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Generalized Convex Functions and Vector Variational Inequalities[1]

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Abstract. In this paper, (α, ϕ, Q)-invexity is introduced, where $\alpha: X \times X \to \operatorname{int} R^m_+$, $\phi: X \times X \to X$, X is a Banach space, Q is a convex cone of R^m. This unifies the properties of many classes of functions, such as Q-convexity, pseudo-linearity, representation condition, null space condition, and V-invexity. A generalized vector variational inequality is considered, and its equivalence with a multi-objective programming problem is discussed using (α, ϕ, Q)-invexity. An existence theorem for the solution of a generalized vector variational inequality is proved. Some applications of (α, ϕ, Q)-invexity to multi-objective programming problems and to a special kind of generalized vector variational inequality are given.

Key Words. Generalized convex functions, generalized vector variational inequalities, multi-objective programming problems, necessary and sufficient conditions.

1. Introduction

Variational inequalities and complementarity problems play an important role in many fields, such as economics, control theory, engineering, etc. (see Ref. 1). Recently, these scalar problems have been generalized to the vector case, based on vector optimization problems and weak minimal element problems (Refs. 2–6). Vector variational inequalities were first introduced in Ref. 2 in a finite-dimensional space. Later, vector variational

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inequalities and quasi-vector variational inequalities were discussed in Ref. 3 in an infinite-dimensional space. Some existence theorems for the solutions of vector variational inequalities and quasi-vector variational inequalities were proved in Refs. 3 and 5.

Let X be a Banach space, R^m an m-dimensional Euclidean space, and let $L(X, R^m)$ be the set of all linear continuous operators from X to R^m. For $l \in L(X, R^m)$ and $x \in X$, $\langle l, x\rangle$ denotes the value of a linear continuous operator l at x. Let P be a convex cone of R^m with int $P \neq \varnothing$. Define the partial orderings of R^m as follows: for all, $y_1, y_2 \in R^m$,

$$y_1 \geq y_2 \Leftrightarrow y_1 - y_2 \in P,$$

$$y_1 \not> y_2 \Leftrightarrow y_1 - y_2 \notin \text{int } P.$$

Note that

$$y_1 \not< y_2 \Leftrightarrow y_2 \not> y_1.$$

A useful relation between $\geq$ and $\not<$ is that

$$a \geq b \not< c \Rightarrow a \not< c,$$

which is derived from the cone equality,

$$\text{int } P + P = \text{int } P;$$

see Ref. 5. Let P^* be the dual cone of P, i.e.

$$P^* = \{y^* \in R^m: y^T y^* \geq 0, \forall y \in P\}.$$

Consider the following vector optimization problem:

(VPC) $\quad$ V-$\min_P f(x)$, $\quad$ s.t. $x \in K$,

where K is a subset of X and $f = (f_1, \ldots, f_m): X \to R^m$. Recall that a point $a \in K$ is called a weak minimum of (VPC) if $f(x) - f(a) \not< 0$, $\forall x \in K$ and that the point $a \in K$ is called a minimum (efficient solution) of (VPC) if there exists no $x \in K$ such that $f(x) - f(a) \in -P\backslash\{0\}$ (Ref. 18). It should be noted that a weak minimum, or a minimum, of (VPC) is just a nondominated solution of (VPC) with a constant domination cone in Ref. 7, i.e., with an open or closed convex cone as domination cone. For the details of nondominated solutions, the reader may refer to Refs. 7 and 8. However, for simplicity, we discuss (VPC) in the sense of weak minimum and minimum in this paper.

Consider the following vector variational inequality (Refs. 3 and 5):

(VVI) $\quad$ find $x \in C$, $\langle T(x), y - x\rangle \not< 0$, $\quad \forall y \in C$,

where C is a convex subset of X and $T = (T_1, \ldots, T_m): X \to L(X, R^m)$. The following results were given in Ref. 3, which shows that (VVI) is closely related to a convex vector optimization problem. Note that f_i' is the Gâteaux derivative to f_i and $f' = (f_1', \ldots, f_m')$.

Theorem 1.1. See Ref. 3. Assume that $K = C$ is a convex subset of X and that $f: X \to R^m$ is differentiable, $T_i = f_i'$, and P is a closed convex cone. Then, we have:

(i) if a is a weak minimum of (VPC), a is a solution of the vector variational inequality (VVI);
(ii) if f is a Gâteaux differentiable P-convex function (see Example 2.1) and a is a solution of the vector variational inequality (VVI), a is a weak minimum of (VPC).

