JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 70, No. 1, JULY 1991

Abort Landing in the Presence of Windshear as a Minimax Optimal Control Problem, Part 1: Necessary Conditions[1,2]

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Abstract. The landing of a passenger aircraft in the presence of windshear is a threat to aviation safety. The present paper is concerned with the abort landing of an aircraft in such a serious situation. Mathematically, the flight maneuver can be described by a minimax optimal control problem. By transforming this minimax problem into an optimal control problem of standard form, a state constraint has to be taken into account which is of order three. Moreover, two additional constraints, a first-order state constraint and a control variable constraint, are imposed upon the model. Since the only control variable appears linearly, the Hamiltonian is not regular. Thus, well-known existence theorems about the occurrence of boundary arcs and boundary points cannot be applied. Numerically, this optimal control problem is solved by means of the multiple shooting method in connection with an appropriate homotopy strategy. The solution obtained here satisfies all the sharp necessary conditions including those depending on the sign of certain multipliers. The trajectory consists of bang-bang and singular subarcs, as well as boundary subarcs induced by the two state constraints. The occurrence of boundary arcs is known to be impossible for regular Hamiltonians and odd-ordered state constraints if the order exceeds two. Additionally, a boundary point also occurs where the third-order state constraint is active. Such a situation is known to be the only possibility for odd-ordered state constraints to be active if the order exceeds two and if the Hamiltonian is regular. Because of the complexity of the optimal control,

[1]An extended abstract of this paper was presented at the 8th IFAC Workshop on Control Applications of Nonlinear Programming and Optimization, Paris, France, 1989 (see Ref. 1).

[2]This paper is dedicated to Professor Hans J. Stetter on the occasion of his 60th birthday.

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0022-3239/91/0700-0001$06.50/0

this single problem combines many of the features that make this kind of optimal control problems extremely hard to solve. Moreover, the problem contains nonsmooth data arising from the approximations of the aerodynamic forces and the distribution of the wind velocity components. Therefore, the paper can serve as some sort of user's guide to solve inequality constrained real-life optimal control problems by multiple shooting.

Key Words. Optimal control, Chebyshev-type optimal control problems, minimax optimal control problems, optimal trajectories, state constraints, state constraints of third order, bang-bang controls, singular controls, multipoint boundary-value problems, multiple shooting methods, flight mechanics, landing, abort landing, windshear problems.

1. Introduction

One of the most dangerous situations for a passenger aircraft in take-off and landing is caused by the presence of low-altitude windshears. This meteorological phenomenon, which is more common in subtropical regions, is usually associated with high ground temperatures leading to a so-called downburst. This downburst involves a column of descending air which spreads horizontally near the ground. Even for a highly-skilled pilot, an inadvertent encounter with a windshear can be a fatal problem, since the aircraft might encounter a headwind followed by a tailwind, both coupled with a downdraft. The transition from headwind to tailwind yields an acceleration so that the resulting windshear inertia force can be as large as the drag of the aircraft, and sometimes as large as the thrust of the engines. This explains why the presence of low-altitude windshears is a threat to safety in aviation. Some 30 aircraft accidents over the past 20 years have been attributed to windshear, and this attests to the perilousness of this occurrence. Among these accidents, the most disastrous ones happened in 1982 in New Orleans, where 153 people were killed, and in 1985 in Dallas, where 137 people were killed (see the references cited in Ref. 2).

This paper is concerned only with the abort landing problem, which is a safer procedure than the penetration landing if the initial altitude is high enough. The paper is strongly influenced by the results obtained by Miele and his coworkers (see Ref. 2 and the other references of this group cited therein). The purpose of the present paper is to give a highly accurate solution of the underlying Chebyshev-type optimal control problem for which an approximate solution was given in Ref. 2. Figure 1 shows the wind flow field and two trajectories. The first trajectory is flown with a maximum angle of attack strategy and hits the ground. The second trajectory is the

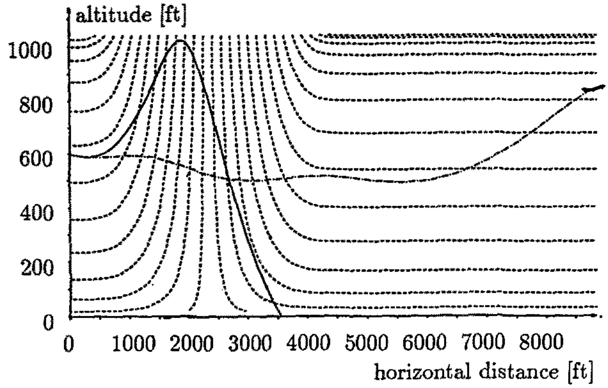

Fig. 1. Maximum angle of attack trajectory (crash trajectory, solid line) and Miele's optimal trajectory (dashed-dotted line).

approximate solution obtained in Refs. 2 and 3 by means of the sequential gradient-restoration algorithm. This solution will be improved in this paper. We show that the optimal trajectory has a very complicated switching structure, which exhibits many of the features that make this kind of optimal control problems extremely hard to solve.

In summary, besides a control constraint and a first-order state constraint, imposed by the model, the problem also includes a third-order state constraint arising from the transformation of the minimax optimal control problem into a standard optimal control problem according to Ref. 4. Additionally, the optimal solution has not only bang-bang subarcs but also singular subarcs—the only control variable appears linearly in the equations of motion. Thus, the Hamiltonian is not regular. Therefore, the theoretical results concerning the existence of boundary arcs and boundary points of state constraints in dependence of their order (see Ref. 5) do not apply here. The order of both state constraints is odd, and the solution has a boundary arc with respect to the first-order state constraint and touch points with respect to the third-order state constraint. Such situations can also appear in the case of a regular Hamiltonian. However, the third-order state constraint leads also to a boundary arc, which is excluded in the case of a regular Hamiltonian and odd-ordered state constraints if the order exceeds two. So, the problem is also of mathematical interest.

Moreover, this problem shows the complexity of the optimal control problems which at present can be solved by the multiple shooting method with a very high precision. For a description of the method, see Ref. 6. Based upon this method, there exists a new FORTRAN code called BNDSCO (see Ref. 7), an offspring of the first author's code OPTSOL (Ref. 8) and its successors (see Refs. 9 and 10). BNDSCO is published in its most recent version in Ref. 11. However, the details of the numerical procedure and the numerical results are preserved for Part 2 of the paper (see Ref. 12). Both parts are based on Ref. 13.

2. Mathematical Model

In the following, we summarize briefly the complete mathematical description of the model according to Ref. 2. If the initial altitude is high enough so that it is safer to abort the landing procedure, the flight maneuver can be modelled as a minimax optimal control problem as follows.

