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TECHNICAL NOTE

Recession Cones and Asymptotically Compact Sets[1]

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Abstract. The present note is concerned with the study of the relations between the notions of asymptotic cones introduced by Dedieu and that of recession cones introduced by Luc. Conditions under which these notions coincide are given, as well as the fact that the compactness condition used by Luc is related (more restrictively) to asymptotic compactness. As an application of these notions, a result on proper efficiency in the sense of Lampe, established by Luc in finite dimensions, is extended to the infinite-dimensional case.

Key Words. Linear topological spaces, asymptotic cones, recession cones, asymptotically compact sets, asymptotically bounded sets, bases of cones, proper efficiency.

1. Introduction

The notion of recession cones has been known for a long time in the study of convex sets. The corresponding notion for nonconvex sets was introduced by Debreu (Ref. 1) in finite-dimensional spaces and by Dedieu (Ref. 2) in infinite-dimensional spaces. Luc (Refs. 3 and 4) introduced another notion of recession cones, similar to that of Debreu. One knows that the pairing between recession cones and compactness gives good results for closedness of some sets (see Refs. 5, 6, 7) as well as stability results in mathematical programming (see Refs. 8 and 9) and formulas for recession cones (see Refs. 7, 8, 9, 3, 4). The above applications are interesting and useful, not only in normed spaces, but also in topological linear spaces. For example, sometimes one requires that the sum of two weakly closed

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sets be also weakly closed. So, the main purpose of this note is to present some situations when these two notions coincide, other than the case of normed linear spaces mentioned by Luc. We also relate the compactness conditions of Luc (Refs. 3 and 4) to asymptotic compactness. As an application of these notions, we extend a result of Luc (Refs. 3 and 4) concerning proper efficiency in the sense of Lampe, stated in finite-dimensional spaces, to the infinite-dimensional case.

2. Preliminary Notions and Notations

Throughout the paper, E is a separated topological linear space and E^* its topological dual. $\mathcal{V}$ denotes the family of balanced closed neighborhoods of the origin in E, while $\mathcal{B}$ denotes the class of bounded subsets of E; when E is a locally convex space (in short, l.c.s.), the elements of $\mathcal{V}$ are also supposed to be convex. For $A \subset E$, cone(A), cl(A), and int(A) denote the generated cone, the closure, and the interior of A, respectively (cone($\emptyset$) $:= \{0\}$). By cone, we mean a nonempty subset C of E such that $tx \in C$ for $t \geq 0$ and $x \in C$. If $U \in \mathcal{V}$, p_U denotes the Minkowski functional associated to U. If C is a cone, we say that $B \subset E$ is a base for C if $C = \text{cone}(B)$ and $0 \notin \text{cl}(B)$. The set $A \subset E$ is relatively compact (r.c.) if cl(A) is compact; A is locally relatively compact (l.r.c.) if, for every $a \in A$, there is $V \in \mathcal{V}$ such that $A \cap (a+V)$ is r.c. A is asymptotically compact (see Ref. 5), in short a.c., if there are $\epsilon > 0$ and $V \in \mathcal{V}$ such that $([0, \epsilon] \cdot A) \cap V$ is r.c., where $\Lambda \cdot A = \{\lambda a \mid \lambda \in \Lambda, a \in A\}$. Note that, in this definition, one may take $\epsilon = 1$ or other fixed $\epsilon > 0$. Similarly, we say that A is locally bounded if, for every $a \in A$, there is $V \in \mathcal{V}$ such that $A \cap (a+V)$ is bounded. A is asymptotically bounded (a.b.) if there are $\epsilon > 0$ and $V \in \mathcal{V}$ such that $([0, \epsilon] \cdot A) \cap V$ is bounded. Note that, in this definition, one may also take $\epsilon = 1$ or other fixed $\epsilon > 0$. Of course, if the origin of E has a bounded neighborhood (e.g., if E is normable), then every subset of E is locally bounded and asymptotically bounded. The asymptotic cone of A (see Ref. 2) is the set

$$A_\infty := \bigcap_{t>0} \text{cl}([0, t] \cdot A).$$

It has been observed in Ref. 9 and elsewhere (and it is easy to show) that $x \in A_\infty$ iff there exist the nets $(t_i) \subset]0, \infty[$, $(x_i) \subset A$ such that $t_i \to 0$, $t_i x_i \to x$. If the topology of E is metrizable, then we can work with sequences instead of nets.

In the next proposition, we collect some properties of asymptotically bounded sets. As its proof is similar to that of Proposition 2.2 of Ref. 9, we omit it.

Proposition 2.1. Let A, $B \subset E$ and $D \subset F$, where F is another separated topological linear space.

(i) A is a.b.$\Leftrightarrow$cl(A) is a.b.; if A is a.b. and $B \subset A$, then B and A_∞ are a.b.

