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# Quasi Interiors, Lagrange Multipliers, and $L^p$ Spectral Estimation with Lattice Bounds[1]

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**Abstract.** Lagrange multipliers useful in characterizations of solutions to spectral estimation problems are proved to exist in the absence of Slater's condition provided a new constraint involving the quasi-relative interior holds. We also discuss the quasi interior and its relation to other generalizations of the interior of a convex set and relationships between various constraint qualifications. Finally, we characterize solutions to the $L^p$ spectral estimation problem with the added constraint that the feasible vectors lie in a measurable strip $[\alpha, \beta]$.

**Key Words.** Lagrange multipliers, convex sets, quasi interiors, duality, spectral estimation.

## 1. Introduction

We use as our motivation two problems from spectral estimation. Let $(K, \nu)$ be a finite measure space; let $A: L^p(K, \nu) \to \mathbb{R}^n$, $1 < p < \infty$, be a continuous linear map; let $b \in \mathbb{R}^n$ be a fixed vector; and suppose that $\alpha$ and $\beta$ are in $L^p(K, \nu)$. We seek to characterize solutions to the positive minimum $L^p$ estimation problem

$$\inf\{(1/p)\|x\|_p^p : Ax = b, 0 \leq x \text{ a.e.}\} \tag{1}$$

and the spectral estimation problem with $L^p$ bounds

$$\inf\{(1/p)\|x\|_p^p : Ax = b, \alpha \leq x \leq \beta \text{ a.e.}\} \tag{2}$$

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by showing that Lagrange multipliers exist for the nonlinear constraints. In more generality, we can suppose that $\alpha$ and $\beta$ are simply measurable and characterize solutions to (2); this includes (1) as a special case $\alpha = 0$, $\beta = +\infty$.

Classically, one would appeal to Slater's condition to assert the existence of Lagrange multipliers. However, in these problems, the interior of the positive cone in $L^p$,

$$C = \{x \in L^p(K, \nu): 0 \leq x \text{ a.e.}\},$$

is empty and Slater's condition fails to hold. A reasonable substitute for the interior would seem to be

$$\{x \in L^p(K, \nu): 0 < x \text{ a.e.}\};$$

this turns out to be the quasi interior of $C$. A natural alternative to Slater's condition would be that there is a feasible function in the quasi interior of the positive cone.

This paper can be logically divided into two parts. Sections 2 and 3 define the quasi interior of a convex set and contrast various generalized interiors and associated constraint qualifications, while Sections 4, 5, 6 use the weaker notion of the quasi-relative interior to produce Lagrange multipliers in the absence of Slater's condition and to characterize solutions to the minimum norm spectral estimation problem in $L^p$.

In Section 2, we define and discuss some of the properties of the quasi interior, particularly those that make it so similar to the true interior. We show how the quasi interior is a generalization to convex sets of several of the quasi interiors of cones that have been defined in the literature.

In Section 3, we then suggest a constraint qualification in the spirit of Slater's constraint qualification, but using the quasi interior, and discuss its relation with other recent qualifications.

Goodrich, Steinhardt and Roberts' work on the positive spectral estimation problem (Refs. 1–2) circumvented the lack of interior by showing that the problem can be formulated in $L^\infty$, where the positive cone has a nonempty interior; thus, Slater's condition holds and classical Lagrange multiplier theorems apply. More recently, Cole and Goodrich (Ref. 3) have characterized the solution to the positive $L^\infty$ bounded spectral estimation problem by imposing conditions to guarantee that Slater's condition holds. Using the duality theory of Borwein and Lewis (Ref. 4), we show in Sections 4 and 5 that Lagrange multipliers exist for these two problems, provided a generalization of Slater's condition using the quasi-relative interior holds. This generalizes the method of Refs. 1–2 and Refs. 3, 5.

Finally, in Section 6, we show how the duality theory can be used to include the cases where $\alpha$ and $\beta$ are extended-valued measurable functions, potentially not in $L^p$.

## 2. Quasi Interior

In this section, we define the notion of the quasi interior and compare this definition with several other similar definitions found in the literature.

**2.1. Definitions and Equivalences.** Let $X$ be a real linear normed space, and let $X^*$ be its topological dual. Let $k$ be a real number, and let

$$H_k(x^*) = \{x \in X: \langle x, x^* \rangle \geq k\}$$

be the closed half-plane associated with $x^*$ and $k$. It can be shown that the interior of $H_k(x^*)$ is

$$H_k^0(x^*) = \{x \in X: \langle x, x^* \rangle > k\}.$$

It follows from the Hahn–Banach theorem that, for a closed convex set $C$,

$$C = \bigcap \{H_k(x^*): H_k(x^*) \supseteq C\}.$$

It can also be shown that, if the interior of $C$ is nonempty, then

$$C^0 = \bigcap \{H_k^0(x^*): H_k(x^*) \supseteq C\}. \tag{3}$$

We are interested in the set on the right of Eq. (3) in the case where the interior of $C$ may be empty.

**Definition 2.1.** For a convex set $C$ in the linear normed space $X$, define the quasi interior of $C$ to be

$$\mathrm{qi}(C) = \bigcap \{H_k^0(x^*): H_k(x^*) \supseteq C\} \cap C. \tag{4}$$

This definition ensures that $\mathrm{qi}(C) \subseteq C$. Note that, if $C$ is closed, then the intersection with $C$ in the definition is unnecessary.

