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Approximate Saddle-Point Theorems in Vector Optimization[1]

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Abstract. The paper contains definitions of different types of nondominated approximate solutions to vector optimization problems and gives some of their elementary properties. Then, saddle-point theorems corresponding to these solutions are presented with an application relative to approximate primal-dual pairs of solutions.

Key Words. Convex optimization problems, approximate nondominated solutions, saddle points, primal-dual pairs of solutions.

1. Introduction

The interest in approximate solutions of optimization problems has revived following recent developments in nondifferentiable optimization. Another area where these results are of considerable importance is approximation theory.

The aim of the present paper is to define different types of approximate efficient solutions to vector optimization problems, and then to develop the corresponding saddle-point theorems along the logic of Refs. 1 and 2. As a consequence of the fact that the notion of approximate solution coincides with that of exact solution in the case when the approximation error is zero, our results reduce to those obtained in the above-mentioned papers. Our definition is in coherence with that of ε-efficiency in Ref. 3. Another base of our theory is that of approximate solutions in the scalar-valued case (as expounded, e.g., in Ref. 4) or in the vectorial case for absolute optimality (Ref. 5).

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Section 2 is devoted to definitions and some basic properties of approximate extremal elements in ordered vector spaces; in Section 3, the main results are proved, i.e., Hurwitz-type saddle-point theorems. Applying our results, in Section 4 we show the equivalence between approximate saddle points and the corresponding primal-dual pairs of solutions.

Throughout the paper, we rely on a knowledge of convex analysis and the theory of ordered vector spaces, and therefore basic notions and facts are used without special explanation. If needed, see, e.g., Refs 6 and 7.

2. Approximate Nondominated Elements

All the vector spaces appearing in the paper are real and ordering cones are supposed to be convex, pointed, and algebraically closed. In the presence of a topological structure, we suppose local convexity and that the ordering cone is closed. We denote by X and V vector spaces and by (Z, K) an ordered vector space with $\operatorname{core}(K) \neq \phi$, where core refers to the algebraic interior. Similarly, rcore denotes the relative algebraic interior. (Y, C) is an order complete space, i.e., a vector lattice where every nonvoid set with a lower bound possesses an infimum. In order to ensure the existence of infima (resp., suprema) for every (i.e., nonbounded) set, we supplement the space (Y, C) with the elements ∞ and $-\infty$ using the notation $\bar{Y} = Y \cup \{-\infty, \infty\}$, and suppose that the usual algebraic and ordering properties hold. Hence, for the set $H \subset Y$, which is not bounded from below, we have $\inf(H) = -\infty$ and $\inf(\phi) = \infty$. The dual space of Y is Y', and the cone of positive functionals with respect to the cone $C \subset Y$ or the dual of C is C^+. The functional $y^* \in Y$ denotes an element of C^+ and $L^+(Z, Y) \subset L(Z, Y)$ stands for the cone of positive linear maps from Z to Y.

We recall now that, for the various ordering relationships between two elements of an ordered vector space, we shall use the following notations, for example, in (Y, C):

$$y_2 \geqq y_1, \qquad \text{iff } y_2 - y_1 \in C,$$

$$y_2 \geq y_1, \qquad \text{iff } y_2 - y_1 \in C \backslash \{0\},$$

$$y_2 > y_1, \qquad \text{iff } y_2 - y_1 \in \operatorname{core}(C).$$

To denote opposite relations, we use symbols like $\not\geq$ and $\not>$. Accordingly,

$$y_2 \geq y_1 \quad \text{or} \quad y_2 \not\geq y_1$$

refer to the fact that $y_1 \in Y$ dominates or does not dominate $y_2 \in Y$ from below, respectively.

The vectors e, $e_n \in Y$ and the scalars ε, $\varepsilon_n \in \mathbb{R}$ represent the approximation error; we suppose that $e \geqq 0$, $e_n \geqq 0$ hold and similarly that ε, ε_n are nonnegative. The above notations and conditions are supposed to be valid throughout the paper and will not be mentioned again.

Definition 2.1. The vector $y \in H$ is a $P(e)$-minimal element of $H \subset \bar{Y}$, or approximately Pareto minimal, in notation

$$y \in P(e) - \min(H),$$

if

$$(y - e - C) \cap H \subset \{y - e\};$$

it is $WP(e)$-minimal, in notation

$$y \in WP(e) - \min(H),$$

if

$$(y - e - \operatorname{core}(C)) \cap H = \phi.$$

Here, of course, we need the condition that $\operatorname{core}(C) \neq \phi$ and, speaking about WP-minimality, we always suppose it.