Performance Index. To avoid crashing on the ground, the ground clearance, or in other words the minimal altitude, has to be maximized,

$$\max_{u \in U} \min_{0 \leq t \leq t_f} h(t).$$

Here, U denotes the set of all admissible control variables u, and $[0, t_f]$ is the fixed flight time interval. Instead, we can also minimize the peak value of the altitude drop, that is, the difference between a constant reference altitude h_R and the instantaneous altitude,

$$J[u] := \Theta \max_{0 \leq t \leq t_f} \{h_R - h(t)\} \stackrel{!}{=} \min; \tag{1}$$

here, Θ denotes a scaling factor to be used in Part 2 for a homotopy strategy, and the reference altitude has to be chosen so as to satisfy

$$h_R \geq h(t), \qquad \text{for all } t \in [0, t_f].$$

Because of the relation between the Hölder norm and the Chebyshev norm, i.e.,

$$\lim_{k \to \infty} \left[\int_0^{t_f} (h_R - h(t))^{2k} \, dt \right]^{1/2k} = \max_{0 \leq t \leq t_f} \{h_R - h(t)\},$$

the optimal trajectory of the Chebyshev functional (1) is approximated in Ref. 2 by the optimal trajectory of the Bolza functional

$$J[u] := \Lambda \int_0^{t_f} (h_R - h(t))^q \, dt \stackrel{!}{=} \min, \tag{2}$$

where q is a relatively large even integer ($q = 6$ in the examples of Ref. 2). Again, Λ is a scaling factor for homotopy purposes. However, in this paper and its Part 2 we mainly stay with the minimax functional (1), and we will solve it.

Differential Equations. To set up the equations of motion, we assume that the aircraft is a particle of constant mass, the flight takes place in a vertical plane, and Newton's law is valid in an Earth-fixed system. Moreover, the wind flow field is assumed to be steady. Under these assumptions, the kinematical and dynamical equations are

$$\dot{x} = V \cos \gamma + W_x, \tag{3a}$$

$$\dot{h} = V \sin \gamma + W_h, \tag{3b}$$

$$\dot{V} = (T/m) \cos(\alpha + \delta) - D/m - g \sin \gamma - (\dot{W}_x \cos \gamma + \dot{W}_h \sin \gamma), \tag{3c}$$

$$\dot{\gamma} = (T/mV) \sin(\alpha + \delta) + L/mV - (1/V) g \cos \gamma + (1/V)(\dot{W}_x \sin \gamma - \dot{W}_h \cos \gamma), \tag{3d}$$

$$\dot{\alpha} = u. \tag{3e}$$

The state variables are the horizontal distance x, the altitude h, the relative velocity V, and the relative path inclination γ. In the formulation above, the relative angle of attack α is regarded as a state variable, too. In fact, its time derivative is chosen as control variable. For the relations between relative quantities and absolute quantities (indicated by a subscript a, see Fig. 2), see Ref. 2.

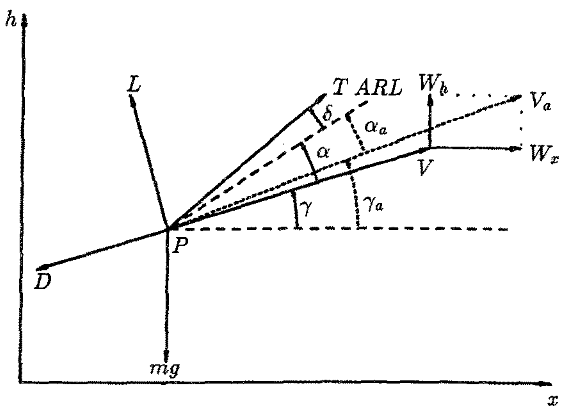

Fig. 2. Coordinate system and explanation of the variables.

These equations are supplemented by the approximations of the aerodynamic forces acting on the aircraft,

$$T = \beta T_*, \tag{4a}$$

$$T_* = A_0 + A_1 V + A_2 V^2, \tag{4b}$$

$$D = (1/2) C_D \rho S V^2, \tag{4c}$$

$$C_D(\alpha) = B_0 + B_1 \alpha + B_2 \alpha^2, \tag{4d}$$

$$L = (1/2) C_L \rho S V^2, \tag{4e}$$

$$C_L(\alpha) = \begin{cases} C_0 + C_1 \alpha, & \alpha \leq \alpha_*, \\ C_0 + C_1 \alpha + C_2 (\alpha - \alpha_*)^2, & \alpha_* \leq \alpha \leq \alpha_{\max}. \end{cases} \tag{4f}$$

Here T, D, L denote thrust, drag, and lift, respectively. The power setting β, normally also a control variable, is specified in advance as in Ref. 2,

$$\beta(t) = \begin{cases} \beta_0 + \dot{\beta}_0 t, & 0 \leq t \leq t_0, \\ 1, & t_0 \leq t \leq t_f. \end{cases} \tag{5}$$

This is justified by the additional hypothesis that, upon sensing the aircraft to be in a windshear, the pilot increases the power setting at a constant time rate until maximum power setting is reached. The maximum power is then held constant afterward.

Next, the windshear model, valid for $h \leq 1000$ ft, is given by the following wind velocity components:

$$W_x = kA(x), \tag{6a}$$

$$W_h = k(h/h_*) B(x), \tag{6b}$$

with

$$A(x) = \begin{cases} -50 + ax^3 + bx^4, & 0 \leq x \leq 500, \\ (1/40)(x - 2300), & 500 \leq x \leq 4100, \\ 50 - a(4600 - x)^3 - b(4600 - x)^4, & 4100 \leq x \leq 4600, \\ 50, & 4600 \leq x, \end{cases} \tag{6c}$$

$$B(x) = \begin{cases} dx^3 + ex^4, & 0 \leq x \leq 500, \\ -51 \exp[-c(x - 2300)^4], & 500 \leq x \leq 4100, \\ d(4600 - x)^3 - e(4600 - x)^4, & 4100 \leq x \leq 4600, \\ 0, & 4600 \leq x. \end{cases} \tag{6d}$$

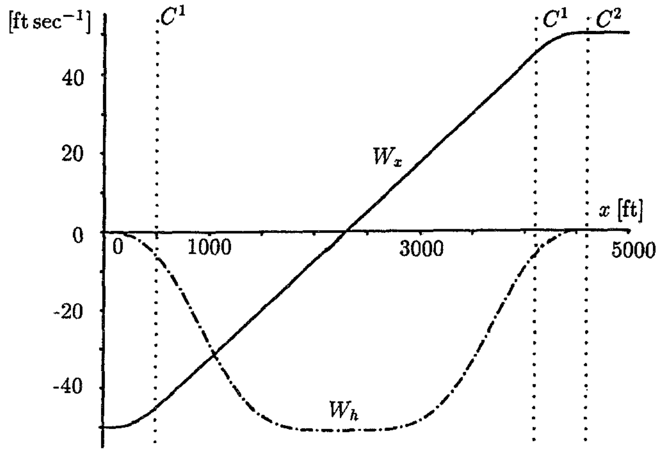

Fig. 3. Wind functions and their differentiability properties.

Here, the parameter k characterizes the intensity of the windshear/downdraft combination. Its value (see Table 1) corresponds to the horizontal wind velocity difference (maximum tailwind minus maximum headwind) of $\Delta W_x = 100$ ft sec^{-1}. The differentiability properties of the wind velocity components at the matching points are indicated in Fig. 3. The continuity of the derivatives of the wind velocity components guarantees that the right-hand side of Eqs. (3) is continuous at these points. Note that our windshear model slightly differs from the one used in Ref. 2 in order to reduce the number of those matching points. Those points must be observed not only

Table 1. Model data for a Boeing B-727 aircraft.