Recall that $x_0 \in C$ is a support point of $C$ if there is a nonzero $x^* \in X^*$ such that

$$\langle x_0, x^* \rangle = \inf_{x \in C} \langle x, x^* \rangle.$$

Also, the cone generated by $K$ with vertex at $\theta$ is

$$\mathrm{cone}(K) = \{\alpha: \alpha = \lambda x, x \in K, \lambda \geq 0\}.$$

Borwein and Lewis (Ref. 4) have defined the quasi-relative interior of a convex set $C$ to be

$$\mathrm{qri}(C) = \{x \in C: \overline{\mathrm{cone}}(C - x) \text{ is a subspace}\}.$$

Several facts are easily checked and summarized in the following theorem.

**Theorem 2.1.** Let $C$ be a convex set with nonempty quasi interior in the linear normed space $X$. Then, the following statements hold:

(a) $\mathrm{qi}(C)=\bigcap\{H_k^0(x^*): H_k(x^*)\supseteq C\}\cap C$ (by definition);
(b) $\mathrm{qi}(C)=\mathcal{N}(C)$, the set of nonsupport points in $C$;
(c) $\mathrm{qi}(C)=\{x\in C:\overline{\mathrm{cone}}(C-x)=X\}$;
(d) $\mathrm{qi}(C)=\mathrm{qri}(C)$;
(e) $\overline{\mathrm{qi}(C)}=\bar{C}$.

If $X$ is finite dimensional, or if $C^0\neq\emptyset$, then $\mathrm{qi}(C)=C^0$.

Klee (Ref. 6) has shown that, in a separable Banach space, any convex set not contained in a hyperplane contains a nonsupport point; thus, its quasi interior is nonempty. However, if the convex set is contained in a hyperplane, then it will still have a nonempty quasi-relative interior (Ref. 4), yet an empty quasi interior.

Borwein and Lewis (Ref. 4) have shown that, if $A:X\to\mathbb{R}^n$ is a continuous linear map and $C$ is a convex set with nonempty quasi-relative interior, then $A(\mathrm{qri}\ C)=\mathrm{ri}(AC)$, the relative interior of $AC$.

We give an example of a nondense convex set $C$ not contained in the intersection of Eq. (3). Denote by $\tilde{C}$ this intersection. Klee (Ref. 7) has given an example of a compact convex set $C$ for which the extreme points properly contain the support points. Let $x_0$ be an extreme point of $C$ that is not a support point, so that $K=C\backslash\{x_0\}$ is convex and $x_0\in\mathrm{qi}(C)$. Then, $\bar{K}=C$, thus $\tilde{K}=\tilde{C}$. However, since $C$ is closed, $\tilde{K}=\mathrm{qi}(C)$, and thus $x_0\notin K\supset\mathrm{qi}(K)$, so $\tilde{K}\neq\mathrm{qi}(K)$.

Given a closed convex cone $S$, we can define orderings on both $X$ and $X^*$. We say

$$x\geq y\Leftrightarrow x-y\in S.$$

On $X^*$, we say

$$x^*\geq y^*\Leftrightarrow\langle x, x^*-y^*\rangle\geq 0,\qquad \forall x\in S.$$

The set of all $x^*\geq\theta$ is denoted $S^+$. One can easily show the following theorem.

**Theorem 2.2.** Let $S\neq X$ be a closed convex cone. Then, $x_0\in\mathrm{qi}(S)$ if and only if

$$\langle x_0, x^*\rangle>0,\qquad \forall x^*\geq\theta, x^*\neq\theta.$$

Other useful examples occur in Banach lattices. If a Banach space $X$ is a Banach lattice, then there is a closed convex cone $S$ such that $S-S=X$, in which case $S^+-S^+=X^*$ (Ref. 8). We define an ordering on $X$ based on this cone $S$. We can then decompose a point $x^*\in X^*$ as $x_+^*-x_-^*$.

**Theorem 2.3.** Let $X$ be a Banach lattice, let $\alpha$ and $\beta$ be in $X$, and set

$$C=\{x\in X:\alpha\leq x\leq\beta\}.$$

Then,

$$\sup\{\langle x, x^*\rangle : x\in C\}=\langle\beta, x^*_+\rangle-\langle\alpha, x^*_-\rangle. \tag{5}$$

Equivalently, $x_0\in C$ is a support point with support functional $x^*$ if and only if

$$\langle x_0, x^*\rangle=\inf\{\langle x, x^*\rangle : x\in C\}=\langle\alpha, x^*_+\rangle-\langle\beta, x^*_-\rangle.$$

The following proof was suggested by A. S. Lewis.