(ii) If A is a.b., then A is locally bounded. If A is radiant in $a \in A$, i.e., there exists $\lambda \in]0, 1]$ such that $[0, \lambda] \cdot (A - a) \subset A - a$, and if there exists $V \in \mathscr{V}$ such that $(a + V) \cap A$ is bounded, then A is a.b. In particular, if A is convex or A is cone, then A is locally bounded iff A is a.b.

(iii) If A and B are a.b., then $A \cup B$ is a.b. Let B be bounded; then A is a.b. iff $A + B$ is a.b.

(iv) If A and D are a.b., then $A \times D$ is a.b.

(v) If A is bounded or asymptotically compact, then A is a.b.

Luc (Refs. 3 and 4) defined the recession cone of A as

$$\text{Rec}(A) := \bigcap_{B \in \mathscr{B}} \text{cl}(\text{cone}(A \backslash B)).$$

As observed in Refs. 3 and 4, $A_\infty \subset \text{Rec}(A)$ and they coincide when E is a normed space.

3. Results

First, we give a characterization of asymptotically compact cones.

Proposition 3.1. Let $C \subset E$ be a cone. The following assertions are equivalent:

(i) C is asymptotically compact;

(ii) C is locally relatively compact;

(iii) C has a relatively compact base;

(iv) cl(C) has a compact base;

(v) cl(C) is locally compact.

Proof.

(i)$\Leftrightarrow$(ii) is known (see Refs. 5 and 9) and is very simple to show.

(ii)$\Rightarrow$(iii). Let C be l.r.c.; then, there exists $U \in \mathscr{V}$ such that $U \cap C$ is r.c. Take $B := C \cap (U \backslash \text{int}(U))$. Of course, B is r.c., being contained in $U \cap C$, and $0 \notin U \backslash \text{int}(U) \supset \text{cl}(B)$. We also have cone$(B) \subset C$. Now, let $x \in C$, $x \neq 0$. The set $\Lambda = \{t\colon t \geq 0, tx \in U\}$ is bounded; otherwise, $\{tx\colon t > 0\} \subset C \cap U$, contradicting the fact that $C \cap U$ is r.c. Take $t_0 = \sup(\Lambda)$; then, $0 < t_0 \in \Lambda$. If $t_0 x \in \text{int}(U)$, then $tx \in \text{int}(U)$, for some $t > t_0$, which is a contradiction. Therefore, $t_0 x \in B$ and so $x \in \text{cone}(B)$. Thus, $C = \text{cone}(B)$.

(iii)$\Rightarrow$(iv). Let $C = \mathrm{cone}(B)$, with $0 \notin \mathrm{cl}(B)$ and B r.c. Of course, $\mathrm{cl}(B)$ is compact and $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{cone}(B)) \supset \mathrm{cone}(\mathrm{cl}(B))$; as $\mathrm{cone}(\mathrm{cl}(B))$ is closed, the other inclusion holds, too. Hence, $\mathrm{cl}(C) = \mathrm{cone}(\mathrm{cl}(B))$.

(iv)$\Rightarrow$(v). Let $\mathrm{cl}(C) = \mathrm{cone}(B)$ with B compact, $0 \notin B$. Then, there exists $U \in \mathscr{V}$ such that $U \cap B = \varnothing$. It follows that $U \cap \mathrm{cl}(C) \subset [0, 1] \cdot B$. Thus, $U \cap \mathrm{cl}(C)$ is compact. Now, for every $0 \neq x \in C$, there exists $t > 0$ such that $x \in \mathrm{int}(tU)$. Therefore, tU is a neighborhood of x and $tU \cap \mathrm{cl}(C)$ is compact.

(v)$\Rightarrow$(i). As $\mathrm{cl}(C)$ is locally compact, there exists $U \in \mathscr{V}$ such that $U \cap \mathrm{cl}(C)$ is compact. Therefore, $U \cap ([0, 1] \cdot C) = U \cap C \subset U \cap \mathrm{cl}(C)$, so that C is asymptotically compact. □

Remark 3.1.

(a) The equivalence of (iv) and (v) is shown in Ref. 10.

(b) Luc (see Refs. 3 and 4) introduced and used the condition

(CB) $\exists B \in \mathscr{B}$ such that $\mathrm{cl}(\mathrm{cone}(X \backslash B))$ has a compact base for a subset X of E.

The equivalence of (iii) and (iv) shows that condition (CB) is equivalent to

(CB′) $\exists B \in \mathscr{B}$ such that $\mathrm{cone}(X \backslash B)$ has an r.c. base.

(c) If $\mathrm{cone}(A)$ is a.c., then A is asymptotically compact. The converse may not be true (see Example 3.1).

(d) If $X \subset E$ satisfies (CB) and every bounded subset of X is relatively compact, then X is asymptotically compact. Such a condition was used in Refs. 3 and 4 for obtaining the fact that a set has the domination property or that the sum of two closed sets is closed.