Recently, in Refs. 9 and 10, the following scalar prevariational inequality was considered:

$$\text{(VI)} \quad \text{find } x \in C_1, F(x)^T \eta(y, x) \geq 0, \qquad \forall y \in C_1,$$

where $F: R^n \to R^n$, $\eta: R^n \times R^n \to R^n$, and C_1 is a subset of R^n. The relations between (VI) and a minimization problem and existence theorems for the solution of (VI) were given in Refs. 9 and 10. It should be noted that this kind of variational inequality is closely related to the invexity (see Refs. 9–12).

In this paper, we introduce (α, ϕ, Q)-invexity, which unifies the properties of many classes of functions, which recently appeared in optimization literature, such as pseudo-linearity (Ref. 13), representation condition (Ref. 14), null space condition (Ref. 15, see also Ref. 16), and V-invexity (Ref. 17). We show that this class of functions is applicable to many optimization problems and that various sufficient conditions for multi-objective programming problems can be obtained. We introduce a generalized vector variational inequality and study its equivalence with a multi-objective programming problem.

The outline of the paper is as follows. In Section 2, we introduce (α, ϕ, Q)-invexity and give various examples and properties of (α, ϕ, Q)-invexity. In Section 3, we study a generalized vector variational inequality and establish its equivalence with a multi-objective programming problem using the (α, ϕ, Q)-invexity condition. We also prove an existence theorem for the solution of the generalized vector variational inequality. In Section 4, we give sufficient conditions for the weak minimum of a multi-objective programming problem. In particular, we derive a sufficient condition for properly efficient solution of an inequality and equality constrained multi-objective programming problem. In Section 5, we present a necessary condition and a sufficient condition for the solution of a vector variational inequality.

2. Generalized Convex Functions

Let us first look at some examples.

Example 2.1. P-Convex Function. See Refs. 8 and 18. The function $g: X \to R^m$ is called P-convex if, for $x_1, \ldots, x_n \in X$, $\sum_{i=1} \lambda_i = 1$, $\lambda_i \geq 0$,

$$\sum_{i=1}^{n} \lambda_i g(x_i) - g\left(\sum_{i=1}^{n} \lambda_i x_i\right) \in P. \tag{1}$$

Assume that P is closed and g is Gâteaux differentiable. Then (1) holds if and only if

$$g(y) - g(x) \in \langle g'(x), y - x \rangle + P, \qquad \forall x, y \in X. \tag{2}$$

It is easy to verify that g is P-convex if and only if $g(X)$ is P-convex, i.e., $g(X) + P$ is convex. It is well known that P-convex functions have many applications in economics and engineering. But examples of functions do exist which are not P-convex.

Example 2.2. Let

$$x = (x_1, x_2) \in D = \{(x_1, x_2): x_1 \geq 1, x_2 \geq 1\} \subset R^2,$$

and let

$$h(x) = ((x_1/x_2)^{n_1}, \ldots, (x_1/x_2)^{n_m}), \qquad n_i \geq 1, \quad i = 1, \ldots, m.$$

Then, it is easy to verify that, for any convex cone P of R^m, $P \neq R^m$, h is not P-convex.

Next, we introduce a class of generalized convex functions which includes the above examples as well as many other classes of functions as we show later. Let $f = (f_1, \ldots, f_m): X \to R^m$. Define $F_x: X \times \cdots \times X \to R^m$ by

$$F_x(y_1, \ldots, y_m) = (\langle f'_1(x), y_1 \rangle, \ldots, \langle f'_m(x), y_m \rangle).$$

Definition 2.1. Let $f: X \to R^m$, $\phi: X \times X \to X$, $\alpha: X \times X \to \text{int } R^m_+$, Q a convex subcone of P. If for any $x, y \in X$, the relation

$$f(y) - f(x) \in F_x(\alpha_1(x, y)\phi(y, x), \ldots, \alpha_m(x, y)\phi(y, x)) + Q \tag{3}$$

holds, then f is called (α, ϕ, Q)-invex on X.

If x is fixed, (3) holds for each $y \in X$, then f is called to be (α, ϕ, Q)-invex at x.

Remark 2.1.

(i) We always assume that $\phi(x, x) = 0$, $\forall x \in X$.

(ii) The relation (3) is equivalent to

$$(f_1(y), \ldots, f_m(y)) - (f_1(x), \ldots, f_m(x))$$
$$\in (\alpha_1(x, y)\langle f'_1(x), \phi(y, x)\rangle, \ldots, \alpha_m(x, y)\langle f'_m(x), \phi(y, x)\rangle) + Q.$$

(iii) If $X = R^m$, $m = 1$, $Q = R_+$, (3) becomes

$$f_1(y) - f_1(x) \geq f'_1(x)^T \eta(y, x),$$

where $\eta(y, x) = \alpha_1(x, y)\phi(y, x)$. Hence f_1 is an invex function with respect to $\eta(y, x)$; see Refs. 11 and 12.