Eqs. (1), (7), (8)	Eqs. (3)	Eqs. (4), (5)
$t_f = 40$ sec	$\rho = 0.2203 \times 10^{-2}$ lb sec^2 ft^{-4}	$A_0 = 0.4456 \times 10^5$ lb
$h_R = 1000$ ft	$S = 0.1560 \times 10^4$ ft^2	$A_1 = -0.2398 \times 10^2$ lb sec ft^{-1}
$u_{max} = 3$ deg sec^{-1}	$g = 3.2172 \times 10^1$ ft sec^{-2}	$A_2 = 0.1442 \times 10^{-1}$ lb sec^2 ft^{-2}
$\alpha_{max} = 17.2$ deg	$mg = 150{,}000$ lb	$\beta_0 = 0.3825$
	$\delta = 2$ deg	$\dot{\beta}_0 = 0.2$ sec^{-1}
		$t_0 = (1 - \beta_0)/\dot{\beta}_0$
Eqs. (4)	**Eqs. (6)**	**Eqs. (9), (10)**
$B_0 = 0.1552$	$k = 1$	$x_0 = 0$ ft
$B_1 = 0.12369$ rad^{-1}	$h_* = 1000$ ft	$\gamma_0 = -2.249$ deg
$B_2 = 2.4203$ rad^{-2}	$a = 6 \times 10^{-8}$ sec^{-1} ft^{-2}	$h_0 = 600$ ft
$C_0 = 0.7125$	$b = -4 \times 10^{-11}$ sec^{-1} ft^{-3}	$\alpha_0 = 7.353$ deg
$C_1 = 6.0877$ rad^{-1}	$c = -\ln(25/30.6) \times 10^{-12}$ ft^{-4}	$V_0 = 239.7$ ft sec^{-1}
$C_2 = -9.0277$ rad^{-2}	$d = -8.02881 \times 10^{-8}$ sec^{-1} ft^{-2}	$\gamma_f = 7.431$ deg
$\alpha_* = 12$ deg	$e = 6.28083 \times 10^{-11}$ sec^{-1} ft^{-3}	

during the integration, but must also function as interior point conditions in optimal control theory. Therefore, a reduction of the matching points, from 8 points in Miele's model to 3 points in our model, makes the derivation of the necessary conditions of optimal control theory much easier. Since Miele's model uses cubic splines, the wind components of his model are C^2-functions everywhere. Moreover, also the gradients of the wind components and their Hessians are different. Nevertheless, the modification of the wind model is negligible as can be seen when comparing Miele's approximate solution, given in Ref. 2, with the corresponding solution verified in Fig. 1 for our wind model using the multiple shooting method.

Inequality Constraints. The angle of attack and its time rate is subject to the inequality constraints

$$|u| \leq u_{\max}, \tag{7}$$

$$\alpha \leq \alpha_{\max}. \tag{8}$$

Boundary Conditions. We have the initial conditions

$$x(0) = x_0, \qquad h(0) = h_0, \tag{9a}$$

$$V(0) = V_0, \qquad \gamma(0) = \gamma_0, \tag{9b}$$

$$\alpha(0) = \alpha_0, \tag{9c}$$

and the terminal condition

$$\gamma(t_f) = \gamma_f. \tag{10}$$

These conditions are a consequence of the assumption of quasi-steady flight prior to the windshear onset, whereas the terminal condition corresponds to the steepest climb condition in quasi-steady flight. In Part 2, we will see that the model assumption of a fixed terminal time should be better replaced by an open-end problem.

Data. The model description is completed by the values of the constants occurring in the previous equations; see Table 1 (some of the data are not available from Refs. 2 and 3). The data refer to a Boeing B-727 aircraft powered by three JT8D-17 turbofan engines. For more details on the model, especially those which are of technical interest, the reader is referred to Refs. 2 and 3.

Transformation of the Minimax Problem into Standard Form. To treat the minimax optimal control problem, it is useful to apply the transformation technique due to Ref. 4. By this transformation, an optimal control problem in standard form is obtained where an additional state constraint has to be taken into account. To this new optimal control problem, the theory of optimal control can be applied conveniently. By the transformation

$$\zeta(t) := \max_{0 \le \hat{t} \le t_f} \{h_R - h(\hat{t})\}, \tag{11}$$

we have

$$I[u] = \Theta\, \zeta(t_f) \overset{!}{=} \min, \tag{12}$$

subject to the additional constraints

$$\dot{\zeta} = 0, \tag{13}$$

$$h_R - h(t) - \zeta(t) \le 0. \tag{14}$$

We end up with an optimal control problem of Mayer's type (12), subject to the constraint (3), (13), (9), (10), (7), (8), (14). To treat the Bolza problem (2) as well as the Mayer problem (12) simultaneously, it is convenient to consider the combined functional

$$\Im[u] := \Lambda \int_0^{t_f} (h_R - h(t))^6\, dt + \Theta\, \zeta(t_f). \tag{15}$$

3. Necessary Conditions

In this section, we apply the well-known theory of optimal control (see, e.g., Ref. 14) to derive necessary conditions for an optimal trajectory of the abort landing in a windshear. It results in a multipoint boundary-value problem, which is well suited for numerical treatment.

State Unconstrained Subarcs. First, we investigate the state unconstrained subarcs of the combined optimal control problem, i.e., we consider the scaling factor combination $\Lambda > 0$ and $\Theta = 0$. For the time being, the state constraints (8) and (14) are left out of consideration.

Defining the Hamiltonian on state unconstrained subarcs by

$$H(y, \lambda, u) := \Lambda(h_R - h)^6 + \lambda^{\mathsf{T}} f$$
$$:= \Lambda(h_R - h)^6 + \lambda_x \dot{x} + \lambda_h \dot{h} + \lambda_V \dot{V} + \lambda_\gamma \dot{\gamma} + \lambda_\alpha u, \tag{16a}$$

$$y^{\mathsf{T}} := (x, h, V, \gamma, \alpha), \tag{16b}$$

$$\lambda^{\mathsf{T}} := (\lambda_x, \lambda_h, \lambda_V, \lambda_\gamma, \lambda_\alpha), \tag{16c}$$

$$f^{\mathsf{T}} := (\dot{x}, \dot{h}, \dot{V}, \dot{\gamma}, \dot{\alpha}), \tag{16d}$$

we can establish the adjoint differential equations if both of the state constraints are assumed to be inactive,

$$\begin{aligned}\dot{\lambda}_x = &-\lambda_x (\partial/\partial x) W_x - \lambda_h (\partial/\partial x) W_h \\ &+ \lambda_V [(\partial/\partial x) \dot{W}_x \cos\gamma + (\partial/\partial x) \dot{W}_h \sin\gamma] \\ &- \lambda_\gamma (1/V)[(\partial/\partial x) \dot{W}_x \sin\gamma - (\partial/\partial x) \dot{W}_h \cos\gamma],\end{aligned} \tag{17a}$$

$$\begin{aligned}\dot{\lambda}_h = &\, 6\Lambda(h_R - h)^5 - \lambda_h (\partial/\partial h) W_h \\ &+ \lambda_V [(\partial/\partial h) \dot{W}_x \cos\gamma + (\partial/\partial h) \dot{W}_h \sin\gamma] \\ &- \lambda_\gamma (1/V)[(\partial/\partial h) \dot{W}_x \sin\gamma - (\partial/\partial h) \dot{W}_h \cos\gamma],\end{aligned} \tag{17b}$$