Example 3.1. Let E be an infinite-dimensional reflexive Banach space endowed with its weak topology, and let $A = \{x \in E : \|x\| \leq 1\}$. This set A is w-compact and therefore a.c., but $\mathrm{cone}(A) = E$ is not a.c. because all w-neighborhoods of the origin are unbounded.

Similar characterizations hold for asymptotically bounded cones.

Proposition 3.2. Let $C \subset E$ be a cone. The following assertions are equivalent:

(i) C is asymptotically bounded;
(ii) C is locally bounded;
(iii) C has a bounded base.

Proof. The equivalence of (i) and (ii) is mentioned in Proposition 2.1. The proof of the implication (ii)$\Rightarrow$(iii) is, practically, the same as that of the corresponding implication from Proposition 3.1. For (iii)$\Rightarrow$(i), let $C = \text{cone}(B)$, with B bounded and $0 \notin \text{cl}(B)$; there exists $V \in \mathscr{V}$ such that $B \cap V = \emptyset$. Then, $V \cap C \subset [0, 1] \cdot B$, and so C is asymptotically bounded. □

Similar to asymptotic compactness, if cone(A) is a.b., then A is a.b., but the converse may not be true; the set A of Example 3.1 is w-bounded, but $\text{cone}(A) = E$ is not w-asymptotically bounded.

The next result gives a condition for having $\text{Rec}(A) = A_\infty$.

Proposition 3.3. Let $A \subset E$ be nonempty, and assume that there exists a bounded subset A_0 of E such that $\text{cone}(A \backslash A_0)$ is locally bounded. Then, $\text{Rec}(A) = A_\infty$. In particular, the conclusion holds if A satisfies (CB).

Proof. By the definition of Rec(A), $\text{Rec}(A) = \text{Rec}(A \backslash A_0)$. So, by Proposition 3.2, we may suppose that $A \subset \text{cone}(B)$ for some bounded set B with $0 \notin \text{cl}(B)$. Let $0 \neq x \in \text{Rec}(A)$; then, for all $V \in \mathscr{V}$ and $n \in \mathbb{N}$, $(x + V) \cap \text{cone}(A \backslash [0, n] \cdot B) \neq \emptyset$. Therefore, for $i = (V, n)$, there exist $t_i \geq 0$, $a_i \in A \backslash [0, n] \cdot B$ such that $x - t_i a_i \in V$. Hence, $\lim(t_i a_i) = x$. As $a_i \in A \backslash [0, n] \cdot B$, $a_i = s_i b_i$ with $s_i \geq n$. Thus, $\lim s_i = \infty$. There exist $M > 0$ and i_0 such that, for $i \geq i_0$, $t_i s_i \leq M$. Otherwise, for a subnet, $t_j s_j \to \infty$. Then, $0 = \lim(t_j s_j)^{-1}(t_j a_j) = \lim b_j$; i.e., $0 \in \text{cl}(B)$, which is a contradiction. From the estimate $t_i s_i \leq M$ and $s_i \to \infty$, we get $t_i \to 0$. Hence, $x \in A_\infty$. □

The same conclusion can be obtained under different conditions.

Proposition 3.4. Suppose that E is a locally convex space and $A \subset E$ is asymptotically bounded. Then, $\text{Rec}(A) = A_\infty$.

Proof. As $A_\infty \subset \text{Rec}(A)$, let us show the converse inclusion. There is $U \in \mathscr{V}$ such that $U \cap ([0, 1] \cdot A) =: B$ is bounded. Let $p = p_U$, $0 \neq x \in \text{Rec}(A)$, and $t > 0$; consider $s > \max\{1, (1 + p(x))/t\}$. For every $V \in \mathscr{V}$, $V \subset U$, $(x + V) \cap \text{cone}(A \backslash sB) \neq \emptyset$, and so there exist $a \in A \backslash sB$, $t' \geq 0$ such that $t'a \in x + V$. As $a \notin sB$ and $s > 1$, $s^{-1}a \notin U$ and so $p(a) > s$. Since $t'a - x \in V \subset U$, $1 \geq p(t'a - x) \geq t'p(a) - p(x) \geq t's - p(x)$; i.e., $t' \leq (1 + p(x))/s < t$. It follows that $(x + V) \cap ([0, t] \cdot A) \neq \emptyset$. Therefore, $x \in \bigcap_{t>0} \text{cl}([0, t] \cdot A) = A_\infty$. □

Note that the hypotheses of Propositions 3.3 and 3.4 cannot be ordered unless we are in particular spaces.

The next result was proved by Goossens (Ref. 7) and Zǎlinescu (Ref. 9) independently for X asymptotically compact and by Luc (Refs. 3 and 4) for E normable and X satisfying condition (CB). A direct proof is possible, but the result is substantially included in Refs. 7 and 9.