$y \in H$ is called $P(y^*, \varepsilon)$-minimal, in notation

$$y \in P(y^*, \varepsilon) - \min(H),$$

if

$$\langle y^*, y \rangle - \varepsilon \leqq \langle y^*, h \rangle, \qquad \forall h \in H.$$

By convention, we say that all kinds of minima of the void set consist of the single element $\infty \in \bar{Y}$. The approximately maximal elements are to be defined in a corresponding manner.

Remark 2.1. Our definitions, in the case $e = 0$ or $\varepsilon = 0$, reproduce the usual exact notions of minimality. Weak approximate minimality means the corresponding approximate minimality with respect to the algebraically nonclosed cone $C' = \{0\} \cup \operatorname{core}(C)$. The notion of $y \in Y$ being $P(y^*, \varepsilon)$-minimal means that $\langle y^*, y \rangle \in \mathbb{R}$ is a $P(\varepsilon)$-minimal element of the set

$$y^*(H) = \{\langle y^*, h \rangle \in \mathbb{R}: h \in H\}.$$

Now, the definition of the convex vector-valued minimization problem and the corresponding vector-valued Lagrangian follows (see, e.g., Ref. 8.). These notions constitute our main object of study.

Definition 2.2. Let

$$f: X \to Y \cup \{\infty\}, \qquad h: X \to Z \cup \{\infty\}$$

be proper convex functions with $\Delta = \operatorname{dom} f \cap \operatorname{dom} h \neq \phi$ and $l \in L(X, V)$. We define the minimization problem (MP) by way of the set of solutions

$$(MP) \qquad \min(MP) = \{x_0 \in F: f(x_0) \in \min\{f(F)\}\},$$

where

$$F = \{x \in X: x \in \Delta, h(x) \leqq 0, l(x) = 0\}$$

is called the feasibility set of problem (MP). Instead of the symbol min, one has to substitute one of the approximate or exact notions of minimality from Definition 2.1. Depending on this choice, we call the elements of $\min(MP)$ the respective type of approximate or exact solutions of the problem (MP).

The Lagrangian of the minimization problem (MP),

$$\Phi_L: X \times L(Z, Y) \times L(V, Y) \to \bar{Y},$$

is defined by the equality

$$\Phi_L(x, R, S) = \begin{cases} \infty, & \text{if } x \notin \Delta, \\ f(x) + R \cdot h(x) + S \cdot l(x) & \text{if } x \in \Delta \text{ and } R \in L^+(Z, Y), \\ -\infty, & \text{if } x \in \Delta \text{ and } R \notin L^+(Z, Y), \end{cases}$$

with the set

$$\operatorname{dom} \Phi_L = \{(x, R, S) \in X \times L(Z, Y) \times L(V, Y): x \in \Delta, R \in L^+(Z, Y)\}$$

called the domain of Φ_L.

The element $(x_0, R_0, S_0) \in X \times L(Z, Y) \times L(V, Y)$ is a respective type of saddle point for the Lagrangian Φ_L if the following conditions are met:

(i) $\Phi_L(x_0, R_0, S_0) \in \min\{\Phi_L(x, R_0, S_0) \in \bar{Y}: x \in X\}$;

(ii) $\Phi_L(x_0, R_0, S_0) \in \max\{\Phi_L(x_0, R, S) \in \bar{Y}: (R, S) \in L(Z, Y) \times L(V, Y)\}$.

Remark 2.2. In the scalar case, these notions coincide and we simply speak of ε-solutions or ε-saddle points.

Definition 2.3. We say that problem (MP) meets the Slater-Uzawa constraint qualification if either

(i) there exists an $x_1 \in \operatorname{rcore}(\Delta)$ with $h(x_1) \in -\operatorname{rcore}(K)$ and $l(x_1) = 0$,

or

(ii) no linear constraint is present and there exists an $x_1 \in \Delta$ with

$$h(x_1) \in -\operatorname{rcore}\{h(x) + k \in Z: x \in \Delta, k \in K\}.$$

For the reader's convenience, now we quote the scalar-valued version of Theorem 5 of Ref. 9.

Theorem 2.1. Suppose that, in the definition of (MP), the space (Y, C) coincides with $(\mathbb{R}, \mathbb{R}^+)$ and that (MP) meets the Slater-Uzawa constraint qualification. If $x_0 \in X$ is an ε-solution of (MP), then there exist functionals $r_0^* \in Z^+$ and $s_0^* \in V$, such that $(x_0, r_0^*, s_0^*) \in \operatorname{dom} \Phi_L$ is an ε-saddle point for the Lagrangian Φ_L.

