq 0}}\{\min[(u+w/u), -w/u]\} = \sup\{\min(1+w, -w) | w \geq 0\}$$
$$= \sup\{-w | w \geq 0\}.$$

Then,

$$F_D(\mu) = \inf\{w - \mu | w \geq 0\} = -\mu, \qquad \text{for all } \mu \in R.$$

Hence, $F_D(\mu)$ is continuous and finite. Obviously, our method converges in one step starting with any $w^0 > 0$.

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