Proposition 3.5. Let X and Y be nonempty subsets of E. Suppose that X satisfies the condition

(CA) $\exists X_0 \in \mathscr{B}$ such that $X \backslash X_0$ is asymptotically compact,

and $X_\infty \cap (-Y_\infty) = \{0\}$. Then, $(X+Y)_\infty \subset X_\infty + Y_\infty$.

Proof. We may take $X_0 \subset X$. Because $X+Y=(X_0 \cup (X\backslash X_0))+Y=(X_0+Y)\cup((X\backslash X_0)+Y)$ and $(X_0+Y)_\infty = Y_\infty$, it is sufficient to suppose that X is a.c. As $\mathrm{cl}(X)$ is a.c. and $(\mathrm{cl}(A))_\infty = A_\infty$ (see Ref. 9), by Corollary 3.12 of Ref. 9, $\mathrm{cl}(X)+\mathrm{cl}(Y)$ is closed [hence $\mathrm{cl}(X+Y)=\mathrm{cl}(X)+\mathrm{cl}(Y)$] and $(X+Y)_\infty \subset X_\infty + Y_\infty$. Condition (C) (see below) was used only to obtain the converse inclusion; see Proposition 2.1 of Ref. 9. □

The first part of the next result is contained in Ref. 7.

Proposition 3.6. Let X, Y be nonempty subsets of E. Suppose that X is asymptotically compact and $X_\infty \cap (-Y_\infty) = \{0\}$.

(i) If Y is asymptotically compact, then so is $X+Y$.
(ii) If Y is asymptotically bounded, then so is $X+Y$.

Proof. Because X is a.c., there exists $U \in \mathscr{V}$ such that $U \cap [0,1]\cdot X =: B$ is r.c. Let us show that

$$\forall \epsilon > 0, \quad W \in \mathscr{V}, \quad \exists \delta > 0, \quad V \in \mathscr{V}:$$

$$V \cap [0,\delta]\cdot(X+Y) \subset (U \cap [0,1]\cdot X) + (W \cap [0,\epsilon]\cdot Y). \tag{1}$$

In the contrary case, there are $\epsilon > 0$ and $W \in \mathscr{V}$ such that

$$\forall \delta > 0, \quad V \in \mathscr{V}:$$

$$V \cap [0,\delta]\cdot(X+Y) \not\subset (U \cap [0,1]\cdot X) + (W \cap [0,\epsilon]\cdot Y) =: A.$$

Without loss of generality we may take $W=U$. Therefore, for every $n \in \mathbb{N}^*$ and every $V = \mathscr{V}$, there exist $t_i \in]0, 1/n]$ and $x_i \in X$, $y_i \in Y$ such that $t_i(x_i + y_i) \in V \backslash A$, where $i=(n, V)$. It follows that $t_i \to 0$ and $t_i(x_i+y_i) \to 0$. Take $p = p_U$. If $(p(t_i x_i))$ has a bounded subnet, then the corresponding subnet $(t_j x_j)$ is contained in tB for some $t>0$; therefore, we may assume that

$t_j x_j \to x \in X_\infty$. It follows that $t_j y_j \to -x$. Hence, $x = 0$. This implies that $t_j(x_j + y_j) \in A$ for $j \geq j_0$, which is a contradiction. Therefore, $p(t_i x_i) \to \infty$. By Proposition 2.2(v) of Ref. 9, there exist a subnet (x_j) of (x_i) and $(s_j) \subset]0, \infty[$ such that $s_j p(x_j) \leq 1$, $s_j x_j \to x \neq 0$. Hence, $s_j(x_j + y_j) = (s_j / t_j)(t_j x_j + t_j y_j) \to 0$, whence $s_j y_j \to -x$. Thus, $x \in X_\infty \cap (-Y_\infty) = \{0\}$, which is a contradiction. Therefore (1) holds.

(i) Suppose that Y is a.c.; then, there exists $W \in \mathscr{V}$ such that $Y \cap [0, 1] \cdot W$ is a.c. Taking δ and V given by (1), we get that $X + Y$ is a.c., since the sum of two r.c. sets is r.c.

(ii) Proceed as for (i) and take into account the fact that the sum of an r.c. set and of a bounded set is a bounded one. □

Conditions like $X_\infty \cap (-Y_\infty) = \{0\}$ are essential in stating the closedness of $X + Y$ for X, Y closed sets. Even in the convex case, one can give examples when $X + Y$ is not closed if $X_\infty \cap (-Y_\infty) \neq \{0\}$ (more exactly if $X_\infty \cap (-Y_\infty)$ is not a linear subspace). From the above results, we see also that this condition is crucial in stating other properties.