**Proof.** We first do the special case $\alpha=0, \beta=e\geq 0$. We have

$$\begin{aligned}\sup\{\langle x, x^*\rangle: x\in C\}&=\sup\{\langle x, x^*\rangle+\langle e-x, 0\rangle: x\geq 0,\ e-x\geq 0\}\\&=\inf\{\langle e, y^*\rangle: y^*\geq 0,\ y^*\geq x^*\}\\&=\inf\{\langle e, y^*\rangle: y^*\geq x^*_+\}\\&=\langle e, x^*_+\rangle,\end{aligned}$$

by Theorem 8.2 in Borwein and Lewis (Ref. 4). To do the general $C$, let

$$D=C-\beta.$$

Then,

$$\sup_{x\in D}\{\langle x, x^*\rangle\}=\langle\beta-\alpha, x^*_+\rangle,$$

so

$$\langle\beta-\alpha, x^*_+\rangle=\sup_{x\in C}\{\langle x-\alpha, x^*\rangle\}=\sup_{x\in C}\{\langle x, x^*\rangle\}-\langle\alpha, x^*\rangle,$$

and thus

$$\sup_{x\in C}\{\langle x, x^*\rangle\}=\langle\beta, x^*_+\rangle-\langle\alpha, x^*_-\rangle.$$

This completes the proof. □

**2.2. Examples.** Let $X=L^p(K)$ for some set $K$ in $n$-dimensional real space, $1\leq p<\infty$, and Lebesgue measure. Let

$$S=\{x\in X: x\geq 0 \text{ a.e.}\}.$$

Then using Theorem 2.2, we can see that

$$\text{qi}(S)=\{x\in X: x>0 \text{ a.e.}\}.$$

Now, let $\alpha$ and $\beta$ be extended-valued measurable functions on $K$ such that $\alpha < \beta$, a.e. Let

$$C = \{x \in X: \alpha \leq x \leq \beta\}.$$

One can check using Theorem 2.3 that

$$\mathrm{qi}(C) = \{x \in X: \alpha < x < \beta\}.$$

**2.3. Comparison of Generalized Interiors.** There have been many attempts to generalize the definition of the interior of a convex set. In this section, we compile and compare some of those definitions.

Recall that the core of a convex set $C$ is

(a) $\mathrm{core}(C) = \{x \in C: \mathrm{cone}(C - x) = X\}$.

Gowda and Teboulle (Ref. 9) have defined the strong quasi-relative interior by

(b) $\mathrm{sqri}(C) = \{x \in C: \mathrm{cone}(C - x) \text{ is a closed subspace}\}$.

For a cone $S$, Peressini (Ref. 8) has defined another quasi interior,

(c) $\mathrm{pqi}(S) = \{x \in S: \mathrm{cone}[-x, x] = X\}$,

where $[-x, x]$ is the order interval $\{y: -x \leq y \leq x\}$ and $S$ is used to define the order on $X$. Fullerton and Braunschweiger's (Refs. 10–11) definition of a quasi interior is

(d) $\mathrm{fbqi}(S) = \{x \in S: S \cap (x - S) \text{ is dense in } X\}$.

Finally, if $X$ is a Banach lattice, Schaefer (Ref. 12) has defined

(e) $\mathrm{sqi}(S) = \{x \in S: \mathrm{span}(S_x) \text{ is dense in } X\}$,

where $S_x$ is the ideal generated by $x$.

If $\overline{S - S} = X$, then (c), (d), and (e) are equivalent. See Borwein and Lewis (Ref. 4) and the original sources for example. In general,

$$C^0 \subseteq \mathrm{core}(C) \subseteq \mathrm{sqri}(C) \subseteq \mathrm{qri}(C),$$

$$C^0 \subseteq \mathrm{core}(C) \subseteq \mathrm{qi}(C) \subseteq \mathrm{qri}(C).$$

Thus, assuming that $\mathrm{qri}(C)$ is not empty is a weaker assumption than assuming any of these other sets is nonempty.

## 3. Constraint Qualification Comparisons

As there have been several attempts to generalize the interior of a convex set, there have also been attempts to generalize the constraint qualifications associated with convex programming. Here, we give a brief comparison of some of the qualifications. For other comparisons, see Refs. 9 and 13.

We consider the following problem. Let $X$ be a Banach space, $f: X \to (-\infty, +\infty]$ a convex function, $A: X \to \mathbb{R}^n$ a linear continuous map, and $C \subseteq X$ a closed convex set. Finally, let $b \in \mathbb{R}^n$ be a fixed vector and suppose that

$$\text{(P)} \quad \mu = \inf\{f(x): Ax = b, x \in C\}$$

is finite.

A common constraint qualification, Slater's condition (Ref. 14), requires that the interior of $C$ be nonempty. For (P), this would read

$$\text{(S)} \quad \exists \hat{x} \in C^0 \text{ such that } A\hat{x} = b.$$

This qualification cannot be met in the examples that we shall explore, for if

$$C = \{x \in L^p(K): x \geq 0 \text{ a.e.}\},$$

then $C^0 = \varnothing$. However, the quasi interior $\mathrm{qi}(C)$ is not empty. The new qualification is then

$$\text{(CQ)} \quad \exists \hat{x} \in \mathrm{qi}(C) \text{ such that } A\hat{x} = b.$$

Borwein and Lewis show that, with their qualification,

$$\text{(BLCQ)} \quad \exists \hat{x} \in \mathrm{qri}(C) \text{ such that } A\hat{x} = b,$$

there is equality between the primal problem (P) and the dual problem (D) discussed below, with dual attainment. Since $\mathrm{qi}(C) \subseteq \mathrm{qri}(C)$, we see that this implies that, with (CQ), we also have this duality relationship.