Let us formulate some simple facts that are easy consequences of the definitions, but are still interesting because they clarify the relationships between the different notions of minimal solution. Omitted proofs are trivial.

Proposition 2.1. Suppose that $e_1 \leqq e_2$ and $\varepsilon_1 \leqq \varepsilon_2$. Then, we have

$$P(e_1)-\min(MP) \subset P(e_2)-\min(MP),$$

$$WP(e_1)-\min(MP) \subset WP(e_2)-\min(MP),$$

$$P(y^*, \varepsilon_1)-\min(MP) \subset P(y^*, \varepsilon_2)-\min(MP).$$

Proposition 2.2. (a) Suppose that we have $\langle y^*, \bar{e}\rangle > 0$ for each $\bar{e} \geq 0$. Then,

$$P(y^*, \varepsilon)-\min(MP) \subset P(e')-\min(MP),$$

with

$$e' = [\varepsilon/\langle y^*, e\rangle] \cdot e;$$

(b) $WP(e)-\min(MP) = \bigcup \{P(y^*, \langle y^*, e\rangle)-\min(MP)\colon y^* \in C^+ \backslash \{0\}\}.$

Proposition 2.3. Suppose that (Y, C) is equipped with such a weakly sequentially complete topology that the ordering cone $C \subset Y$ is normal and $\operatorname{int}(C) \neq \phi$. Consider a sequence $\{e_n \in C\colon n \in N\}$ decreasing to $e \in C$. Then,

$$P(e)-\min(MP) \subset \bigcap \{P(e_n)-\min(MP)\colon n \in N\} \subset WP(e)-\min(MP),$$

and

$$\bigcap \{WP(e_n)-\min(MP)\colon n \in N\} = WP(e)-\min(MP)$$

Proof. The first inclusion is obvious. For the second, let us reason by contradiction, and suppose that the element $x_0 \in F$ is not $WP(e)$-minimal. This means that we can find another $x_1 \in F$ with

$$f(x_1) < f(x_0) - e. \tag{1}$$

As $\operatorname{int}(C) \neq \phi$, the formula under (1) is equivalent to

$$f(x_0) - e - f(x_1) \in \operatorname{int}(C).$$

As a consequence of Corollary 3.5, Chapter 2, Ref. 6, for the sequence we have

$$\lim\{f(x_0)-e_n-f(x_1)\colon n\in N\}=f(x_0)-e-f(x_1)\in \operatorname{int}(C);$$

and so, there exists an $m\in N$ with

$$f(x_0)-e_m-f(x_1)\in \operatorname{int}(C).$$

This means that $f(x_1)$ dominates the element $f(x_0)-e_m\in Y$ from below. The proof of the second statement is analogous. □

Proposition 2.4. Suppose that the sequence $\{\varepsilon_n\in \mathbb{R}^+\colon n\in N\}$ decreases to $\varepsilon\in\mathbb{R}^+$. Then,

$$\bigcap\{P(y^*,\varepsilon_n)-\min(MP)\colon n\in N\}=P(y^*,\varepsilon)-\min(MP).$$

3. Saddle-Point Theorems

Proposition 3.1. The element $(x_0, R_0, S_0)\in X\times L(Z, Y)\times L(V, Y)$ is a $P(e)$-saddle point of the Lagrangian Φ_L, iff

(a) $\Phi_L(x_0, R_0, S_0)\in P(e)-\min\{\Phi_L(x, R_0, S_0)\in \bar{Y}\colon x\in X\}$;

(b) $x_0\in F$;

(c) $-e\not\geq R_0\cdot h(x_0)\leqq 0$.

Proof. Condition (a) is identical with the first part of the definition. Suppose now that $(x_0, R_0, S_0)\in X\times L(Z, Y)\times L(V, Y)$ is a $P(e)$-saddle point By the conditions set on problem (MP), $-\infty\neq\Phi_L(x_0, R_0, S_0)\neq\infty$, implying $x_0\in\Delta$ and $R_0\in L^+(Z, Y)$, so we have

$$\Phi_L(x_0, R_0, S_0)=f(x_0)+R_0\cdot h(x_0)+S_0\cdot l(x_0).$$

From the definition of $P(e)$-saddle point, we also know that

$$\Phi_L(x_0, R, S)\not\geq\Phi_L(x_0, R_0, S_0)+e, \tag{2}$$

for each $(R, S)\in L(Z, Y)\times L(V, Y)$. Selecting $S=S_0$ and $R=R_0$, we obtain

$$(R-R_0)\cdot h(x_0)\not\geq e, \qquad \forall R\in L^+(Z, Y), \tag{3}$$

and

$$(S-S_0)\cdot l(x_0)\not\geq e, \qquad \forall S\in L(V, Y), \tag{4}$$

respectively.