Of course, it is interesting to have sufficient conditions for $(X + Y)_\infty \supset X_\infty + Y_\infty$. Goossens (see Ref. 7) obtained this relation under the hypothesis that X is radiant, while we obtained it under the condition

$$\text{(C)} \quad \forall x \in X_\infty, \quad \forall (t_i) \subset]0, \infty[, \quad t_i \to 0, \quad \exists (x_i) \subset X: t_i x_i \to x.$$

Luc (Refs. 3 and 4) got the corresponding relation for recession cones under the hypothesis

$$\text{(CD)} \quad \forall a \in \operatorname{Rec}(X), \quad \exists A \in \mathscr{B}, \quad \forall t \geq 0: (ta + A) \cap X \neq \varnothing.$$

We observed in Ref. 9 that every radiant set satisfies (C). The next proposition gives the relation between (CD) and (C), and provides another situation when $\operatorname{Rec}(X) = X_\infty$.

Proposition 3.7. Let X be a nonempty subset of E. If X satisfies (CD) then $\operatorname{Rec}(X) = X_\infty$ and X also satisfies (C). The converse is not true.

Proof. Let X satisfy (CD), and take $a \in \operatorname{Rec}(X)$. There exists a bounded set A such that $(ta + A) \cap X \neq \varnothing$ for every $t > 0$. Let $(t_i) \subset]0, \infty[$, $t_i \to 0$. Then, there exist $(a_i) \subset A$ and $(x_i) \subset X$ such that $(1/t_i)a + a_i = x_i$, i.e., $t_i x_i = a + t_i a_i$, for every i. As (a_i) is bounded and $t_i \to 0$, we have $a = \lim(t_i x_i)$. It follows that $a \in X_\infty$ (taking $t_n = 1/n$) and (C) holds. The other asssertion is proved by the following example. □

Example 3.2. Let $E = \mathbb{R}^2$ and $X = \{(x, x^2) \in \mathbb{R}^2 : x \ge 0\}$. X satisfies (C) but does not satisfy (CD).

Indeed, $X_\infty = \{(0, u): u \ge 0\}$ and, for $u \ge 0$ and $0 < t_i \to 0$, there exists $((x_i, y_i)) = (((u/t_i)^{1/2}, u/t_i)) \subset X$, $t_i(x_i, y_i) \to (0, u)$, i.e., (C) is satisfied. Let $A \subset \mathbb{R}^2$ be bounded and $M > 0$ such that $\|(x, y)\| \le M$ for $(x, y) \in A$. Then, $(0, M^2 + M + 1) \notin X - A$. If not, $(0, M^2 + M + 1) = (x, x^2) - (a, b)$, with $(a, b) \in A$, $x \ge 0$, so that $x = a \le M$. Therefore, $M^2 + M + 1 = x^2 - b \le M^2 + M$, which is a contradiction. Hence, (CD) is not satisfied.

4. Application

Let $C \subset E$ be a convex cone, and let X be a nonempty subset of E. The element $x \in X$ is a proper efficient point (in the sense of Lampe) of X with respect to C if there exists a convex cone $K \subset E$ such that $X \cap (x - K) = \{x\}$ and $C \setminus \{0\} \subset \operatorname{int}(K)$; we denote by $\operatorname{PropE}(X, C)$ the set of those elements $x \in E$ satisfying the above condition. Note that, for $X_0 \subset X$, one has that $X_0 \cap \operatorname{PropE}(X, C) \subset \operatorname{PropE}(X_0, C)$, but the converse is not generally true, even if X_0 is a lower section of X.

Luc (Refs. 3 and 4) established the following result. Let E be a finite-dimensional space, let $C \subset E$ be a convex cone with nonempty interior such that $\operatorname{cl}(C) \cap (-\operatorname{cl}(C)) = \{0\}$, and let $X \subset E$ be nonempty. Suppose that $X_\infty \cap (-\operatorname{cl}(C)) = \{0\}$. Then, $x \in \operatorname{PropE}(X, C)$ iff there exists $e \in x + \operatorname{int}(C)$ such that $x \in \operatorname{PropE}(X \cap (e - C), C)$. We want to extend this result to general locally convex spaces and so, throughout this section, E is such a space. In order to do this, we need the following result that does not seem to be new, but we cannot find a reference for it.

Proposition 4.1. Let $\{0\} \ne C \subset E$ be a convex cone.

(i) If C has a convex base B, then there exist $x^* \in E^* \setminus \{0\}$, $\tilde{B} \subset \ker(x^*) =: \tilde{E}$ a convex set, and $\tilde{x} \in E$, $\langle \tilde{x}, x^* \rangle = 1$, such that $C = \operatorname{cone}(\tilde{x} + \tilde{B})$. The set $\tilde{B}$ is bounded if B is. Moreover, $x \in C \setminus \{0\}$ iff $\langle x, x^* \rangle > 0$ and $\langle x, x^* \rangle^{-1} x \in \tilde{x} + \tilde{B}$.