$$\begin{aligned}\dot{\lambda}_V = &-\lambda_x \cos\gamma - \lambda_h \sin\gamma \\ &- \lambda_V [(1/m) T_V \cos(\alpha + \delta) - (1/m) D_V \\ &\qquad - ((\partial/\partial V) \dot{W}_x \cos\gamma + (\partial/\partial V) \dot{W}_h \sin\gamma)] \\ &- \lambda_\gamma (1/V)[(1/m)(T_V - T/V) \sin(\alpha + \delta) \\ &\qquad + (1/m)(L_V - L/V) + (g/V) \cos\gamma \\ &\qquad - (1/V)(\dot{W}_x \sin\gamma - \dot{W}_h \cos\gamma) \\ &\qquad + ((\partial/\partial V) \dot{W}_x \sin\gamma - (\partial/\partial V) \dot{W}_h \cos\gamma)],\end{aligned} \tag{17c}$$

$$\begin{aligned}\dot{\lambda}_\gamma = &\, \lambda_x V \sin\gamma - \lambda_h V \cos\gamma \\ &+ \lambda_V [g \cos\gamma + (\partial/\partial\gamma) \dot{W}_x \cos\gamma - \dot{W}_x \sin\gamma \\ &\qquad + (\partial/\partial\gamma) \dot{W}_h \sin\gamma + \dot{W}_h \cos\gamma] \\ &- \lambda_\gamma (1/V)[g \sin\gamma + (\partial/\partial\gamma) \dot{W}_x \sin\gamma + \dot{W}_x \cos\gamma \\ &\qquad - (\partial/\partial\gamma) \dot{W}_h \cos\gamma + \dot{W}_h \sin\gamma],\end{aligned} \tag{17d}$$

$$\begin{aligned}\dot{\lambda}_\alpha = &\, \lambda_V (1/m)[T \sin(\alpha + \delta) + D_\alpha] \\ &- \lambda_\gamma (1/mV)[T \cos(\alpha + \delta) + L_\alpha].\end{aligned} \tag{17e}$$

From the minimum principle, we obtain a bang-bang expression for the optimal control,

$$u=\begin{cases} u_{\max}, & \lambda_\alpha<0, \\ -u_{\max}, & \lambda_\alpha>0. \end{cases} \tag{18}$$

The adjoint variable λ_α plays the role of the so-called natural switching function. Its isolated zeros,

$$\lambda_\alpha(t_{\text{bang}})=0, \tag{19}$$

mark the switches of the bang-bang structure. If nonisolated zeros of λ_α occur, the following relation holds:

$$\lambda_\alpha(t)\equiv 0, \qquad t_{\text{entry}}\leq t\leq t_{\text{exit}}, \qquad t_{\text{entry}}<t_{\text{exit}}. \tag{20}$$

Repeating twice the differentiation of this identity with respect to time and substituting Eq. (3) and (17), we obtain an expression for the optimal control on singular subarcs,

$$u_{\text{sing}}=-A_{\text{sing}}/B_{\text{sing}}, \tag{21}$$

where

$$\begin{aligned} A_{\text{sing}}(t)=\dot\lambda_V[&T\sin(\alpha+\delta)+D_\alpha] \\ &-(1/V)[\dot\lambda_\gamma-\lambda_\gamma(\dot V/V)][T\cos(\alpha+\delta)+L_\alpha] \\ &+\lambda_V[\dot T\sin(\alpha+\delta)+D_{\alpha V}\dot V] \\ &-\lambda_\gamma(1/V)[\dot T\cos(\alpha+\delta)+L_{\alpha V}\dot V], \end{aligned}$$

$$B_{\text{sing}}(t)=\lambda_V[T\cos(\alpha+\delta)+D_{\alpha\alpha}]-\lambda_\gamma(1/V)[L_{\alpha\alpha}-T\sin(\alpha+\delta)].$$

Hence, the order of the singular control is $p=1$. Therefore, u is either discontinuous or continuously differentiable at the junction points t_{entry} and t_{exit}, respectively; see Ref. 15. The generalized strong Legendre–Clebsch condition (see Ref. 16) is to be read in this case as follows:

$$B_{\text{sing}}<0, \qquad t_{\text{entry}}\leq t\leq t_{\text{exit}}. \tag{22}$$

The two junction points are determined by the so-called entry or tangency conditions

$$\lambda_\alpha(t_{\text{entry}})=0, \qquad \dot\lambda_\alpha(t_{\text{entry}})=0. \tag{23}$$

More switching points t_i are induced by the nonsmooth approximations of the aerodynamic forces and the wind functions. They are determined by the specification of so-called interior-point conditions of the form

$$N(y(t_i), t_i) = 0. \tag{24}$$

These conditions imply the following additional necessary conditions:

$$\lambda^{\mathrm{T}}(t_i^+) = \lambda^{\mathrm{T}}(t_i^-) - \kappa^{\mathrm{T}} N_y, \qquad \kappa \in \mathbb{R}^n, \tag{25}$$

$$H(t_i^+) = H(t_i^-) + \kappa_0 N_t, \qquad \kappa_0 \in \mathbb{R}. \tag{26}$$

We now discuss the results obtained from these so-called jump conditions.

(i) Due to the approximation of the power setting in Eq. (5), we have the interior-point condition

$$N_1(y(t_1), t_1) = t_1 - t_0 = 0. \tag{27}$$

Thus, λ is continuous, but H may be discontinuous at t_1. On the other hand, H can be discontinuous only if $\dot{\alpha} = u$ is discontinuous. Since either u is continuous in the interior of a bang-bang subarc or λ_α identically vanishes on a singular subarc, the Hamiltonian H is continuous at t_1. However, because of the discontinuity of $\dot{T}$, the control u is generally discontinuous at t_1, if t_1 is within a singular subarc.

(ii) The switching point, induced by the approximation of the lift coefficient (4) via

$$N_2(y(t_2), t_2) = \alpha(t_2) - \alpha_* = 0, \tag{28}$$

implies a possible discontinuity of only λ_α. Because of the continuity of the Hamiltonian, we have

$$\lambda_\alpha(t_2^-) u(t_2^-) = \lambda_\alpha(t_2^+) u(t_2^+).$$

If t_2 lies in the interior of a bang-bang subarc as well as if t_2 lies within a singular subarc, the above equation implies that λ_α is continuous at t_2. In the first case, this holds because $u(t_2^-) = u(t_2^+) = \pm u_{\max} \neq 0$, and in the second case because $\lambda_\alpha(t_2^-) = \lambda_\alpha(t_2^+) = 0$. However, again the singular control is generally discontinuous at t_2 because of the discontinuity of $L_{\alpha\alpha}$.

(iii) The windshear approximation (6) leads to three interior-point conditions,

$$N_3(y(t_3), t_3) = x(t_3) - 500 = 0, \tag{29a}$$

$$N_4(y(t_4), t_4) = x(t_4) - 4100 = 0, \tag{29b}$$

$$N_5(y(t_5), t_5) = x(t_5) - 4600 = 0. \tag{29c}$$

Because of the continuity of f and H, there holds

$$\lambda_x(t_i^-)\dot{x}(t_i) = \lambda_x(t_i^+)\dot{x}(t_i), \qquad i = 3, 4, 5,$$

which implies, since $\dot{x} \neq 0$, that λ_x must be continuous at t_i. However, $\dot{\lambda}_x$ is discontinuous at t_3 and t_4; here, the wind functions are of class C^1 only (see Fig. 3).

In summary, all the adjoint variables as well as the Hamiltonian are continuous at the switching points induced by the nonsmoothness of the data. Nevertheless, the singular control is generally discontinuous at t_1 and t_2, the two switching points induced by the power setting and the lift approximation, respectively.