Borwein and Wolkowicz (Ref. 15) introduced another constraint qualification that applied in the positive $L^p$ problem. In this case, the set $C$ is the positive cone which we shall denote by $S$. If we let

$$F = \{x \in S: Ax = b\},$$

then their qualification reads

$$\text{(BWCQ)} \quad \overline{\mathrm{cone}}(F - S) = X.$$

We now show that this is our (CQ) when $\mathrm{qi}(S) \neq \varnothing$.

**Lemma 3.1.** The Borwein and Wolkowicz constraint qualification,

$$\overline{\mathrm{cone}}(F - S) = X,$$

implies $b \in \mathrm{ri}(A(S))$.

For a proof, see Gowda and Teboulle (Ref. 9).

**Theorem 3.1.** Suppose $\mathrm{qi}(S) \neq \varnothing$. Then, $\overline{\mathrm{cone}}(F-S) = X$ if and only if there is an $x_0 \in \mathrm{qi}(S)$ such that $Ax_0 = b$; i.e., there is an $x_0 \in \mathrm{qi}(S) \cap F$. That is, (BWCQ) if and only if (CQ).

**Proof.** For $x_0 \in \mathrm{qi}(S) \cap F$, since

$$\overline{\mathrm{cone}}(F-S) \supseteq \overline{\mathrm{cone}}(x_0 - S) = X,$$

by Theorem 2.1, we have that

$$\overline{\mathrm{cone}}(F-S) = X.$$

To show the other implication, note that $\overline{\mathrm{cone}}(F-S) = X$ implies $b \in \mathrm{ri}(A(S))$ from Lemma 3.1. Also, since

$$\mathrm{ri}(A(S)) = A(\mathrm{qi}(S)),$$

$\overline{\mathrm{cone}}(F-S) = X$ implies there is an $x_0 \in \mathrm{qi}(S)$ such that $Ax_0 = b$. □

Micchelli and Utreras (Ref. 16) consider the problem

$$\mu = \min\{(1/2)\|x\|_2^2 \colon x \in S,\ Ax = b\},$$

where $X$ is a Hilbert space. This is our general problem (P) with the objective functional $(1/2)\|x\|_2^2$. Another constraint qualification is introduced,

$$\text{(MCQ)} \quad \{A^*y^* \colon \langle b, y^* \rangle \geq 0\} \cap S^- = \{\theta\}.$$

Here, $S^- = (-S)^+$. We show that (CQ) implies this condition.

**Proposition 3.1.** If $S \subseteq X$ is a closed convex cone, $\mathrm{qi}(S) \neq \varnothing$, $A\colon X \to \mathbb{R}^n$ is a continuous linear map, and $F = S \cap A^{-1}[b] \neq \varnothing$, then (CQ) implies (MCQ).

**Proof.** Let $x^* = A^*y^*$, for some $y^* \in (\mathbb{R}^n)^*$ such that $\langle b, y^* \rangle \geq 0$, and let $\langle x, x^* \rangle \leq 0$ for all $x \in S$. By (CQ), there is an $x_0 \in \mathrm{qi}(S)$ such that $Ax_0 = b$, so $\langle Ax_0, y^* \rangle \geq 0$, which implies $\langle x_0, A^*y^* \rangle \geq 0$, and thus $\langle x_0, x^* \rangle \geq 0$. Since $\langle x, x^* \rangle \leq 0$ for all $x \in S$, $\langle x_0, x^* \rangle = 0$ which implies that $x_0$ is a support point of the cone $S$. This contradicts the assumption that $x_0 \in \mathrm{qi}(S)$. □

Another constraint qualification is given in Irvine and Smith (Ref. 17). For $X$ the dual space of $Y$, $S$ a convex cone in $X$, they investigate the problem

$$\mu = \inf\{\|x\| \colon x \in S,\ Ax = b\},$$

where $Ax = (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle)^T$ for vectors $y_1, \ldots, y_n \in Y$. Their constraint qualification is that there exists $\alpha^* \in \mathbb{R}^n$ such that $\langle \alpha^*, b\rangle = 1$ and

$$\text{(ISCQ)}\quad \rho\left(\sum_{i=1}^{n} \alpha_i^* y_i\right) = \inf\left\{\rho\left(\sum_{i=1}^{n} \alpha_i y_i\right) : \langle \alpha, b\rangle = 1, \alpha \in \mathbb{R}^n\right\},$$

where

$$\rho(y) = \sup\{\langle y, x\rangle : x \in S, \|x\| \leq 1\}.$$

It is shown that, if $b$ is in the relative interior of $A(S)$, then there is such a vector $\alpha^*$.

**Corollary 3.1.** (BLCQ) implies (ISCQ).

## 4. Duality Theory

Borwein and Lewis (Ref. 4) have developed a Fenchel duality theory for the quasi-relative interior. For brevity, we only state the result.

**Theorem 4.1.** Suppose $f: X \to (-\infty, +\infty]$ is convex, $C \subseteq X$ is convex, $\mu$ is finite in problem (P), and $b \in \mathrm{ri}(A(C))$. If

$$\text{(D)}\quad d \equiv \max_{\lambda \in \mathbb{R}^n}\left\{\langle b, \lambda\rangle + \inf_{x \in C}\{f(x) - \langle Ax, \lambda\rangle\}\right\},$$

then $\mu = d$, where the maximum is attained at some $\bar{\lambda} \in \mathbb{R}^n$.