(ii) Let $C = \operatorname{cone}(\tilde{x} + \tilde{B})$ with $\tilde{x}$ and $\tilde{B}$ as in (i); then, $\tilde{x} + \tilde{b} \in \operatorname{int}(C)$ iff $\tilde{b} \in \operatorname{int}_{\tilde{E}} \tilde{B}$.

Proof.

(i) Let $C = \operatorname{cone}(B)$, with B convex and $0 \notin \operatorname{cl}(B)$. By a separation theorem (cf. Ref. 1), there exists $x^* \in E^*$ such that $\langle x, x^* \rangle \ge 1$ for all $x \in B$. Take $B_0 = \{x \in C : \langle x, x^* \rangle = 1\}$, $\tilde{x} \in B_0$, and $\tilde{B} = B_0 - \tilde{x} \subset \tilde{E}$. It is easy to see that $C = \operatorname{cone}(\tilde{x} + \tilde{B})$ and $x \in C \setminus \{0\}$ iff $\langle x, x^* \rangle > 0$ and $\langle x, x^* \rangle^{-1} x \in \tilde{x} + \tilde{B}$.

Assume now that B is bounded. If $x \in B_0$, then $x = tb$ with $t > 0$ and $b \in B$. Thus, $1 = \langle x, x^* \rangle = t\langle b, x^* \rangle \geq t$. Therefore, $B_0 \subset [0, 1] \cdot B$, and so $\tilde{B}$ is bounded.

(ii) Let $C = \text{cone}(\tilde{x} + \tilde{B})$, with $\tilde{B} \subset \tilde{E}$ convex and $\langle \tilde{x}, x^* \rangle = 1$, and let $\tilde{x} + \tilde{b} \in \text{int}(C)$. There exist $U \in \mathscr{V}$ such that $\tilde{x} + \tilde{b} + U \subset \text{cone}(\tilde{x} + \tilde{B})$. Then, $\tilde{b} + U \cap \tilde{E} \subset \tilde{B}$. Indeed, if $u \in U \cap \tilde{E}$, then $\tilde{x} + \tilde{b} + u \in \text{cone}(\tilde{x} + \tilde{B})$, so that there exist $t \geq 0$, $b \in \tilde{B}$ with $\tilde{x} + \tilde{b} + u = t(\tilde{x} + b)$. Hence, $1 = t\langle \tilde{x} + b, x^* \rangle = t$. It follows that $\tilde{b} + u \in \tilde{B}$. Hence, $\tilde{b} \in \text{int}_{\tilde{E}}\, \tilde{B} \neq \varnothing$. Conversely, let $\tilde{b} \in \text{int}_{\tilde{E}}\, \tilde{B}$. Then, there exists $U \in \mathscr{V}$ such that $\tilde{b} + U \cap \tilde{E} \subset \tilde{B}$. As the mapping $\varphi: \{u: \langle u, x^* \rangle > -1\} \to E$, $\varphi(u) = (1 + \langle u, x^* \rangle)^{-1}(\tilde{x} + \tilde{b} + u) - \tilde{x} - \tilde{b}$ is continuous, and $\varphi(0) = 0 \in U$, there exists $V \in \mathscr{V}$ such that $\varphi(u) \in U$ for $u \in V$. As $\varphi(u) \in \tilde{E}$ for every u, we see that $\varphi(u) \in U \cap \tilde{E}$ for $u \in V$, so that $\tilde{b} + \varphi(u) \in \tilde{B}$ for $u \in V$. It follows that $\tilde{x} + \tilde{b} + u \in \text{cone}(\tilde{x} + \tilde{B}) = C$ for $u \in V$, and so $\tilde{x} + \tilde{b} \in \text{int}(C)$. □

First we give the following partial extension of Luc's result.

Proposition 4.2. Let $C \subset E$ be a cone with convex base and nonempty interior, and let $X \subset E$ be a bounded set. Suppose that $x \in \text{PropE}(C \cap (e - C), C)$ for some $e \in x + \text{int}(C)$. Then, $x \in \text{PropE}(X, C)$.

Proof. By hypothesis, there exists a convex cone K such that $C \backslash \{0\} \subset \text{int}(K)$ and $X \cap (e - C) \cap (x - K) = \{x\}$, and we must show that there exists a convex cone $\tilde{K}$ such that $C \backslash \{0\} \subset \text{int}(\tilde{K})$ and $X \cap (x - \tilde{K}) = \{x\}$. Without loss of generality, we may assume that $x = 0$ and replace X by $-X$. Thus, our hypothesis becomes

$$C \backslash \{0\} \subset \text{int}(K), \qquad X \cap (C - e) \cap K = \{0\}, \tag{2}$$

and the conclusion

$$C \backslash \{0\} \subset \text{int}(\tilde{K}), \qquad X \cap \tilde{K} = \{0\}, \tag{3}$$