(iv) In Example 2.2, h is (α, ϕ, R^m_+)-invex on D if we choose

$$\alpha_i((x_i, x_2), (y_1, y_2)) = x_2/y_2, \qquad i = 1, \ldots, m,$$
$$\phi((y_1, y_2), (x_1, x_2)) = (y_1 - x_1, y_2 - x_2).$$

The following examples show that (α, ϕ, Q)-invexity includes many classes of functions.

Example 2.3. V-Invexity. See Ref. 17. Let $P = Q = R^m_+$. Then (3) becomes

$$f_i(y) - f_i(x) \geq \alpha_i(x, y)\langle f'_i(x), \phi(y, x)\rangle, \qquad i = 1, \ldots, m, \quad \forall x, y \in X.$$

This means that f is a V-invex function; that is, each f_i is invex with respect to $\eta_i(y, x) = \alpha_i(x, y)\phi(y, x)$; see Remark 2.1 (iii). If $\alpha_1(x, y) = \cdots = \alpha_m(x, y) = \alpha^*(x, y)$, we say that f is (α, ϕ, P)-invex with respect to the same $\alpha^*(x, y)$.

Example 2.4. Representation Condition. See Ref. 14. Let

$$\phi(y, x) = z_{x,y} - x,$$
$$\alpha_i(x, y) \in \text{int } R_+, \quad i = 1, \ldots, m, \qquad Q = \{0\}.$$

From (3), we get

$$f_i(y) - f_i(x) = \alpha_i(x, y)\langle f'_i(x), z_{x,y} - x\rangle, \qquad i = 1, \ldots, m, \quad \forall x, y \in X.$$

Then, f satisfies the representation condition in Ref. 14.

Example 2.5. Null Space Condition. See Refs. 15 and 16. Let

$$\alpha_i(x, y) \in \text{int } R_+, \quad i = 1, \ldots, m, \qquad P = R^m_+, \qquad Q = \{0\}.$$

It follows from (3) that

$$f_i(y) - f_i(x) = \alpha_i(x, y)\langle f_i'(x), \phi(y, x)\rangle, \qquad i = 1, \ldots, m, \quad \forall x, y \in X.$$

This means that f satisfies a null space condition. At this time, we say that f is $(\alpha, \phi, 0)$-invex. If $\alpha_1(x, y) = \cdots = \alpha_m(x, y) = \alpha^*(x, y)$, we say that f is $(\alpha, \phi, 0)$-invex with respect to the same $\alpha^*(x, y)$.

Example 2.6. Pseudo-Linearity. See Ref. 13. Let

$$X = R^n, \qquad \phi(y, x) = y - x,$$
$$\alpha_i(x, y) \in \operatorname{int} R_+, \quad i = 1, \ldots, m, \qquad P = R_+^m, \qquad Q = \{0\}.$$

Then, (3) becomes

$$f_i(y) - f_i(x) = \alpha_i(x, y)\langle f_i'(x), y - x\rangle, \qquad i = 1, \ldots, m, \quad \forall x, y \in X.$$

Hence, each f_i is a pseudo-linear function.

The following theorem summarizes the relations among these properties of a function:

Theorem 2.1. We have the following relations:

$$\text{pseudo-linearity} \Rightarrow \text{representation condition} \Rightarrow \text{null space condition}$$
$$\Rightarrow \text{V-invexity} \Rightarrow (\alpha, \phi, P)\text{-invexity}.$$

Proof. The proof follows easily from the definitions and is omitted. □

Next, we show some properties of the (α, ϕ, Q)-invex functions.

Theorem 2.2. If for each $i = 1, \ldots, k$, $g_i = (f_{i1}, \ldots, f_{im})$ satisfies the (α, ϕ, Q_i)-invexity condition, $\lambda_i \geq 0$, $i = 1, \ldots, k$, then $\sum_{i=1}^k \lambda_i g_i$ satisfies the (α, ϕ, Q)-invexity condition with $Q = \operatorname{co} \bigcup_{i=1}^k Q_i$.