Finally, the missing boundary conditions are given by the transversality or natural boundary conditions,

$$\lambda_x(t_f) = \lambda_h(t_f) = \lambda_V(t_f) = \lambda_\alpha(t_f) = 0. \tag{30}$$

Up to now, the necessary conditions of the unconstrained problem for $\Lambda > 0$ and $\Theta = 0$ form a boundary-value problem with 10 differential equations (3), (17) and the same number of two-point boundary conditions (9), (10), (30). If we assume that the switching structure is known, each bang-bang switching point is determined by an equation of type (19), whereas each interior singular subinterval is determined by two entry conditions (23). If there are singular subarcs adjacent to the endpoints of the interval $[0, t_f]$, one of the conditions (9), (10), (23), or (30) must be linearly dependent. It is important that the switching structure can be guessed by means of the switching function λ_α with the help of an appropriate homotopy procedure (see Part 2). Moreover, the discontinuity points t_i, $i = 1, \ldots, 5$, due to the nonsmooth data are determined by the interior-point conditions (27)–(29). Not only does this procedure enable the computation of the piecewise-defined relations for the aerodynamic forces and the wind functions, but it also marks the points of discontinuity in the higher derivatives of both the state and adjoint variables. These discontinuities must be carefully obeyed to preserve the order of convergence of the numerical integration routines. Because of these interior-point conditions, we speak of multipoint boundary-value problems.

State Constrained Subarcs. Now, we are concerned with the state variable inequality constraints (8) and (14), especially with the scaling factor combination $\Lambda = 0$ and $\Theta > 0$. In spite of the additional auxiliary state variable ζ, we may still use the notations of Eq. (16) with $\Lambda = 0$. Later, we will see that ζ and its Lagrange multiplier λ_ζ can be eliminated. For λ_ζ, we have

$$\dot{\lambda}_\zeta = 0 \tag{31a}$$

on unconstrained subarcs and

$$\lambda_\zeta(0) = 0, \qquad \lambda_\zeta(t_f) = \Theta. \tag{31b}$$

Thus, λ_ζ is piecewise constant at least on unconstrained subarcs.

Before deriving the necessary conditions in the state-constrained case for this special problem, we cite here a set of sharp necessary conditions from Ref. 17, since this paper is not easily accessible. These necessary conditions may be very helpful for the rejection of nonoptimal solutions. The following theorem is due to Maurer.

Theorem 3.1. Let $y(t)$ and $u(t)$ be an optimal solution of a state-constrained optimal control problem which, without loss of generality, is assumed to be of the Bolza type and autonomous. Let the state constraint be

$$S(y(t)) \leq 0 \tag{32}$$

and of order q. Let f and S be C^{2q}-functions. Let the control defined by $S^{(q)} \equiv 0$ on $]t_{\text{entry}}, t_{\text{exit}}[$ be denoted by u^{bound}; and let u^{bound} be a C^q-function. Here, $S^{(q)}$ is the qth total time derivative of S. Finally, let

$$S_u^{(q)}(y, u) \neq 0, \qquad \text{on }]t_{\text{entry}}, t_{\text{exit}}[.$$

Then, for $i \in \{1, \ldots, q\}$, there exist functions $\lambda^i: [0, t_f] \to \mathbb{R}^n$ and $\mu^i: [0, t_f] \to \mathbb{R}$ such that the adjoint variables satisfy the system of differential equations

$$\dot{\lambda}^i = -H_y^{i\mathrm{T}}, \tag{33a}$$

$$H^i := H^{i,\text{free}} + \mu^i S^{(i)}, \tag{33b}$$

where $H^{i,\text{free}}$ denotes the Hamiltonian of the unconstrained problem built with the multipliers λ^i. Transversality conditions analogous to those for the unconstrained case hold for each function λ^i. Furthermore, there holds that, at an entry point of a boundary arc,

$$\lambda^i(t_{\text{entry}}^+) = \lambda^i(t_{\text{entry}}^-) - \sum_{j=0}^{i-1} \sigma_j S_y^{(j)}(y(t_{\text{entry}}))^{\mathrm{T}}, \qquad \sigma_j \geq 0; \tag{34}$$

at an exit point of a boundary arc,

$$\lambda^i(t_{\text{exit}}^+) = \lambda^i(t_{\text{exit}}^-); \tag{35}$$

at a contact point (not necessarily a touch point) of the state constraint,

$$\lambda^i(t_{\text{contact}}^+) = \lambda^i(t_{\text{contact}}^-) - \sigma S_y(y(t_{\text{contact}}))^{\mathrm{T}}, \qquad \sigma \geq 0. \tag{36}$$

Moreover, the functions μ^i satisfy

$$\mu^i S(y(t)) \equiv 0, \qquad \text{on } [0, t_f].$$

They are C^i-functions on $]t_{\text{entry}}, t_{\text{exit}}[$ and are given by

$$\mu^i = (-1)^{q+1-i} \lambda^{i\mathrm{T}} \chi_{q-i} / S_u^{(q)}, \qquad 1 \leq i \leq q, \tag{37}$$

where

$$\chi_0 := f_u, \qquad \chi_{j+1} := \dot{\chi}_j - f_y \chi_j, \qquad j = 0, 1, \ldots, q-1.$$

For these adjoint functions, there hold the sign conditions

$$(-1)^k (d^k/dt^k) \mu^i(t) = \mu^{i-k}(t) \geq 0, \qquad t \in]t_{\text{entry}}, t_{\text{exit}}[, \qquad k = 0, \ldots, i, \tag{38}$$

and

$$(d^k/dt^k) u^i(t_{\text{exit}}^-) = 0, \qquad k = 0, \ldots, i-2, \quad \text{if } i \geq 2.$$

Finally, the optimal control satisfies the minimum principle,

$$u^{\text{optimal}} = \arg\min_{u \in U} H^i(y, u, \lambda^i, \mu^i), \tag{39}$$

and

$$H^i(y, u^{\text{optimal}}, \lambda^i, \mu^i) \equiv \text{const.}, \qquad t \in [0, t_f]. \tag{40}$$

Remark 3.1. For $i = q$, Theorem 3.1 contains the well-known necessary conditions of Bryson, Denham, and Dreyfus (see Ref. 18), namely,

$$\mu^q(t) \geq 0 \quad \text{and} \quad \mu^q(t_{\text{exit}}) = 0.$$

However, Eqs. (38) describe a strengthening by which nonoptimal trajectories can be rejected. Morever, the theorem distinguishes between entry and exit points. Therefore, by reversing the time scale, i.e., using $\bar{t} := t_f - t$, we obtain an additional set of necessary conditions which, in general, is independent of the first one and which may be useful to eliminate nonoptimal trajectories.

First-Order State Constraint. The inequality constraint (8) is of the first order, since

$$\dot{S}(t) = \dot{\alpha} = u, \qquad \text{with } S(t) := \alpha - \alpha_{\max} \leq 0. \tag{41}$$

Lemma 3.1. As for a regular Hamiltonian, the state constraint $S \leq 0$ can be active only on nonvanishing intervals.

Proof. We proceed by contradiction. Suppose that an active contact point $t_{\text{contact}} \in]0, t_f[$ exists; i.e., the solutions of the unconstrained and the constrained problem are different. Then, there holds that $\alpha(t_{\text{contact}}) = \alpha_{\max}$. Two cases must be considered.

Case 1. t_{contact} lies within a singular subarc. According to Theorem 3.1, especially Eq. (36), this implies $\sigma = 0$, since there holds that $\lambda_\alpha \equiv 0$ on the singular subarc.