Suppose now that $h(x_0)\leqq 0$ does not hold. Then, by the strict algebraic separation theorem [see Ref. 10, Section 17.5(2)] applied for the sets

$\{h(x_0)\} \subset Z$ and $-K \subset Z$, the existence of a functional $z^* \in K^+$ is guaranteed with the property

$$\langle z^*, h(x_0)\rangle > 0.$$

Let $c \geq 0$, $c \in Y$ be an arbitrary, fixed element, and define the map $R \in L(Z, Y)$ as

$$R: z \to [\langle z^*, z\rangle / \langle z^*, h(x_0)\rangle](e + c) + R_0 z.$$

For the operator R, we obviously have $R \in L^+(Z, Y)$ and

$$(R - R_0) \cdot h(x_0) = e + c,$$

in contradiction with (3).

A similar argument leads to contradiction with (4), if we suppose $l(x_0) \neq 0$. Here, we define an operator $S \in L(V, Y)$ as

$$S: v \to [\langle v^*, v\rangle / \langle v^*, l(x_0)\rangle] \cdot (e + c) + S_0 v.$$

The last inequality in (c) is a consequence of $x_0 \in F$ and $R_0 \in L^+(Z, Y)$ while the first follows from (2), if we choose $(R, S) = (0, 0)$.

To prove the reverse implication, suppose that (a), (b), (c) are valid. From the last two, we have the following relations:

$$f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) + e \not\leq f(x_0) \geqq f(x_0) + R \cdot h(x_0) + S \cdot l(x_0),$$

for each $(R, S) \in L^+(Z, Y) \times L(V, Y)$, implying the missing relationship for $(x_0, R_0, S_0) \in \operatorname{dom} \Phi_L$ to be a $P(e)$-saddle point. □

Remark 3.1. The property stated in Proposition 3.1 is as much negative as positive; therefore it is a first sign of the problems to be seen in the sequel. Point (c), namely, turns into the well-known complementarity condition

$$R_0 \cdot h(x_0) = 0$$

in the case of exact saddle points. In general, however, it only means that

$$R_0 \cdot h(x_0) \in (-C \backslash \{-e - C\}) \cup \{-e\},$$

and the right-hand side here is an unbounded set.

The proof of the following statement is analogous.

Proposition 3.2. The element $(x_0, R_0, S_0) \in X \times L(Z, Y) \times L(V, Y)$ is a $WP(e)$-saddle point of the Lagrangian Φ_L iff

(a) $\Phi_L(x_0, R_0, S_0) \in WP(e) - \min\{\Phi_L(x, R_0, S_0) \in \bar{Y}: x \in X\}$;

(b) $x_0 \in F$;

(c) $-e \not> R_0 \cdot h(x_0) \leqq 0$.

Theorem 3.1. Suppose that the point $(x_0, R_0, S_0) \in \operatorname{dom} \Phi_L$ is a $P(e)$-saddle point [resp., $WP(e)$-saddle point or $P(y^*, \varepsilon)$-saddle point] of the Lagrangian Φ_L. Then, $x_0 \in X$ is an approximate solution of the minimization problem (MP) in the respective sense where the approximation error is

$$e' = e - R_0 \cdot h(x_0)$$

in the first and second cases and

$$\varepsilon' = 2 \cdot \varepsilon \tag{5}$$

in the last case.

Proof. By Propostion 3.1, $x_0 \in X$ is a feasible point. If $x \in F$ is another feasible point, then

$$f(x) \geqq f(x) + R_0 \cdot h(x) + S_0 \cdot l(x) \not\leq f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) - e,$$

and this means that

$$f(x) \not\leq f(x_0) - (e - R_0 \cdot h(x_0) - S_0 \cdot l(x_0)).$$

By feasibility, $l(x_0) = 0$, and so the first case is proved. The proof of the rest is analogous, with the additional use in the last case of the transitivity of the relation $\leqq$ on $\mathbb{R}$. □

Remark 3.2. Instead of the relation (5) for the approximation error $e' \in Y$, we have

$$0 \leqq e' \not\geq 2 \cdot e \quad \text{and} \quad 0 \leqq e' \not> 2 \cdot e,$$

as a consequence of the points (c) in Propositions 3.1 and 3.2 respectively. However, unlike the scalarized case, transitivity for the relation of nondomination or weak nondomination does not hold, and so we cannot claim in Theorem 3.1 that $x_0 \in X$ is a $P(2 \cdot e)$-solution or $WP(2 \cdot e)$-solution.