Proof. For $x, y \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i g_i(y) - \sum_{i=1}^k \lambda_i g_i(x) \\
&\in \sum_{i=1}^k \lambda_i F_x^i(\alpha_1(x, y)\phi(y, x), \ldots, \alpha_m(x, y)\phi(y, x)) + \sum_{i=1}^k \lambda_i Q_i \\
&\subset \sum_{i=1}^k \lambda_i(\langle f_{i1}'(x), \alpha_1(x, y)\phi(y, x)\rangle, \ldots, \\
&\qquad \langle f_{im}'(x), \alpha_m(x, y)\phi(y, x)\rangle) + \sum_{i=1}^k \lambda_i Q_i
\end{aligned}$$

$$\subset \left(\left\langle \left(\sum_{i=1}^{k} \lambda_i f_{i1} \right)'(x), \alpha_1(x, y)\phi(y, x) \right\rangle, \ldots, \right.$$
$$\left. \left\langle \left(\sum_{i=1}^{k} \lambda_i f_{im} \right)'(x), \alpha_m(x, y)\phi(y, x) \right\rangle \right) + Q$$
$$= F_x(\alpha_1(x, y)\phi(y, x), \ldots, \alpha_m(x, y)\phi(y, x)) + Q,$$

where

$$Q = \mathrm{co} \bigcup_{i=1}^{k} Q_i,$$
$$F_x(\cdot) = \left(\left\langle \left(\sum_{i=1}^{k} \lambda_i f_{i1} \right)'(x), . \right\rangle, \ldots, \left\langle \left(\sum_{i=1}^{k} \lambda_i f_{im} \right)'(x), . \right\rangle \right).$$

Then, $\sum_{i=1}^{k} \lambda_i g_i$ is an (α, ϕ, Q)-invex function. □

Theorem 2.3. Let $f = h \circ g$. If $h: R^n \to R^m$ is $(1, y - x, P)$-invex, $g: X \to R^n$ is $(\alpha, \phi, 0)$-invex, then f is (α, ϕ, P)-invex.

Proof. We have

$$\begin{aligned} f(y) - f(x) &= h(g(y)) - h(g(x)) \in \langle h'(g(x)), g(y) - g(x) \rangle + P \\ &= \langle h'(g(x)), (\langle g_1'(x), \alpha_1(x, y)\phi(y, x) \rangle, \ldots, \\ &\qquad \langle g_m'(x), \alpha_m(x, y)\phi(y, x) \rangle) \rangle + P \\ &= (\langle (h_1 \circ g)'(x), \alpha_1(x, y)\phi(y, x) \rangle, \ldots, \\ &\qquad \langle (h_m \circ g)'(x), \alpha_m(x, y)\phi(y, x) \rangle) + P \\ &= F_x(\alpha_1(x, y)\phi(y, x), \ldots, \alpha_m(x, y)\phi(y, x)) + P, \end{aligned}$$

where

$$F_x(\cdot) = (\langle f_1'(x), . \rangle, \ldots, \langle f_m'(x), . \rangle).$$

The proof is completed. □

3. Generalized Vector Variational Inequalities

Let X be a Banach space, and let P be a convex cone of R^m with $\mathrm{int}\, P \neq \varnothing$, $T = (T_1, \ldots, T_m): X \to L(X, R^m)$, $\phi: X \times X \to X$. Consider the following generalized vector variational inequality problem:

(GVVI) find $x \in K$, $F_x(\phi(y, x), \ldots, \phi(y, x)) \nless 0, \quad \forall y \in K,$

where K is a subset of X, $F_x(z) = (\langle T_1(x), z_1 \rangle, \ldots, \langle T_m(x), z_m \rangle)$, $z = (z_1, \ldots, z_m) \in X \times \cdots \times X$.

Remark 3.1.

(i) If $\phi(y, x) = y - x$, then the generalized vector variational inequality (GVVI) becomes

$$\text{(VVIm)} \quad (\langle T_1(x), y - x \rangle, \ldots, \langle T_m(x), y - x \rangle) \not< 0, \qquad \forall x \in K.$$

This is the vector variational inequality studied in Refs. 4 and 5 when K is convex.

(ii) If $m = 1$, $X = R^n$, this was considered in Refs. 9 and 10; see Section 1.

Consider the following generalized multi-objective programming problem:

$$\text{(GVPC)} \quad \text{V-min}_P\,(f_1(x), \ldots, f_m(x)), \qquad \text{s.t. } x \in K,$$

where $f_i: X \to R$ is Gâteaux differentiable, $i = 1, \ldots, m$, P is a convex cone of R^m with int $P \neq \emptyset$.

Theorem 3.1. If K is open and a is a weak minimum of (GVPC), then a is a solution of the generalized vector variational inequality (GVVI).