Note that

$$d = \max\{\langle b, \lambda\rangle - f_C^*(A^*\lambda)\},$$

where $f_C = f + i_C$, $f_C^*$ is the convex conjugate of $f_C$, and $i_C$ is the indicator function of $C$ as defined below. Since $\mathrm{ri}(AC) = A(\mathrm{qri}\, C)$ when $\mathrm{qri}(C) \neq \emptyset$, we can restate this as follows: if (BLCQ) holds, then $\mu = d$.

## 5. Lagrange Multipliers

We consider the following problem. Let $Z$ be a linear normed space with a positive cone $P$. Define

$$\text{(P)}\quad \mu = \inf\{f(x): Ax = b, G(x) \leq \theta\},$$

where $G: X \to Z$ is $P$-convex,

$$\lambda G(x) + (1-\lambda)G(y) - G(\lambda x + (1-\lambda)y) \in P,$$

for any $x, y \in Z$, $\lambda \in (0, 1)$. A vector $z^* \in Z^*$ is a *Lagrange multiplier* if

$$\mu = \inf\{f(x) + \langle G(x), z^* \rangle: Ax = b\}.$$

The paper by Goodrich and Steinhardt (Ref. 1) characterized the solution to the problem in $X = L^p([-\pi, \pi])$,

$$\mu = \inf\{\|x\|_p^p: Ax = b, x \geq 0 \text{ a.e.}\}, \tag{6}$$

where $A: X \to \mathbb{R}^n$ and $(Ax)_j$ is the $j$th Fourier coefficient of $x$. The characterization was obtained by showing that the solution to (6) was in $L^\infty$ and that there is a feasible point $\hat{x}$ such that $\hat{x} \geq \epsilon > 0$, a.e. In this case, Slater's condition holds for the problem in $L^\infty$, and there is a $x^* \geq 0$, $x^* \in (L^\infty)$; such that

$$\mu = \inf_{x \in L^\infty}\{\|x\|_p^p - \langle x, x^* \rangle: Ax = b\};$$

that is, there is a Lagrange multiplier for this problem. See Luenberger (Ref. 18) for example. The linear constraints are eliminated via another multiplier under a regularity assumption on $A$.

More recently, Cole and Goodrich (Ref. 3) addressed the same problem with an $L^\infty$ bound,

$$\mu = \inf\{\|x\|_p^p: Ax = b, 0 \leq x \leq \beta, \text{ a.e.}\}, \tag{7}$$

with $\beta \in L^\infty$, and it is assumed that there is an $\epsilon > 0$ and $\hat{x}$ feasible such that $\varepsilon \leq \hat{x} \leq \beta - \epsilon$, which again is Slater's condition. Using Ref. 18, they characterized solutions again with a regularity condition on $A$.

We shall derive two Lagrange multiplier theorems in a more general setting that imply generalizations of the above examples without demanding that Slater's condition (S) hold. With these theorems, the techniques used in Refs. 1, 3 are more direct. We begin with a few definitions.

Let $C$ be a convex set; the normal cone to $C$ at $x_0 \in C$ is

$$N_C(x_0) = \{x^* \in X^*: \langle x - x_0, x^* \rangle \leq 0, x \in C\}.$$

The indicator function of $C$ is

$$i_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Recall that, for a convex function $f$, $x^* \in X^*$ is a *subgradient* of $f$ at $x_0$ if

$$\langle x - x_0, x^* \rangle \leq f(x) - f(x_0), \qquad \forall x \in X. \tag{8}$$

Relation (8) is referred to as the subgradient inequality. The set of all subgradients of $f$ at $x$ is denoted $\partial f(x)$. A convex function $f: X \to (-\infty, +\infty]$

is proper if it is somewhere finite. It is a standard result from convex analysis that, if $\text{cont}(f)$ is the set of points of continuity of $f$ and

$$\text{(RC)} \quad \text{cont}(f) \cap C \neq \emptyset,$$

then for $x \in \text{dom}(f) \cap C$,

$$\partial(f+i_C)(x) = \partial f(x) + N_C(x).$$

See Rockafellar (Ref. 19) or Holmes (Ref. 20) for example. We call (RC) our regularity condition. Recall our constraint qualification,

$$\text{(BLCQ)} \quad \exists \hat{x} \in \text{qri}(C) \text{ such that } A\hat{x} = b.$$

**Lemma 5.1.** Let $X$ be a reflexive Banach space, let $f: X \to (-\infty, +\infty]$ be convex and proper, let $A: X \to \mathbb{R}^n$ be a continuous linear map, suppose that $x_0 \in C$, $Ax_0 = b$, and

$$f(x_0) = \inf\{f(x): Ax = b, x \in C\}$$

is finite, and assume (RC) and (BLCQ). Then, there is an $x^* \in N_C(x_0)$ such that

$$f(x) + \langle x - x_0, x^* \rangle \geq f(x_0),$$

for all $x$ such that $Ax = b$.