Proposition 3.3. Suppose that problem (MP) meets the Slater-Uzawa constraint qualification. If $x_0 \in X$ is a $P(y^*, \varepsilon)$-approximate solution of the problem, then there exist operators $R_0 \in L^+(Z, Y)$ and $S_0 \in L(V, Y)$ such that $(x_0, R_0, S_0) \in \operatorname{dom} \Phi_L$ is a $P(y^*, \varepsilon)$-saddle point of the Lagrangian Φ_L.

Proof. It is supposed that $x_0 \in X$ is an ε-solution of the scalar-valued optimization problem

$$\min\{\langle y^*, f(x)\rangle \in \mathbb{R}: x \in \Delta, h(x) \leqq 0, l(x) = 0\}.$$

By Theorem 2.1, there exist functionals $r_0^* \in K^+$ and $s_0^* \in V$ ensuring that (x_0, r_0^*, s_0^*) is an ε-saddle point for the Lagrangian corresponding to the above scalar problem; i.e.,

$$\langle y^*, f(x_0)\rangle + \langle r^*, h(x_0)\rangle + \langle s^*, l(x_0)\rangle - \varepsilon$$
$$\leqq \langle y^*, f(x_0)\rangle + \langle r_0^*, h(x_0)\rangle + \langle s_0^*, l(x_0)\rangle$$
$$\leqq \langle y^*, f(x)\rangle + \langle r_0^*, h(x)\rangle + \langle s_0^*, l(x)\rangle + \varepsilon.$$

If $c \in C$ is an element with $\langle y^*, c\rangle = 1$, then, defining $R_0 \in L^+(Z, Y)$ and $S_0 \in L(V, Y)$ with the following correspondences:

$$R_0 : z \to c \cdot \langle r_0^*, z\rangle, \qquad S_0 : v \to c \cdot \langle s_0^*, v\rangle,$$

the statement is proved. □

Theorem 3.2. Suppose that problem (MP) meets the Slater–Uzawa constraint qualification and $\text{core}(C) \neq \phi$. If $x_0 \in X$ is a $WP(e)$-solution of problem (MP), then there exist operators $R_0 \in L^+(Z, Y)$ and $S_0 \in L(V, Y)$ such that $(x_0, R_0, S_0) \in \text{dom}\, \Phi_L$ is a $WP(e)$-saddle point of the Lagrangian Φ_L.

Proof. By point (b) in Proposition 2.2, there exists a $y^* \in C^+$ such that $x_0 \in X$ is a $P(y^*, \langle y^*, e\rangle)$-solution of (MP), and so Proposition 3.3 implies that there exist a $P(y^*, \langle y^*, e\rangle)$-saddle point for Φ_L. Now, obviously for $y^* \in C^+$, we have that $\langle y^*, c_0\rangle > 0$ for each $c_0 \in \text{core}(C)$. From an easy argument, now we can conclude that this $P(y^*, \langle y^*, e\rangle)$-saddle point is a $WP(e)$-saddle point as well. □

Remark 3.3. A respective theorem concerning $P(e)$-solutions cannot be stated, since a $y^* \in C^+$, which is strictly positive for the whole cone $C \subset Y$, does not always exist.

4. Primal and Dual Functions

In this final section, we only deal with the scalarized case [i.e., $P(y^*, \varepsilon)$-type minimality], as otherwise the solution of the respective approximate primal problem carries little information, as is indicated in Remark 4.1.