Case 2. t_{contact} lies on a bang-bang subarc. Then, there hold that

$$\dot{\alpha}(t^-_{\text{contact}}) = +u_{\max}, \qquad \dot{\alpha}(t^+_{\text{contact}}) = -u_{\max},$$

which are, according to Eq. (18), equivalent to

$$\lambda_\alpha(t^-_{\text{contact}}) \leq 0, \qquad \lambda_\alpha(t^+_{\text{contact}}) \geq 0.$$

Therefore, by the jump condition in Eq. (36), we have $\sigma \leq 0$; but the sign condition in Eq. (36) requires $\sigma \geq 0$. Hence, $\sigma = 0$.

Since $\sigma = 0$ holds in both cases, the solutions of the unconstrained and the constrained problem do not differ, a contradiction to t_{contact} being an active contact point. □

Therefore, we can assume that the state constraint is active on a nonvanishing interval $[t_{\text{entry}}, t_{\text{exit}}]$, i.e.,

$$\alpha \equiv \alpha_{\max}, \quad \text{on } [t_{\text{entry}}, t_{\text{exit}}], \qquad t_{\text{entry}} < t_{\text{exit}}. \tag{42}$$

We apply Theorem 3.1 for $i = q = 1$. For simplicity, we omit the superscripts i in the following. The control variable is determined along the constrained subarcs by

$$u^{\text{bound}} \equiv 0, \qquad \text{on } [t_{\text{entry}}, t_{\text{exit}}]. \tag{43}$$

Define the Hamiltonian by

$$H := H^{\text{free}} + \mu \dot{S}(t) = \lambda_x \dot{x} + \cdots + \lambda_\gamma \dot{\gamma} + (\lambda_\alpha + \mu) u, \tag{44}$$

where H^{free} stands for the Hamiltonian in Eq. (16). Then from Eq. (37), we have

$$\mu = -\lambda_\alpha, \qquad \text{on } [t_{\text{entry}}, t_{\text{exit}}]. \tag{45}$$

Since H is continuous at t_{entry} as well as at t_{exit}, and since there holds that $H \equiv \text{const}$, for $t \geq t_0$ [see Eq. (5)], we have

$$\lambda_\alpha(t_{\text{entry}}^-)u(t_{\text{entry}}^-) = 0, \qquad \lambda_\alpha(t_{\text{exit}}^+)u(t_{\text{exit}}^+) = 0.$$

Note that, as the only multiplier, λ_α may be discontinuous at t_{entry} or at t_{exit}, according to Eq. (34). However, if t_{entry} lies within a bang-bang subarc as well as if t_{entry} lies within a singular subarc, the above equations imply that (the same holds for t_{exit}, too)

$$\lambda_\alpha(t_{\text{entry}}^-) = 0, \qquad \lambda_\alpha(t_{\text{exit}}^+) = 0. \tag{46a}$$

Together with the entry condition

$$\alpha(t_{\text{entry}}) = \alpha_{\max}, \tag{46b}$$

we have three interior-point conditions to fix the unknowns t_{entry}, t_{exit}, and the jump σ_0 of the jump condition according to (34),

$$\lambda_\alpha(t_{\text{entry}}^+) = \lambda_\alpha(t_{\text{entry}}^-) - \sigma_0. \tag{47}$$

The jump σ_0 enters the differential equations only via this jump condition. In this case, λ_α is continuous at t_{exit}. The adjoint differential equations need not be modified, because $\dot{S}_y = 0$.

Moreover, the sign conditions (34) and (38) give additional information. Here, by these equations and (45), we have

$$\sigma_0 \geq 0, \quad \lambda_\alpha \leq 0, \quad \dot{\lambda}_\alpha \geq 0, \qquad \text{on }]t_{\text{entry}}, t_{\text{exit}}[. \tag{48}$$

Incidentally, if these sign conditions are satisfied, then the generally independent sign conditions for entry and exit point interchanged are also satisfied.

In summary, on a constrained subarc of the first-order state constraint (8), the optimal control is determined by Eq. (43). Besides, there are three interior-point conditions (46) to determine the unknowns t_{entry}, t_{exit}, and σ_0. Additionally, we have a jump condition (47), too. Therefore, the necessary conditions define now a multipoint boundary-value problem with jump conditions. Finally, there are some sign conditions (48), which may lead to rejection of nonoptimal solutions of that multipoint boundary-value problem by means of an a posteriori check.

A combination of all the necessary conditions obtained so far define the multipoint boundary-value problem to compute a candidate for the optimal control problem of Ref. 2, i.e., for $\Lambda = 1$ and $\Theta = 0$.

Third-Order State Constraint. By virtue of the transformation of the minimax problem into standard form, we also have to take into consideration the state constraint (14), which is of order 3; thus, $\bar{q}=3$ and

$$\bar{S}=h_R-h(t)-\zeta(t), \tag{49}$$

$$\dot{\bar{S}}=-V\sin\gamma-W_h(x,h), \tag{50}$$

$$\ddot{\bar{S}}=(D/m)\sin\gamma-(L/m)\cos\gamma+g-(T/m)\sin(\alpha+\delta+\gamma), \tag{51}$$

$$\dddot{\bar{S}}=a+bu, \tag{52}$$

with

$$\begin{aligned} mA(t) &:= D_V\dot{V}\sin\gamma+D\dot{\gamma}\cos\gamma-L_V\dot{V}\cos\gamma+L\dot{\gamma}\sin\gamma \\ &\qquad -\dot{T}\sin(\alpha+\delta+\gamma)-T\dot{\gamma}\cos(\alpha+\delta+\gamma), \\ mB(t) &:= D_\alpha\sin\gamma-L_\alpha\cos\gamma-T\cos(\alpha+\delta+\gamma). \end{aligned}$$

Since the Hamiltonian is not regular, the well-known existence theorem of Ref. 5 (which says that, for odd-ordered state constraints, boundary arcs are excluded if the order exceeds two) cannot be applied here. Boundary points (here, touch points) as well as boundary arcs may occur; therefore, both cases are investigated here.

Touch Points. If the optimal solution touches the constraint at a point t_{touch}, we have two interior-point conditions for each touch point, namely,

$$h(t_{\text{touch}})=h_R-\zeta(t_{\text{touch}}), \tag{53}$$

$$\dot{h}(t_{\text{touch}})=0; \tag{54}$$

according to Theorem 3.1, Eq. (36), we also have the jump conditions

$$\lambda_h(t_{\text{touch}}^+)=\lambda_h(t_{\text{touch}}^-)+\bar{\sigma}, \qquad \bar{\sigma}\geq 0, \tag{55}$$

$$\lambda_\zeta(t_{\text{touch}}^+)=\lambda_\zeta(t_{\text{touch}}^-)+\bar{\sigma}. \tag{56}$$

By Eqs. (53) and (54), the two unknowns t_{touch} and $\bar{\sigma}$ can be determined. Again, $\bar{\sigma}$ is coupled with the differential equations, here only by Eq. (55), since λ_ζ does not enter the system of differential equations.

In general, the above equations make it possible to eliminate some of the variables. For example, if a unique minimum of the altitude exists, the jump $\bar{\sigma}$ and λ_ζ can be obtained directly from Eqs. (31) and (56). Because $\bar{\sigma}=\Theta>0$, this yields a fixed jump condition for (55). Moreover, ζ is decoupled. Therefore, Eqs. (13) and (53) can be omitted, and ζ can be computed from the solution afterward. This procedure goes with the general framework of

Refs. 19 and 20 on how to formulate the multipoint boundary-value problem for minimax optimal control problems.