for the same K and $\tilde{K}$. By using Proposition 4.1, there exist $x^* \in E^* \backslash \{0\}$, $\tilde{B} \subset \tilde{E} := \ker(x^*)$ a convex set, and $\tilde{x} \in E$, $\langle \tilde{x}, x^* \rangle = 1$, such that $C = \text{cone}(\tilde{x} + \tilde{B})$. As $\text{int}(C) \neq \varnothing$, $\text{int}_{\tilde{E}}\, \tilde{B} \neq \varnothing$. Moreover, we may assume that $0 \in \text{int}_{\tilde{E}}\, \tilde{B}$ and $e = \tilde{t}\tilde{x}$ with $\tilde{t} > 0$; otherwise, $e = \tilde{t}(\tilde{x} + \tilde{b})$, $\tilde{t} > 0$, $\tilde{b} \in \text{int}_{\tilde{E}}\, \tilde{B}$, and replace $\tilde{B}$ by $\tilde{B} - \tilde{b}$ and $\tilde{x}$ by $\tilde{x} + \tilde{b}$. As X is bounded, there exists $M > 0$ such that $|\langle x, x^* \rangle| \leq M$ for every $x \in X$. Let us consider the Minkowski functional $p = p_{\tilde{B}}$ of $\tilde{B}$ in $\tilde{E}$. Then,

$$\text{int}_{\tilde{E}}\, \tilde{B} = \{x \in \tilde{E}: p(x) < 1\} \subset \tilde{B} \subset \{x \in \tilde{E}: p(x) \leq 1\} = \text{cl}(\tilde{B}).$$

For $s \in]0, \tilde{t}/M[$, let $B_s = \{x \in \tilde{E}: p(x) \leq 1 + s\}$ and $C_s = \text{cone}(\tilde{x} + B_s)$. Of course C_s is a convex cone with $C\backslash\{0\}\subset \text{int}(C_s)$. Moreover,

$$\begin{aligned} &C_s\backslash(C-e)\\ &=\{x: \exists t\geq 0, b\in B_s, x=t(\tilde{x}+b), x\notin C-e\}\\ &=\{x: \exists t\geq 0, b\in B_s, x=t(\tilde{x}+b), (t+\tilde{t})\tilde{x}+tb\notin C\}\\ &=\{x: \exists t> 0, b\in B_s, x=t(\tilde{x}+b), \tilde{x}+(t/(t+\tilde{t}))b\notin \tilde{x}+\tilde{B}\}\\ &\subset\{x: \exists t> 0, b\in B_s, x=t(\tilde{x}+b), p(b)\geq 1+\tilde{t}/t\}\\ &\subset\{x: \exists t> 0, b\in B_s, x=t(\tilde{x}+b), s\geq \tilde{t}/t\}\\ &\subset\{x\in E: \langle x, x^*\rangle\geq \tilde{t}/s\}. \end{aligned}$$

So, $X\cap(C_s\backslash(C-e))=\varnothing$. Therefore, by (2),

$$\begin{aligned} &X\cap(C_s\cap K)\\ &=X\cap((C_s\backslash(C-e))\cup(C_s\cap(C-e)))\cap K\\ &=(X\cap K\cap(C_s\backslash(C-e)))\cup(X\cap C_s\cap(C-e)\cap K)\\ &=C_s\cap\{0\}=\{0\}. \end{aligned}$$

Taking $\tilde{K}=C_s\cap K$, (3) holds. The proof is complete. □

Remark 4.1. The result of Proposition 4.2 is true if $x^*(X)$ is bounded in $\mathbb{R}$.

The result stated in Proposition 4.2 can be extended to unbounded sets.

Proposition 4.3. Let $C\subset E$ be a cone with convex base and nonempty interior, and let $X\subset E$ be a nonempty set satisfying condition (CA). Assume that $X_\infty\cap(-\text{cl}(C))=\{0\}$. If $x\in \text{PropE}(X\cap(e-C), C)$ for some $e\in x+\text{int}(C)$, then $x\in\text{PropE}(X, C)$.

For the proof of this result, we need the following lemma.

Lemma 4.1. Let $C=\text{cone}(\tilde{x}+\tilde{B})$, where $x^*\in E^*\backslash\{0\}$, $0\in\tilde{B}\subset \ker(x^*)=:\tilde{E}$ is convex, $\tilde{x}\in E$, with $\langle\tilde{x}, x^*\rangle=1$, and take $C_n=\text{cone}(\tilde{x}+(1+1/n)\,\text{cl}(\tilde{B}))$ for $n\in\mathbb{N}^*$. Then, $\text{cl}(C)=\bigcap_{n\geq 1}\text{cl}(C_n)$.