Proof. For any $x \in K$, there exists $t_0 > 0$, such that $0 < t < t_0$, $a + t\phi(x, a) \in K$, and

$$f(a + t\phi(x, a)) - f(a) \not< 0.$$

Then,

$$(\langle f'_1(a), \phi(x, a) \rangle, \ldots, \langle f'_m(a), \phi(x, a) \rangle) \not< 0.$$

Let

$$T_i = f'_i, \qquad i = 1, \ldots, m,$$

$$F_a(z) = (\langle T_1(a), z_1 \rangle, \ldots, \langle T_m(a), z_m \rangle),$$

$$z = (z_1, \ldots, z_m) \in X \times \cdots \times X.$$

We get

$$F_a(\phi(x, a), \ldots, \phi(x, a)) \not< 0, \qquad \forall x \in K.$$

Then, a is a solution of the generalized vector variational inequality (GVVI). □

Remark 3.2. In Theorem 3.1, other than the condition that K is open, we may assume that K has the ϕ-connectedness; i.e., $\forall x, y \in K$, $t \in [0, 1]$, we have $x + t\phi(y, x) \in K$; see Ref. 10.

Theorem 3.2. Assume that f is (α, ϕ, Q)-invex and Q is a convex subcone of P. If a is a solution of the generalized vector variational inequality (GVVI), then a is a weak minimum of (GVPC).

Proof. Since a is a solution of the generalized vector variational inequality (GVVI), we get

$$F_a(\phi(x, a), \ldots, \phi(x, a)) \nless 0, \qquad \forall x \in K.$$

Equivalently,

$$F_a(\phi(x, a), \ldots, \phi(x, a)) \notin -\operatorname{int} P, \qquad \forall x \in K,$$

i.e.,

$$(F_a(\phi(x, a), \ldots, \phi(x, a)) + Q) \cap -\operatorname{int} P = \emptyset.$$

It follows from the (α, ϕ, Q)-invexity of $f(x)$ that

$$f(x) - f(a) \notin -\operatorname{int} P, \qquad \forall x \in K,$$

$$f(x) - f(a) \nless 0, \qquad \forall x \in K.$$

So, a is a weak minimum of (GVPC). □

Corollary 3.1. Assume that $T = f' = (f'_1, \ldots, f'_m)$, f is an (α, ϕ, P)-invex function with the same $\alpha^*(a, x) = 1$ and $\phi(x, a) = x - a$. If $a \in K$ is a solution of the vector variational inequality (VVIm), then a is a weak minimum of (GVPC).

Next, we consider the existence for the solution of the generalized vector variational inequality (GVVI).

Theorem 3.3. Let K be a compact convex set of X, and let T, ϕ be continuous. If the function $\varphi(y) = F_x(\phi(y, x), \ldots, \phi(y, x))$ is P-convex for each fixed $x \in K$, then the generalized vector variational inequality (GVVI) is solvable. Furthermore, the solution set of (GVVI) is a compact set.

Proof. Let

$$G(y) = \{z \in K: F_z(\phi(y, z), \ldots, \phi(y, z)) \nless 0\}, \qquad y \in K.$$

Let

$$\{y_1, \ldots, y_n\} \subset K, \qquad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0.$$

Then,

$$y=\sum_{i=1}^{n}\lambda_i y_i\in\bigcup_{i=1}^{n}G(y_i).$$

Otherwise, $y\notin G(y_i)$, $\forall i$; thus,

$$F_y(\phi(y_i,y),\ldots,\phi(y_i,y))<0,\qquad \forall i.$$

Then,

$$\sum_{i=1}^{n}\lambda_i F_y(\phi(y_i,y),\ldots,\phi(y_i,y))<0.$$

By the P-convexity of

$$\varphi(y)=F_x(\phi(y,x),\ldots,\phi(y,x)),$$

we have

$$0=F_y(\phi(y,y),\ldots,\phi(y,y))\leq\sum_{i=1}^{n}\lambda_i F_y(\phi(y_i,y),\ldots,\phi(y_i,y))<0,$$

a contradiction. So,

$$y=\sum_{i=1}^{n}\lambda_i y_i\in\bigcup_{i=1}^{n}G(y_i).$$

From the continuity of T and ϕ, $G(y)$ is closed. By using the Knaster, Kuratowski, and Mazurkiewicz theorem (Ref. 5), we get

$$\bigcap_{y\in K}G(y)\neq\emptyset.$$

Let

$$x\in\bigcap_{y\in K}G(y).$$

Then

$$F_x(\phi(y,x),\ldots,\phi(y,x))\not<0,\qquad \forall y\in K.$$

Since the intersection of closed sets is closed, the solution set of (GVVI), $\bigcap_{y\in K}G(y)$, is closed, so it is compact. □

Remark 3.3. In order to prove the existence for the solution of the generalized vector variational inequality (GVVI) for the case where K is not compact, we require additional conditions, such as the coercitivity conditions (Refs. 9 and 10).