Definition 4.1. We call the following set-valued maps the approximate primal and dual functions of the minimization problem (MP):

$$P(y^*, \varepsilon) : X \to 2^{\bar{Y}},$$
$$P(y^*, \varepsilon) : x \to P(y^*, \varepsilon) - \max\{\Phi_L(x, R, S):$$
$$(R, S) \in L(Z, Y) \times L(V, Y)\},$$

and

$$D(y^*, \varepsilon): L(Z, Y) \times L(V, Y) \to 2^{\bar{Y}},$$
$$D(y^*, \varepsilon): (R, S) \to P(y^*, \varepsilon) - \min\{\Phi_L(x, R, S): x \in X\}.$$

The approximate primal and dual problems $(P(y^*, \varepsilon))$ and $(D(y^*, \varepsilon))$ are defined in terms of the functions $P(y^*, \varepsilon)$ and $D(y^*, \varepsilon)$. Accordingly, $x_0 \in X$ or $(R_0, S_0) \in L^+(Z, Y) \times L(V, Y)$ is a solution of the approximate primal or dual problems, if

$$P(y^*, \varepsilon)(x_0) \cap P(y^*, 3\varepsilon) - \min\{\bigcup P(y^*, \varepsilon)(x): x \in X\} \neq \phi$$

or

$$D(y^*, 2\varepsilon)(R_0, S_0) \cap P(y^*, \varepsilon) - \max\{\bigcup D(y^*, \varepsilon)(R, S):$$
$$R \in L(Z, Y), S \in L(V, Y)\} \neq \phi,$$

respectively.

Proposition 4.1. We have

$$P(y^*, \varepsilon)(x) \subset P(y^*, \varepsilon) - \max\{f(x) - C\}, \qquad \forall x \in F,$$
$$P(y^*, \varepsilon)(x) = \{\infty\}, \qquad \forall x \in X \backslash F.$$

Proof. For $x \in F$, we have

$$\{\Phi_L(x, R, S) \in Y \cup \{-\infty, \infty\}: R \in L^+(Z, Y), S \in L(V, Y)\}$$
$$= \{f(x) + R \cdot h(x): R \in L^+(Z, Y)\} \subset f(x) - C,$$

as well as

$$\sup\{\langle y^*, \Phi_L(x, R, S)\rangle \in \mathbb{R}: R \in L^+(Z, Y), S \in L(V, Y)\}$$
$$= \sup\{\langle y^*, f(x) - c\rangle: c \in C\} = \langle y^*, f(x)\rangle,$$

implying the statement if $x \in F$. The rest is obvious. □

Remark 4.1. If we define, e.g., the approximate primal problem $(P(e))$ in a corresponding manner to Definition 4.1, then the analogue of Proposition 4.1, is valid, and in such a way that the set $P(e)(x)$ is not bounded from below if $x \in F$ and $h(x) \neq 0$. As a consequence, it would have only $-\infty$ as a solution. As we know from, e.g., Ref. 11, this irregularity disappears if $e = 0$.

Proposition 4.2. (a) If $x_0 \in X$ is a $P(y^*, \varepsilon)$-solution of problem (MP), then it is a solution of problem $(P(y^*, \varepsilon))$.

(b) If $x_0 \in X$ is a solution of problem $(P(y^*, \varepsilon))$, then it is a $P(y^*, 4\varepsilon)$ solution of problem (MP).

Proof. (a) By Propostion 4.1, we have, for all $x \in F$, that

$$f(x) \in P(y^*, \varepsilon)(x).$$

Therefore, it is sufficient to prove that

$$f(x_0) \in P(y^*, 3\varepsilon) - \min\{\bigcup \{P(y^*, \varepsilon)(x) \colon x \in X\}\}. \tag{6}$$

Again, by the last proposition,

$$\bigcup \{P(y^*, \varepsilon)(x) \colon x \in F\} \subset \bigcup \{P(y^*, \varepsilon) - \max\{f(x) - C\}\}$$
$$= \{y \in \bar{Y} \colon \exists x \in F, y \leqq f(x), \langle y^*, y\rangle \geqq \langle y^*, f(x)\rangle - \varepsilon\}.$$

As we supposed that $x_0 \in X$ is a $P(y^*, \varepsilon)$-solution, we also have the inequality

$$\langle y^*, f(x_0)\rangle - 3\varepsilon \leqq \langle y^*, f(x)\rangle - \varepsilon, \qquad \forall x \in F.$$

Hence, by the definition of $P(y^*, 3\varepsilon) - \min$, the validity of (6) now follows.