**Proof.** Let $i_C$ be the indicator function of $C$, and let $f_C = f + i_C$. Then,

$$f(x_0) = \inf\{f_C(x): Ax = b\}.$$

By the duality theory, Theorem 4.1,

$$\begin{aligned} f(x_0) &= \max_{\lambda \in \mathbb{R}^n} \left\{ \langle b, \lambda \rangle + \inf_{x \in C} \{ f(x) - \langle x, A^*\lambda \rangle \} \right\} \\ &= \langle b, \bar{\lambda} \rangle + \inf_{x \in C} \{ f(x) - \langle x, A^*\bar{\lambda} \rangle \} \\ &\leq \langle b, \bar{\lambda} \rangle + f(x_0) - \langle x_0, A^*\bar{\lambda} \rangle \\ &= f(x_0), \end{aligned}$$

so $x_0$ minimizes

$$f(x) - \langle x, A^*\bar{\lambda} \rangle + i_C(x),$$

and thus

$$0 \in \partial(f - A^*\bar{\lambda} + i_C)(x_0) = \partial f(x_0) - A^*\bar{\lambda} + N_C(x_0),$$

by our regularity condition. Therefore, there is an $x^* \in N_C(x_0)$ and $y^* \in \partial f(x_0)$ such that

$$x^* + y^* = A^*\bar{\lambda}.$$

Now, take any $x$ such that $Ax = b$,

$$\langle x - x_0, x^* \rangle = \langle x - x_0, A^*\bar{\lambda} - y^* \rangle = \langle x - x_0, -y^* \rangle \geq f(x_0) - f(x),$$

and the proof is complete. □

Our first Lagrange multiplier theorem applies to the problem (6). In that case, $G(x) = -x$, and $C = S$ is the positive cone.

**Theorem 5.1.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be convex, let $S \subseteq X$ be a closed positive cone, let $A: X \to \mathbb{R}^n$ be a continuous linear map, and let $b \in \mathbb{R}^n$. Suppose that (RC) holds as well as (BLCQ). If there is feasible attainment for

$$\mu = \inf\{f(x): Ax = b, x \in S\}$$

at $x_0$, then there is an element $z^* \in S^+$ such that

$$\mu = \inf\{f(x) - \langle x, z^* \rangle: Ax = b\}.$$

Furthermore, $\langle x_0, z^* \rangle = 0$.

This is equivalent to saying that $z^*$ is a Lagrange multiplier.

**Proof.** By Lemma 5.1, there is an $x^* \in N_C(x_0) \subset -S^+$ such that

$$f(x_0) = \inf\{f(x) + \langle x - x_0, x^* \rangle: Ax = b\}.$$

Since $x^* \in N_C(x_0)$ and $x_0 \in S$ implies $\langle x_0, x^* \rangle = 0$, we have

$$\mu = \inf\{f(x) + \langle x, x^* \rangle: Ax = b\}.$$

Let $z^* = -x^*$. □

The other case that we consider is the generalization of the $L^p$ bounded problem discussed earlier. For this, we let $X$ be a Banach lattice, $\alpha, \beta \in X$,

$$C = \{x: \alpha \leq x \leq \beta\}.$$

To formulate a Lagrange multiplier theory, we let

$$G(x) = \begin{pmatrix} \alpha - x \\ x - \beta \end{pmatrix},$$

so that

$$\mu = \inf\{f(x): x \in C, Ax = b\} = \inf\{f(x): G(x) \leq \theta, Ax = b\}.$$

**Theorem 5.2.** Assume (RC), (BLCQ), and feasible attainment at $x_0$. There is a $z^* \in S^+ \times S^+$ such that

$$\mu = \inf\{f(x) + \langle G(x), z^* \rangle : Ax = b\}.$$

Also, at the solution $x_0$, we have

$$\langle G(x_0), z^* \rangle = 0,$$

i.e., complementary slackness.

**Proof.** By Lemma 5.1, we have an $x^* \in -N_C(x_0)$ such that

$$f(x_0) = \min_{Ax=b} \{f(x) - \langle x - x_0, x^* \rangle\}.$$

Now, $x^* \in -N_C(x_0)$ implies

$$\langle x - x_0, x^* \rangle \geq 0, \qquad \text{for all } x \in C,$$

which implies that $x_0$ is a support point of $C$ with support functional $x^*$, so by Theorem 2.3,

$$\langle x_0, x^* \rangle = \langle \alpha, x_+^* \rangle - \langle \beta, x_-^* \rangle,$$

so

$$\langle x - x_0, x^* \rangle = \langle x - \alpha, x_+^* \rangle - \langle x - \beta, x_-^* \rangle.$$

If we let

$$z^* = (x_+^*, x_-^*)^*,$$

then we have our Lagrange multiplier,

$$\begin{aligned} \mu &= \min_{Ax=b} \{f(x) + \langle \alpha - x, x_+^* \rangle + \langle x - \beta, x_-^* \rangle\} \\ &= \min_{Ax=b} \{f(x) + \langle G(x), z^* \rangle\}. \end{aligned}$$

□

These two Lagrange multiplier theorems can be used to characterize solutions to the problems in Eqs. (6) and (7), provided the linear constraints satisfy a regularity condition, for example, that the functions $\phi_i$ defining the linear constraints are linearly independent. In fact, these theorems make the details of the characterization easier than in Refs. 1 and 3, since we can work with $L^q$ as the dual of $L^p$, $1/p + 1/q = 1$, instead of $(L^\infty)^*$, the space of finitely additive measures.