Proof. As $\tilde{B}\subset(1+1/n)\cdot\text{cl}(\tilde{B})$, $C\subset C_n$, and so $\text{cl}(C)\subset\bigcap_{n\geq 1}\text{cl}(C_n)$. Conversely, let $x\in\bigcap_{n\geq 1}\text{cl}(C_n)$ and $U\in\mathscr{V}$ be such that $|\langle u, x^*\rangle|\leq 1$ for $u\in U$. Then, for every $V\in\mathscr{V}$ with $V\subset U$ and $n\in\mathbb{N}^*$, $(x+V)\cap C_n\neq\varnothing$; i.e., there are $v_i\in V$, $t_i\geq 0$, $b_i\in(1+1/n)\cdot\text{cl}(\tilde{B})$ such that $x+v_i=t_i(\tilde{x}+b_i)$, where $i=(n, V)$. Hence, $\langle x+v_i, x^*\rangle=t_i$, whence $|t_i|\leq|\langle x, x^*\rangle|+1$. As $n(n+1)^{-1}b_i\in\text{cl}(\tilde{B})$, there exists $w_i\in V$ such that $n(n+1)^{-1}b_i+w_i=:\tilde{b}_i\in\tilde{B}$. Thus,

$$\begin{aligned} x+v_i&=t_i(\tilde{x}+(n+1)n^{-1}\tilde{b}_i-(n+1)n^{-1}w_i)\\ &=t_i(n+1)n^{-1}(\tilde{x}+\tilde{b}_i)-t_in^{-1}\tilde{x}-t_i(n+1)n^{-1}w_i. \end{aligned}$$

Hence, $x+v_i+t_i(n+1)n^{-1}w_i+t_i n^{-1}\tilde{x} \in \mathrm{cone}(\tilde{x}+\tilde{B})=C$. Taking the limit, we find that $x \in \mathrm{cl}(C)$. □

Proof of Proposition 4.3. Let $X_0 \subset X$ be bounded such that $X \backslash X_0$ is a.c. Of course, we may suppose that $x=0$. Let x^*, $\tilde{x}$, and $\tilde{B}$ be as in the proof of Proposition 4.2. and take $C_n=\mathrm{cone}(\tilde{x}+(1+1/n)\cdot \mathrm{cl}(\tilde{B}))$. There exists $\bar{n} \in \mathbb{N}^*$ such that $x^*(X \cap (-C_{\bar{n}}))$ is bounded. Otherwise, for every n, there exists $x_n \in X \cap (-C_n)$ such that $|\langle x_n, x^* \rangle| \geq n$. Thus the sequence (x_n) is unbounded. Because X_0 is bounded, we may assume that $(x_n) \subset X \backslash X_0$. As $X \backslash X_0$ is a.c., by Proposition 2.2(v) of Ref. 9, there exist a subnet (y_j) of (x_n) and $(t_j) \subset]0, \infty[$ such that $t_j \to 0$ and $t_j y_j \to y \neq 0$. Of course $y \in X_\infty$. As $C_{n+1} \subset C_n$, it follows that $-y \in \mathrm{cl}(C_n)$ for every n. By the preceding lemma, we get that $-y \in \mathrm{cl}(C)$; i.e., $y \in X_\infty \cap (-\mathrm{cl}(C))$, which is a contradiction.

Let $\bar{n}$ be such that $x^*(X \cap (-C_{\bar{n}}))$ is bounded. As $0 \in \mathrm{PropE}(X \cap (e-C), C)$, it follows that $0 \in \mathrm{PropE}(X \cap (-C_{\bar{n}}) \cap (e-C), C)$. As $x^*(X \cap (-C_{\bar{n}}))$ is bounded, by Proposition 4.2 (see also Remark 4.1), $0 \in \mathrm{PropE}(X \cap (-C_{\bar{n}}), C)$. Therefore, there exists a convex cone K such that $C \backslash \{0\} \subset \mathrm{int}(K)$ and $X \cap (-C_{\bar{n}}) \cap (-K)=\{0\}$. But $K \cap C_n$ is a convex cone with $C \backslash \{0\} \subset \mathrm{int}(C_{\bar{n}}) \cap \mathrm{int}(K)=\mathrm{int}(K \cap C_{\bar{n}})$. Hence, $x=0 \in \mathrm{PropE}(X, C)$. □

Note that, for E finite-dimensional, the result stated in Proposition 4.3. is a little bit more general than Luc's result, because we do not suppose that C is acute, i.e., $\mathrm{cl}(C) \cap (-\mathrm{cl}(C))=\{0\}$.

The above notion of proper efficiency with respect to C is exactly Lampe's minimality. Other notions of proper efficiency are also known. Among them, there are those due to Slater, Gerstewitz, and Iwanow-Nehse (denoted by S, G, E in Ref. 12). It is easy to see that Propositions 4.2 and 4.3 still hold for G-minimality and E-minimality (in fact, with the same proof), but they do not hold for S-minimality.

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