Boundary Arcs. If the optimal solution stays on the boundary of the constraint for a nonvanishing interval $[\bar{t}_{\text{entry}}, \bar{t}_{\text{exit}}]$, we first have to modify the Hamiltonian according to Theorem 3.1, case $i = \bar{q} = 3$,

$$H := H^{\text{free}} + \bar{\mu}\dddot{S} = H^{\text{free}} + \bar{\mu}(A + Bu). \tag{57}$$

Now, on constrained subarcs, we have to alter the right-hand sides of the adjoint differential equations. Denoting the right-hand side of Eqs. (17) by $\dot{\lambda}^{\text{free}}$, we obtain the rather complicated equations

$$\dot{\lambda}_\eta = \dot{\lambda}_\eta^{\text{free}} - \bar{\mu}(\partial/\partial\eta)\{A + Bu\}, \qquad \text{with } \eta = x, h, V, \gamma, \alpha, \tag{58}$$

where the multiplier $\bar{\mu}$ is defined according to Theorem 3.1, Eq. (37), by

$$\bar{\mu} = \begin{cases} -\lambda_\alpha/B, & \text{on } [\bar{t}_{\text{entry}}, \bar{t}_{\text{exit}}], \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

In order to give an impression of the complex structure of the right-hand sides defined on a constrained subarc, we give the differential equation for λ_γ,

$$\begin{aligned}
\dot{\lambda}\gamma = \dot{\lambda}_\gamma^{\text{free}} &+ \{\lambda_\alpha/[D_\alpha \sin\gamma - L_\alpha \cos\gamma - T\cos(\alpha+\delta+\gamma)]\} \\
&\times \{[D_V \cos\gamma + L_V \sin\gamma]\dot{\gamma} + [D_V \sin\gamma - L_V \cos\gamma] \\
&\times [-g\cos\gamma + \dot{W}_x \sin\gamma - \dot{W}_h \cos\gamma \\
&\quad - (\partial/\partial\gamma)\dot{W}_x \cos\gamma - (\partial/\partial\gamma)\dot{W}_h \sin\gamma] \\
&+ [-D\sin\gamma + T\sin(\alpha+\delta+\gamma)]\dot{\gamma} \\
&+ (1/V)[D\cos\gamma + L\sin\gamma - T\cos(\alpha+\delta+\gamma)] \\
&\times [g\sin\gamma + \dot{W}_x \cos\gamma + \dot{W}_h \sin\gamma \\
&\quad + (\partial/\partial\gamma)\dot{W}_x \sin\gamma - (\partial/\partial\gamma)\dot{W}_h \cos\gamma] \\
&- \dot{T}\cos(\alpha+\delta+\gamma) \\
&+ [D_\alpha \cos\gamma + L_\alpha \sin\gamma + T\sin(\alpha+\delta+\gamma)]u^{\text{bound}}\},
\end{aligned}$$

where $u^{\text{bound}} = -A/B$ has to be substituted as defined by $\dddot{S} \equiv 0$ [see Eq. (52)] according to Theorem 3.1. Here, it is assumed that

$$-u_{\max} < u^{\text{bound}} < u_{\max}$$

holds in the interior of any boundary arc. Therefore, the boundary control is singular in the sense of the minimum principle, i.e., the switching function vanishes on boundary subarcs (see Ref. 21). Furthermore, the differential equation for λ_ζ remains unchanged. Thus, λ_ζ is piecewise constant over the entire interval $[0, t_f]$.

The jump conditions are

$$\lambda(\bar{t}^+_{\text{entry}}) = \lambda(\bar{t}^-_{\text{entry}}) - N_y^{\mathrm{T}} \begin{bmatrix} \bar{\sigma}_0 \\ \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix}, \tag{60}$$

$$\lambda_\zeta(\bar{t}^+_{\text{entry}}) = \lambda_\zeta(\bar{t}^-_{\text{entry}}) + \bar{\sigma}_0, \tag{61}$$

with

$$N_y := (\partial/\partial y)(\bar{S}, \dot{\bar{S}}, \ddot{\bar{S}})^{\mathrm{T}}$$

$$= \begin{bmatrix} 0 & -\partial W_h/\partial x & 0 \\ -1 & -\partial W_h/\partial h & 0 \\ 0 & -\sin\gamma & (1/m)[(\partial D/\partial V)\sin\gamma \\ & & -(\partial L/\partial V)\cos\gamma - (\partial T/\partial V)\sin(\alpha+\delta+\gamma)] \\ 0 & -V\cos\gamma & (1/m)[D\cos\gamma + L\sin\gamma - T\cos(\alpha+\delta+\gamma)] \\ 0 & 0 & (1/m)[(\partial D/\partial\alpha)\sin\gamma \\ & & -(\partial L/\partial\alpha)\cos\gamma - T\cos(\alpha+\delta+\gamma)] \end{bmatrix}^{\mathrm{T}}.$$

The five unknowns, the two switching points $\bar{t}_{\text{entry}}$, $\bar{t}_{\text{exit}}$ and the three jump parameters $\bar{\sigma}_0$, $\bar{\sigma}_1$, $\bar{\sigma}_2$ are determined by three tangency conditions and the continuity of the Hamiltonian at $\bar{t}_{\text{entry}}$ and $\bar{t}_{\text{exit}}$. Hence,

$$h(\bar{t}_{\text{entry}}) = h_R - \zeta(\bar{t}_{\text{entry}}), \tag{62}$$

$$\dot{h}(\bar{t}_{\text{entry}}) = \ddot{h}(\bar{t}_{\text{entry}}) = 0, \tag{63}$$

$$\lambda_\alpha(\bar{t}^-_{\text{entry}})[u(\bar{t}^+_{\text{entry}}) - u(\bar{t}^-_{\text{entry}})] = 0, \tag{64}$$

$$\lambda_\alpha(\bar{t}^-_{\text{exit}})u(\bar{t}^-_{\text{exit}}) = \lambda_\alpha(\bar{t}^+_{\text{exit}})u(\bar{t}^+_{\text{exit}}). \tag{65}$$

Finally, the sign conditions are

$$\bar{\sigma}_0 \geq 0, \qquad \bar{\sigma}_1 \geq 0, \qquad \bar{\sigma}_2 \geq 0, \tag{66a}$$

$$\bar{\mu} \geq 0, \qquad \dot{\bar{\mu}} \leq 0, \qquad \ddot{\bar{\mu}} \geq 0, \qquad \dddot{\bar{\mu}} \leq 0. \tag{66b}$$

Here, the first derivative of $\bar{\mu}$ can be still computed analytically, whereas for the second and third derivatives numerical approximations by difference quotients are advisable.

In the case where there exist only one flat minimum of the altitude, we can in the same way eliminate the variables ζ, λ_ζ, $\bar{\sigma}_0$ from Eqs. (31), (61), (13), (62); again, compare Refs. 19 and 20.

In view of the numerical results, we now discuss the elimination procedure in the case where one boundary arc and one touch point exist for an optimal trajectory. The multiplier λ_ζ can be eliminated by Eqs. (31), (56), (61), resulting in the relation

$$\bar{\sigma}+\bar{\sigma}_0=\Theta, \tag{67}$$

from which, say, $\bar{\sigma}_0$ can be eliminated. There must hold that

$$\bar{\sigma}_0=\Theta-\bar{\sigma}>0.$$

Moreover, the auxiliary variable ζ can be still omitted by the new interior-point condition

$$h(\bar{t}_{\text{entry}})=h(t_{\text{touch}}). \tag{68}$$

The auxiliary variable ζ can be computed afterwards.