In addition, the approach previously taken with both of the model problems required the existence of a function that is bounded away from the bounds, whereas with these theorems we just need a function not equal to the bounds almost everywhere.

Rather than show the details of these calculations, we now show how the duality theory of Borwein and Lewis can further generalize these problems. Note that the assumption of linear independence on the linear constraints is not particularly restrictive, since any dependence lessens the amount of information to be gotten from the data. The following results can be obtained via a case-by-case study using the Lagrange multiplier theorems above, but we find the duality arguments to be more unifying.

## 6. Minimum Norm Problems with Bounds

Let $\alpha$ and $\beta$ be extended-valued measurable functions on $K$, and consider the closed convex set

$$C = [\alpha, \beta] \cap L^p(K).$$

Let

$$\mu = \inf\{\|x\|_p^p \colon Ax = b, x \in C\}$$

be finite; i.e., there is an $x \in C$ such that $Ax = b$. Since $L^p$ is strictly convex, $1 < p < \infty$, there is a unique solution $x_0$ such that $f(x_0) = \mu$.

The duality theory requires that there is a feasible vector in the quasi-relative interior of $C$. That is, there is an $\hat{x}$ such that $A\hat{x} = b$ and $\alpha < \hat{x} < \beta$. This in turn requires that we assume $\alpha < \beta$, which is really no restriction, for if

$$E = \{t \colon \alpha(t) = \beta(t)\},$$

then

$$\int_K \phi_i x = b_i \Leftrightarrow \int_{E^c} \phi_i x = b_i - \int_E \phi_i \alpha = b_i'.$$

So, we assume

(BLCQ) $\exists \hat{x} \in L^p(K),\ A\hat{x} = b,\ \alpha < \hat{x} < \beta$, a.e.

This is equivalent to (CQ) in this context.

We shall need to know how to go from a solution of the dual problem to a solution of the primal problem. Let $f \colon X \to (-\infty, +\infty]$ be strictly convex, and suppose that

$$\mu = \inf\{f(x) \colon Ax = b\}$$

by finite. By Theorem 4.1,

$$\mu = \max_{\lambda}\{\langle \lambda, b\rangle - f^*(A^*\lambda)\},$$

where $f^*: X^* \to \mathbb{R}$ is defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x, x^*\rangle - f(x)\},$$

is the convex conjugate functional (Ref. 21). Under the assumption that $\mathrm{cont}(f) \neq \emptyset$, the maximum in the dual is attained at $\bar{\lambda}$ if and only if

$$b \in A(\partial f^*(A^*\bar{\lambda})) \tag{9}$$

by the chain rule for subgradients (Ref. 21).

Let $x_0 \in \partial f^*(A^*\bar{\lambda})$ and $Ax_0 = b$; we now show $f(x_0) = \mu$. Since $X$ is reflexive, $x_0 \in X$. By Rockafellar (Ref. 19), $A^*\bar{\lambda} \in \partial f(x_0)$, so $f(x_0) \in \mathbb{R}$ and, by the subgradient inequality (8),

$$\langle x - x_0, A^*\bar{\lambda}\rangle \leq f(x) - f(x_0). \tag{10}$$

Take any $x$ such that $Ax = b$; then, (10) implies $f(x_0) \leq f(x)$, therefore $f(x_0) = \mu$. By strict convexity of $f$, there is only one solution to the minimization problem, and thus

$$\partial f^*(A^*\bar{\lambda}) \cap \{x: Ax = b\} = \{x_0\}.$$

In fact, it can be shown that, when

$$f(x) = \|x\|_p^p,$$

$f^*$ is Frechet differentiable and thus the subgradient is a singleton. We have shown the following lemma.

**Lemma 6.1.** Let $f: X \to (-\infty, +\infty]$ be strictly convex and proper with differentiable conjugate $f^*$, let $A: X \to \mathbb{R}^n$ be continuous and linear, let $b \in \mathbb{R}^n$ such that (BLCQ) and our regularity condition (RC) hold. Suppose that

$$\mu = \inf\{f(x): Ax = b\}$$

is attained at the feasible point $x_0$. Then,

$$x_0 \in \partial f^*(A^*\bar{\lambda}), \qquad \text{for some } \bar{\lambda} \in \mathbb{R}^n,$$

and $x_0$ is unique.

We need to calculate $N_C(x_0)$.

**Lemma 6.2.** For $C = [\alpha, \beta] \cap L^p(K)$, $x_0 \in C$, $x^* \in N_C(x_0)$ implies

$$x^*(t)\begin{cases} = 0, & \beta(t) < x_0(t) < \beta(t), \\ \geq 0, & x_0(t) = \beta(t), \\ \leq 0, & x_0(t) = \alpha(t). \end{cases}$$

**Proof.** Recall that

$$N_C(x_0)=\{x^*\in X^*: \langle x-x_0, x^*\rangle\leq 0\}.$$

Let

$$E_1=\{t: \alpha(t)<x_0(t)<\beta(t)\}.$$

Let

$$x(t)=\begin{cases} x_0(t)+\epsilon(t), & t\in E_1,\\ x_0(t), & t\in E_1^c,\end{cases}$$

where $\epsilon>0$ is chosen so that $x\in C$. Then, $x>x_0$ on $E_1$, and

$$\int_{E_1} xx^*\leq\int_{E_1} x_0x^*,$$

so $x^*\leq 0$ on $E_1$. Pick $\epsilon(t)<0$ in this argument to get $x^*\geq 0$ on $E_1$. Thus, $x^*=0$ on $E_1$.