(b) Let us suppose now that $x_0 \in X$ solves $(P(y^*, \varepsilon))$, i.e., there exists an

$$y_0 \in P(y^*, \varepsilon)(x_0) \cap P(y^*, 3\varepsilon) - \min\{\bigcup \{P(y^*, \varepsilon)(x) \colon x \in X\}\}.$$

Belonging to the first set means that

$$y_0 = f(x_0) - c_0, \tag{7}$$

where

$$c_0 \in C \quad \text{and} \quad 0 \leqq \langle y^*, c_0\rangle \leqq \varepsilon.$$

As we have, for all $x \in X \backslash F$, that

$$P(y^*, \varepsilon)(x) = \{\infty\},$$

it is enough to consider $x \in F$, implying that

$$f(x) \in P(y^*, \varepsilon)(x).$$

Hence, belonging to the second set implies that

$$\langle y^*, y_0\rangle - 3\varepsilon \leqq \langle y^*, f(x)\rangle, \qquad \forall x \in X,$$

and by (7)

$$\langle y^*, f(x_0)\rangle - 4\varepsilon \leqq \langle y^*, f(x)\rangle, \qquad \forall x \in X.$$

□

Definition 4.2. The element $(x_0, R_0, S_0) \in X \times L(Z, Y) \times L(V, Y)$ is called a $P(y^*, \varepsilon)$-dual pair of solutions if these conditions hold:

(i) $x_0 \in X$ is a solution of the problem $(P(y^*, \varepsilon))$;

(ii) $f(x_0) \in D(y^*, 2\varepsilon)(R_0, S_0) \cap P(y^*, \varepsilon) - \max\{\bigcup \{D(y^*, \varepsilon)(R, S) \colon R \in L(Z, Y), S \in L(V, Y)\}$.

Remark 4.2. The definition could equivalently be formulated as: $x_0 \in X$ and $(R_0, S_0) \in L(Z, Y) \times L(V, Y)$ is a solution of the primal and the dual problem, respectively, where the latter is valid by way of $f(x_0) \in Y$.

Theorem 4.1. (a) If $(x_0, R_0, S_0) \in \text{dom}\, \Phi_L$ is a $P(y^*, \varepsilon)$-saddle point of the Lagrangian Φ_L, then it is a $P(y^*, \varepsilon)$-dual pair of solutions.

(b) If $(x_0, R_0, S_0) \in X \times L(Z, Y) \times L(V, Y)$ is a $P(y^*, \varepsilon)$-dual pair of solutions, then it is a $P(y^*, 2\varepsilon)$-saddle point of the Lagrangian Φ_L.

Proof. (a) On one hand, by Proposition 4.1, we have

$$f(x_0) \in P(y^*, \varepsilon)(x_0).$$

On the other hand, by Theorem 3.1, we know that $x_0 \in X$ is a $P(y^*, 2\varepsilon)$-solution of problem (MP). Together with Proposition 4.1, this yields the relation

$$\langle y^*, f(x_0)\rangle - 3\varepsilon \leqq \langle y^*, y\rangle, \qquad \forall y \in P(y^*, \varepsilon)(x),$$

i.e.,

$$f(x_0) \in P(y^*, 3\varepsilon) - \min\{P(y^*, \varepsilon) \colon x \in X\}.$$

This proves the first requirement of $(x_0, R_0, S_0) \in \text{dom}\, \Phi_L$ being a $P(y^*, \varepsilon)$-dual pair of solutions. If $(x_0, R_0, S_0) \in \text{dom}\, \Phi_L$ is a $P(y^*, \varepsilon)$-saddle point, then, by (c) Proposition 3.2, we have

$$\langle y^*, f(x_0)\rangle - \varepsilon \leqq \langle y^*, \Phi_L(x_0, R_0, S_0)\rangle;$$

and also, by the definition of saddle point,

$$\Phi_L(x_0, R_0, S_0) \in P(y^*, \varepsilon) - \min\{\Phi_L(x, R_0, S_0) \colon x \in X\}.$$

If we combine these two relations, we obtain

$$\langle y^*, f(x_0)\rangle - 2\varepsilon \leqq \langle y^*, \Phi_L(x, R_0, S_0)\rangle, \qquad \forall x \in X;$$

and as a consequence,

$$f(x_0) \in D(y^*, 2\varepsilon)(R_0, S_0).$$

We also have to prove that

$$f(x_0) \in P(y^*, \varepsilon) - \max\{D(y^*, \varepsilon)(R, S) \colon R \in L(Z, Y), S \in L(V, Y)\}.$$