4. Sufficiency of the Generalized Karush–Kuhn–Tucker Conditions

In this section, we continue to present the applications of the (α, ϕ, Q)-invexity. We give various sufficient conditions for the generalized Karush–Kuhn–Tucker conditions of multi-objective programming problems.

Consider the following multi-objective programming problem:

$$\text{(GVP)} \quad \text{V-min}_P \quad f(x) = (f_1(x), \ldots, f_m(x)),$$
$$\text{s.t.} \quad x \in X,$$
$$-g(x) = (-g_1(x), \ldots, -g_n(x)) \in S,$$
$$h(x) = (h_1(x), \ldots, h_l(x)) = 0,$$

where $f_i, g_j, h_k: X \to R$ are Gâteaux differentiable, P and S are convex cones of R^m and R^n, respectively, and $\operatorname{int} P \neq \varnothing$.

By using some regularity conditions, such as local solvability, the following generalized Karush–Kuhn–Tucker necessary conditions for (GVP) were given (Ref. 20), which are a generalization of the well-known Lagrange multiplier theorem (see Ref. 19): if a is a weak minimum of (GVP), then there exist $\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\}$, $\tau = (\tau_1, \ldots, \tau_n) \in S^*$, $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that

$$\sum_{i=1}^{m} \zeta_i f_i'(a) + \sum_{j=1}^{n} \tau_j g_j'(a) + \sum_{k=1}^{l} \lambda_k h_k'(a) = 0, \tag{4}$$

$$\tau_j g_j(a) = 0, \qquad j = 1, \ldots, n. \tag{5}$$

In this, a condition which is called a constraint qualification is used to confirm that $\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\}$.

Next, we consider the sufficiency of the above generalized Karush–Kuhn–Tucker necessary conditions using the (α, ϕ, Q)-invexity.

Theorem 4.1. If f is (α, ϕ, P)-invex with respect to the same $\alpha^*(a, x) > 0$, g is (β, ϕ, S)-invex with respect to the same $\beta^*(a, x) > 0$, h is $(\gamma, \phi, 0)$-invex with respect to the same $\gamma^*(a, x) > 0$, and the generalized Karush–Kuhn–Tucker conditions (4)–(5) hold at a, then a is a weak minimum of (GVP).

Proof. Suppose that a is not a weak minimum of (GVP). Then, there exists a feasible point x such that

$$(f_1(x), \ldots, f_m(x)) - (f_1(a), \ldots, f_m(a)) \in -\operatorname{int} P.$$

From the (α, ϕ, P)-invexity of f and

$$\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\},$$

we have

$$0 > \sum_{i=1}^{m} \zeta_i (f_i(x) - f_i(a)) \geq \sum_{i=1}^{m} \zeta_i \alpha^*(a, x) \langle f'_i(a), \phi(x, a) \rangle. \tag{6}$$

From the generalized Karush–Kuhn–Tucker conditions (4)–(5), we get

$$\begin{aligned} &\sum_{i=1}^{m} \zeta_i \langle f'_i(a), \phi(x, a) \rangle \\ &= - \sum_{j=1}^{n} \tau_j \langle g'_j(a), \phi(x, a) \rangle - \sum_{k=1}^{l} \lambda_k \langle h'_k(a), \phi(x, a) \rangle \\ &\geq - \sum_{j=1}^{n} [1/\beta^*(a, x)] \tau_j (g_j(x) - g_j(a)) \\ &\quad - \sum_{k=1}^{l} [1/\gamma^*(a, x)] \lambda_k (h_k(x) - h_k(a)) \\ &\geq 0. \end{aligned}$$

Hence,

$$0 > \sum_{i=1}^{m} \zeta_i (f_i(x) - f_i(a)) \geq 0,$$

a contradiction. Then, a is a weak minimum of (GVP). □

Remark 4.1. The (α, ϕ, P)-invexity of f can be replaced by the (α, ϕ, Q)-invexity of f, where Q is a convex subcone of P since (6) still holds for this case.

When $P = R^m_+$, $S = R^n_+$, (GVP) becomes the following multi-objective programming problem:

$$\begin{aligned} \text{(GVP1)} \quad &\text{V-min}_{R^m_+} \quad (f_1(x), \ldots, f_m(x)) \\ &\text{s.t.} \qquad x \in X, \\ &\qquad\qquad g_j(x) \leq 0, \quad j = 1, \ldots, n, \\ &\qquad\qquad h_k(x) = 0, \quad k = 1, \ldots, l. \end{aligned}$$

Now, we obtain a sufficient condition for a minimum of (GVP1) using less restrictive assumptions than Theorem 4.1.