Let

$$E_2=\{t: x_0(t)=\alpha(t)\}.$$

Then, for any $x\in C$ such that $x=x_0$ on $E_2^c$,

$$\int_{E_2} xx^*\leq\int_{E_2}\alpha x^*$$

implies $x^*\leq 0$ on $E_2$, since $\alpha<\beta$.

A similar argument works for

$$E_3=\{t: x_0(t)=\beta(t)\}.$$ □

We now characterize solutions to the general $L^p$ spectral estimation problem with bounds. We use the notation, for $a$ and $b$ extended real numbers,

$$a\vee b=\max\{a, b\},\qquad a\wedge b=\min\{a, b\}.$$

**Theorem 6.1.** Let $K$ be a finite measure space with measure $\nu$, $C=[\alpha,\beta]\cap L^p(K)$, where $\alpha$ and $\beta$ are extended real-valued functions, $p\in(1,\infty)$. Let $\{\phi_i\}$ be a finite set of functions in $L^q$ for $q$ conjugate to $p$; let $A: L^p(K)\to\mathbb{R}^n$ by

$$(Ax)_i=\int_K x(t)\phi_i(t)\,d\nu;$$

and let $b\in\mathbb{R}^n$. The feasible set is

$$F=C\cap\{x: Ax=b\}.$$

Provided there is a feasible point $\hat{x}$ such that $\alpha < \hat{x} < \beta$, a.e., the solution to the minimization problem

$$\inf\{\|x\|_p^p: Ax = b, x \in C\}$$

is the unique feasible point of the form

$$x_0 = \alpha \vee (\mathrm{sign}(a^*)|a^*|^{q-1}) \wedge \beta, \qquad \text{for some } a^* \in L^p(K).$$

**Proof.** Let

$$f(x) = (1/p)\|x\|_p^p,$$

and consider the problem

$$\mu = \inf\{f(x): Ax = b, x \in C\} = \inf\{f_C(x): Ax = b\},$$

where $f_C = f + i_C$. Then by Lemma 6.1, the solution is $x_0 \in \partial f_C^*(A^*\lambda)$, so $A^*\lambda \in \partial f_C(x_0)$ and

$$a^* = A^*\lambda \in \partial f(x_0) + \partial i_C(x_0) = \partial((1/p)\|x_0\|^p) + N_C(x_0).$$

By Lemma 6.2,

$$a^*(t)\begin{cases} = x_0|^{p-1}\,\mathrm{sign}(x_0), & \alpha(t) < x_0(t) < \beta(t), \\ \leq |x_0|^{p-1}\,\mathrm{sign}(x_0), & \alpha(t) = x_0(t), \\ \geq |x_0|^{p-1}\,\mathrm{sign}(x_0), & \beta(t) = x_0(t). \end{cases}$$

We now solve for $x_0$.

On $E_1$,

$$\mathrm{sign}(a^*) = \mathrm{sign}(x_0) \quad \text{and} \quad x_0 = |a^*|^{q-1}\,\mathrm{sign}(a^*).$$

On $E_2$, simple calculations yield

$$\alpha \geq \mathrm{sign}(a^*)|a^*|^{q-1},$$

and thus

$$x_0 = \alpha \vee (\mathrm{sign}(a^*)|a^*|^{q-1}),$$

on $E_1 \cup E_2$. Finally, upon consideration of $E_3$, we can show that

$$x_0 = \alpha \vee (\mathrm{sign}(a^*)|a^*|^{q-1}) \wedge \beta$$

on $E_1 \cup E_2 \cup E_3 = K$ and where

$$a^* = A^*\lambda = \sum \lambda_i \phi_i.$$

□

Some special cases are now discussed.

Positive Spectral Estimation. If $\alpha = 0$ and $\beta = +\infty$, then we are in the case considered in Goodrich and Steinhardt (Ref. 1). The improvement here is that no regularity of the linear constraints are needed. The solution is

$$x_0 = (\sum \lambda_i \phi_i)_+^{q-1}.$$

Bounded Spectral Estimation. If $\alpha, \beta \in L^p$, then we are in the case considered by Limber (Ref. 22), again with the improvement that the linear constraints do not have to be regular. This includes the $L^\infty$ bounded case in Cole and Goodrich (Ref. 3) if $\alpha = 0$ and $0 < \epsilon \leq \beta \in L^\infty$.

General Lattice Bounds. By appropriate choices of $\alpha$ and $\beta$, we can allow the feasible functions to be free on subsets of $K$. The solution is still the truncated function. See Dontchev (Ref. 23) for applications to constrained interpolation.

More general integral objective functions with lattice bounds will be explored in a future work (Ref. 24), and numerical investigations will be reported in Ref. 25. See also Ref. 26.

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