4. Conclusions

In summary, the whole collection of necessary conditions defines a multipoint boundary-value problem of the following type:

$$\dot{z}(t)=F(t, z(t))=\begin{cases} F_0(t, z(t)), & 0\leq t\leq \tau_1, \\ \vdots & \\ F_s(t, z(t)), & \tau_s\leq t\leq t_f, \end{cases} \tag{69a}$$

$$z(\tau_k^+)=\Sigma_k\,(\tau_k, z(\tau_k^-)), \qquad 1\leq k\leq s, \tag{69b}$$

$$r_i(z(0), z(t_f))=0, \qquad 1\leq i\leq 2n, \tag{69c}$$

$$r_i(\tau_{k_i}, z(\tau_{k_i}^-))=0, \qquad 2n+1\leq i\leq N, \tag{69d}$$

where F combines the right-hand sides of Eqs. (3) and (17) or (58) as well as some so-called trivial equations of the form $\dot{\sigma}=0$ for each jump parameter. Depending on the type of the subarc, the control u and the multiplier $\bar{\mu}$ are chosen according to Eqs. (18), (21), (43) or $\dddot{S}\equiv 0$ [see Eq. (52)] and (59). The first $2n$ components of r_i contain the two-point boundary conditions (9), (10), (30), whereas the other components stand for the interior conditions such as (19), (23), (27)–(29), (46), (54), (63)–(65) as well as (68) if need be. Finally, Σ_k collects all jump conditions such as (47), (55), (60) considering eventually Eq. (67). In this form, the optimal control problem is accessible for a numerical treatment by the multiple shooting code BNDSCO.

After a solution is obtained, the observance of the inequality constraints and of some sign conditions must be checked. If an inequality constraint (7), (8), or (14), or the switching condition (18) is violated, the assumed switching structure is false. A graphical output usually gives valuable hints about modifying the formulation of the multipoint boundary-value problem. The occurrence of singular subarcs and the branching of a boundary point into two boundary points or one boundary arc can be detected by this procedure too. For example, if the necessary condition for a touch point of the third-order state constraint at a local minimum of the altitude is violated, i.e., $\ddot{h}(t_{touch}) > 0$, we have an indication that this touch point may split either into two touch points or a boundary arc. Corresponding to that, we have to modify the boundary-value problem. Moreover, the other sign conditions such as (22), (48), (55), (66) may reject nonoptimal solutions.

Complete numerical results and the description of how to get them will be presented in Part 2 of this paper (see Ref. 12). That part of the paper contains especially the homotopy strategy to overcome the laborious drawback of indirect methods in obtaining both appropriate initial values and the switching structure. A brief summary of the multiple shooting method will be also given there.

References

1. Bulirsch, R., Montrone, F., and Pesch, H. J., *Optimal Control in Abort Landing of a Passenger Aircraft*, Proceedings of the 8th IFAC Workshop on Control Applications of Nonlinear Programming and Optimization, Paris, France, 1989.
2. Miele, A., Wang, T., and Melvin, W. W., *Optimal Abort Landing Trajectories in the Presence of Windshear*, Journal of Optimization Theory and Applications, Vol. 55, pp. 165–202, 1987.
3. Miele, A., Wang, T., and Melvin, W. W., *Quasi-Steady Flight to Quasi-Steady Flight Transition for Abort Landing in a Windshear: Trajectory Optimization and Guidance*, Journal of Optimization Theory and Applications, Vol. 58, pp. 165–207, 1988.
4. Warga, J., *Minimizing Variational Curves Restricted to a Preassigned Set*, Transactions on the American Mathematical Society, Vol. 112, pp. 432–455, 1964.
5. Jacobson, D. H., Lele, M. M., and Speyer, J. L., *New Necessary Conditions of Optimality for Control Problems with State-Variable Inequality Constraints*, Journal of Mathematical Analysis and Application, Vol. 35, pp. 255–284, 1971.
6. Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, Springer-Verlag, New York, New York, 1980.
7. Oberle, H. J., *Numerische Berechnung optimaler Steuerungen von Heizung und Kühlung für ein realistisches Sonnenhausmodell*, Habilitationsschrift, Munich University of Technology, Munich, Germany, 1982.

8. BULIRSCH, R., *Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung*, Report of the Carl-Cranz Gesellschaft, DLR, Oberpfaffenhofen, Germany, 1971.
9. DEUFLHARD, P., *A Modified Newton Method for the Solution of Ill-Conditioned Systems of Nonlinear Equations with Application to Multiple Shooting*, Numerische Mathematik, Vol. 22, pp. 289–315, 1974.
10. DEUFLHARD, P., *A Relaxation Strategy for the Modified Newton Method*, Optimization and Optimal Control, Edited by R. Bulirsch *et al.*, Springer-Verlag, Berlin, Germany, pp. 59–73, 1975.
11. OBERLE, H. J., and GRIMM, W., *BNDSCO—A Program for the Numerical Solution of Optimal Control Problems*, Internal Report No. 515-89/22, Institute for Flight Systems Dynamics, DLR, Oberpfaffenhofen, Germany, 1989.
12. BULIRSCH, R., MONTRONE, F., and PESCH, H. J., *Abort Landing in the Presence of Windshear as a Minimax Optimal Control Problem, Part 2: Multiple Shooting and Homotopy*, Journal of Optimization Theory and Applications, Vol. 70, pp. 221–252, 1991.
13. MONTRONE, F., *Berechnung der optimalen Steuerung einer Boeing 727 beim Auftreten eines Scherwindes*, Diploma Thesis, Department of Mathematics, Munich University of Technology, Munich, Germany, 1989.
14. BRYSON, A. E., and HO, Y. C., *Applied Optimal Control*, Ginn and Company, Waltham, Massachusetts, 1969.
15. MCDANELL, J. P., and POWERS, W. F., *Necessary Conditions for Joining Optimal Singular and Nonsingular Subarcs*, SIAM Journal on Control, Vol. 9, pp. 161–173, 1971.
16. KELLEY, H. J., KOPP, R. E., and MOYER, H. G., *Singular Extremals*, Topics in Optimization, Edited by G. Leitmann, Academic Press, New York, New York, pp. 63–101, 1967.
17. MAURER, H., *Optimale Steuerprozesse mit Zustandsbeschränkungen*, Habilitationsschrift, University of Würzburg, Würzburg, Germany, 1976.
18. BRYSON, A. E., DENHAM, W. F., and DREYFUS, S. E., *Optimal Programming Problems with Inequality Constraints, I*, AIAA Journal, Vol. 1, pp. 2544–2550, 1963.
19. OBERLE, H. J., *Numerical Treatment of Minimax Optimal Control Problems with Application to the Reentry Flight Path Problem*, Journal of the Astronautical Sciences, Vol. 36, pp. 159–178, 1988.
20. OBERLE, H. J., *Numerical Solution of Minimax Optimal Control Problems by Multiple Shooting Technique*, Report No. 85/3, Department of Mathematics, University of Hamburg, Hamburg, Germany, 1985.
21. MAURER, H., *On Optimal Control Problems with Bounded State Variables and Control Appearing Linearly*, SIAM Journal on Control and Optimization, Vol. 15, pp. 345–362, 1977.