Theorem 4.2. If f is (α, ϕ, R^m_+)-invex, g is (β, ϕ, R^m_+)-invex, h is $(\gamma, \phi, 0)$-invex, and the generalized Karush–Kuhn–Tucker conditions (4)–(5) hold with $\zeta = (\zeta_1, \ldots, \zeta_m) \in \operatorname{int} R^m_+$ at a, then a is a minimum of (GVP1).

Proof. If a is not a minimum of (GVP1), there there exists a feasible point x such that $f_i(x) \leq f_i(a)$, $i = 1, \ldots, m$, and strict inequality holds for at least one i. Since $\zeta_i > 0$, $i = 1, \ldots, m$, we get

$$\zeta_i(f_i(x) - f_i(a)) \geq 0, \qquad i = 1, \ldots, m,$$

and strict inequality holds for at least one i. On the other hand, since f is (α, ϕ, R^m_+)-invex, g is (β, ϕ, R^m_+)-invex, h is $(\gamma, \phi, 0)$-invex, we have

$$\begin{aligned} 0 &> \sum_{i=1}^{m} [\zeta_i/\alpha_i(a, x)](f_i(x) - f_i(a)) \\ &\geq \sum_{i=1}^{m} \zeta_i \langle f'_i(a), \phi(x, a)\rangle \\ &= -\sum_{j=1}^{n} \tau_j \langle g'_j(a), \phi(x, a)\rangle - \sum_{k=1}^{l} \lambda_k \langle h'_k(a), \phi(x, a)\rangle \\ &\geq -\sum_{j=1}^{n} [\tau_j/\beta_j(a, x)](g_j(x) - g_j(a)) \\ &\quad - \sum_{k=1}^{l} [\lambda_k/\gamma_k(a, x)](h_j(x) - h_j(a)) \\ &\geq 0, \end{aligned}$$

a contradiction. The proof is completed. □

At this time, if we strengthen the hypotheses in Theorem 4.2, we obtain a sufficient condition for properly efficient solution of (GVP1). Recall that $a \in X$ is a properly efficient solution of (GVP1) (Ref. 22) if it is an efficient solution (minimum) of (GVP1) and there exists a real number $M > 0$ such that, for each i,

$$[f_i(a) - f_i(x)]/[f_i(x) - f_j(a)] < M,$$

for some j satisfying $f_j(a) < f_j(x)$ whenever $x \in X$ is a feasible solution satisfying $f_i(x) < f_i(a)$.

Theorem 4.3. Assume that the generalized Karush–Kuhn–Tucker necessary conditions (4)–(5) hold at a with $\zeta = (\zeta_1, \ldots, \zeta_m) \in \text{int } R^m_+$. If f is (α, ϕ, R^m_+)-invex with respect to the same $\alpha^*(a, x)$, g is (β, ϕ, R^n_+)-invex with respect to the same $\beta^*(a, x)$, h is $(\gamma, \phi, 0)$-invex with respect to the same $\gamma^*(a, x)$, then a is a properly efficient solution of (GVP1).

Proof. From Theorem 4.2, a is an efficient solution of (GVP1). Consider the following scalar minimization problem:

$$\text{(GPV1s)} \quad \min \quad \sum_{i=1}^{m} \zeta_i f_i(x),$$
$$\text{s.t.} \quad x \in X,$$
$$g_j(x) \leqslant 0, \qquad j = 1, \ldots, n,$$
$$h_k(x) = 0, \qquad k = 1, \ldots, l.$$

If x is a feasible solution of (GVP1), then x is also a feasible solution of (GVP1s). It follows from the invexity of f, g, h that

$$\begin{aligned}
&\sum_{i=1}^{m} \zeta_i (f_i(x) - f_i(a)) \\
&\geq \alpha^*(a, x) \sum_{i=1}^{m} \zeta_i \langle f_i'(a), \phi(x, a) \rangle \\
&= \alpha^*(a, x)\left(\sum_{j=1}^{n} \tau_j \langle g_i'(a), \phi(x, a) \rangle - \sum_{k=1}^{l} \lambda_k \langle h_k'(a), \phi(x, a) \rangle \right) \\
&\geq -[\alpha^*(a, x)/\beta^*(a, x)] \sum_{j=1}^{n} \tau_j (g_j(x) - g_j(a)) \\
&\quad - [\alpha^*(a, x)/\gamma^*(a, x)] \sum_{k=1}^{l} \lambda_k (h_k(x) - h_k(a)) \\
&\geq 0.
\end{aligned}$$

Then a is an optimal solution of (GVP1s). It follows from a theorem in Ref. 21 that a is a properly efficient solution of (GVP1). □

Remark 4.2. In comparison with other sufficient conditions for the properly efficient solution, the conditions in Theorem 4.3 subsume the ones used in Ref. 16 in which $\alpha^*(a, x) = \beta^*(a, x) = \gamma^*(a, x)$ were assumed.