If this is not so, then there exist $R \in L(Z, Y)$, $S \in L(V, Y)$, $y_1 \in D(y^*, \varepsilon)(R, S)$ such that

$$\langle y^*, y_1\rangle > \langle y^*, f(x_0)\rangle + \varepsilon. \tag{8}$$

Here, it is necessary that $R \in L^+(Z, Y)$ be valid, because otherwise $D(y^*, \varepsilon)(R, S) = \{-\infty\}$, and consequently $\langle y^*, y_1 \rangle = -\infty$. Therefore, it is the finite values of Φ_L that define $D(y^*, \varepsilon(R, S)$, i.e.,

$$D(y^*, \varepsilon)(R, S) = P(y^*, \varepsilon) - \min\{f(x) + R \cdot h(x) + S \cdot l(x): x \in \Delta\}, \quad (9)$$

implying that

$$y_1 = f(x_1) + R \cdot h(x_1) + S \cdot l(x_1),$$

for some $x_1 \in \Delta$. Using (c) in Proposition 3.2 and the formula under (8), we obtain

$$\langle y^*, y_1 \rangle > \langle y^*, f(x_0) + R \cdot h(x_0) + S \cdot l(x_0) \rangle + \varepsilon.$$

This and $y_1 \in D(y^*, \varepsilon)(R, S)$, however, contradict (9). So, the second requirement is proved.

(b) By the first part of the definition of the $P(y^*, \varepsilon)$-dual pair of solutions, the conditions imply that $\infty \notin P(y^*, \varepsilon)(x_0)$; therefore, by Proposition 4.1, $x_0 \in F$ holds. By the second part, we know that $-\infty \notin D(y^*, 2\varepsilon)(R_0, S_0)$; therefore, $R_0 \in L^+(Z, Y)$, implying that $(x_0, R_0, S_0) \in \operatorname{dom} \Phi_L$. As a consequence of $x_0 \in F$, we have

$$\langle y^*, \Phi_L(x_0, R_0, S_0) \rangle \leqq \langle y^*, f(x_0) \rangle,$$

and so

$$f(x_0) \in D(y^*, 2\varepsilon)(R_0, S_0) \quad (10)$$

implies that

$$\Phi_L(x_0, R_0, S) \in P(y^*, 2\varepsilon) - \min\{\Phi_L(x, R_0, S_0): x \in X\}. \quad (11)$$

From (10), it also follows that

$$\langle y^*, f(x_0) \rangle - 2\varepsilon \leqq \langle y^*, f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \rangle,$$

i.e.,

$$-2\varepsilon \leqq \langle y^*, R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \rangle. \quad (12)$$

By (11), (12) and the relation $x_0 \in F$, Proposition 3.2 can be applied; therefore, $x_0 \in F$ is a $P(y^*, 2\varepsilon)$-saddle point of Φ_L. □

References

1. TANINO, T., and SAWARAGI, Y., *Duality Theory in Multiobjective Programming*, Journal of Optimization Theory and Applications, Vol. 31, pp. 509–529, 1980.
2. CORLEY, H. W., *Duality Theory with Respect to Cones*, Journal of Mathematical Analysis and Applications, Vol. 84, pp. 560–568, 1981.

3. LORIDAN, P., *ε-Solutions in Vector Minimization Problems*, Journal of Optimization Theory and Applications, Vol. 43, pp. 265–276, 1984.
4. STRODIOT, J. J., NGUYEN, V. H., and HEUKEMES, N., *ε-Optimal Solutions in Nondifferentiable Convex Programming and Some Related Questions*, Mathematical Programming, Vol. 25, pp. 307–328, 1983.
5. KUTATELADZE, S. S., *ε-Subdifferentials and ε-Optimality*, Sibirskii Matematicheskii Zhurnal, Vol. 21, pp. 120–130, 1980 (in Russian).
6. PERESSINI, A. L., *Ordered Topological Vector Spaces*, Harper and Row, New York, New York, 1967.
7. HOLMES, R. B., *Geometric Functional Analysis*, Springer, Berlin, Germany, 1975.
8. ZOWE, J., *Konvexe Funktionen und Konvexe Dualitätstheorie in Geordneten Vektorräumen*, Habilitationsschrift dem Naturwissenschaftlichen Fachbereich IV der Bayerischen Julius-Maximilians-Universität Würzburg, Würzburg, Germany, 1976.
9. VÁLYI, I., *On Duality Theory Related to Approximate Solutions of Vector Optimization Problems*, Nondifferentiable Optimization: Motivations and Applications, Edited by V. F. Demyanov and D. Pallaschke, Springer, Berlin, Germany, 1985.
10. KÖTHE, G., *Topologische Lineare Räume, I*, Springer, Berlin, Germany, 1966.
11. LUC, D. T., *On Duality Theory in Multiobjective Programming*, Journal of Optimization Theory and Applications, Vol. 43, pp. 557–582, 1984.