5. Vector Variational Inequality

In this section, we study a vector variational inequality which is a special case of the generalized vector variational inequality given in Section 3 and present a necessary condition and a sufficient condition for the solution of the vector variational inequality problem. We intend to generalize the results in Refs. 22 and 23 to the vector case.

Let $T = (T_1, \ldots, T_m): X \to L(X, R^m)$ be continuous, and let g_j, h_k be as in (GVP). P is a convex cone of R^m with $\operatorname{int} P \neq \varnothing$. Consider the following vector variational inequality:

(VVIk) find $a \in K$, $(\langle T_1(a), x-a\rangle, \ldots, \langle T_m(a), x-a\rangle) \nleq 0$, $\forall x \in K$,

where

$$K = \{x \in X: g_j(x) \leq 0, j = 1, \ldots, n, h_k(x) = 0, k = 1, \ldots, l\}$$

is a feasible region. Then, we have the following result.

Theorem 5.1. If a is a solution of the vector variational inequality (VVIk) and if $g'_j(a), j \in I(a), h'_k(a), k = 1, \ldots, l$, are linearly independent, then there exist $\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\}$, $\tau = (\tau_1, \ldots, \tau_n) \geq 0$, $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that

$$\sum_{i=1}^{m} \zeta_i T_i(a) + \sum_{j=1}^{n} \tau_j g'_j(a) + \sum_{k=1}^{l} \lambda_k h'_k(a) = 0, \tag{7}$$

$$\tau_j g_j(a) = 0, \qquad j = 1, \ldots, n. \tag{8}$$

Proof. Consider the multi-objective programming problem with the linear objective function and nonlinear constraints:

(GVPk) V-$\min_P$ $(k_1(x), \ldots, k_m(x)) = (\langle T_1(a), x-a\rangle, \ldots, \langle T_m(a), x-a\rangle)$,

s.t. $x \in K$.

Then, a is a solution of (GVPk). From the generalized Karush–Kuhn–Tucker necessary conditions, there exist $\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\}$, $\tau = (\tau_1, \ldots, \tau_n) \geq 0$, $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that

$$\sum_{i=1}^{m} \zeta_i k'_i(a) + \sum_{j=1}^{n} \tau_j g'_j(a) + \sum_{k=1}^{l} \lambda_k h'_k(a) = 0,$$

$$\tau_j g_j(a) = 0, \qquad j = 1, \ldots, n.$$

Since

$$k'_i(a) = T_i(a), \qquad i = 1, \ldots, m,$$

the proof is completed. □

Theorem 5.2. If g is $(\beta, y-x, R^n_+)$-invex, h is $(\gamma, y-x, 0)$-invex, (7) and (8) hold at a, then a is a solution of the vector variational inequality (VVIk).

Proof. It follows from (7) and (8) and the invexity of g and h that

$$\begin{aligned}\sum_{i=1}^{m} \zeta_i \langle T_i(a), x - a \rangle &= \left\langle \sum_{j=1}^{n} \tau_j g_j'(a) - \sum_{k=1}^{l} \lambda_k h_k'(a), x - a \right\rangle \\ &\geq - \sum_{j=1}^{n} [\tau_j / \beta_j(a, x)](g_j(x) - g_j(a)) \\ &\quad - \sum_{k=1}^{l} [\lambda_k / \gamma_k(a, x)](h_k(x) - h_k(a)) \\ &\geq 0, \qquad \forall x \in K.\end{aligned}$$

Since $\zeta = (\zeta_1, \ldots, \zeta_m) \in P^* \backslash \{0\}$, we get

$$(\langle T_1(a), x - a \rangle, \ldots, \langle T_m(a), x - a \rangle) \not< 0, \qquad \forall x \in K.$$

Therefore, a is a solution of the vector variational inequality (VVIk). □

6. Conclusions

In this paper, a generalized vector variational inequality was considered and its equivalent relations with a generalized multi-objective programming problem were established. Various sufficient conditions for the solutions (i.e., weak minimum, minimum, and properly efficient solution) of multi-objective programming problems were given. A necessary condition and a sufficient condition for the solution of a vector variational inequality were presented. These results were obtained using the (α, ϕ, Q)-invexity, introduced and studied in Section 2, which unifies P-convexity, pseudo-linearity, representation condition, null space condition, and V-invexity. An existence theorem for the solution of the generalized vector variational inequality on a compact convex set was also proved.

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