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Geodesic Convexity in Nonlinear Optimization[1]

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Abstract. The properties of geodesic convex functions defined on a connected Riemannian C^2 k-manifold are invesigated in order to extend some results of convex optimization problems to nonlinear ones, whose feasible region is given by equalities and by inequalities and is a subset of a nonlinear space.

Key Words. Generalized convexity, nonconvex optimization, Riemannian manifold, geodesic convexity.

1. Introduction

The concept of convexity plays an important role in mathematical optimization theory. The usual set convexity in linear topological spaces is based upon the possibility of connecting any two points of the space, which has led to the convex and generalized convex functions as well as to the convex optimization. Since convexity is often not enjoyed by real problems, various approaches to the generalizations of the usual line segment have been proposed recently (Refs. 1–16). In these order of ideas, we propose here a generalization which differs from the other ones in the use of a k-dimensional Riemannian manifold of R^n, $k \leq n$, as a definition domain.

In the sequel, only the geodesic convexity will be considered, which will be denoted as g-convexity. In this case, the linear space is replaced by a Riemannian manifold, i.e., by a nonlinear space; the line segment is replaced by a geodesic arc. The advantage of this approach, motivated first of all by Luenberger's works (Refs. 17 and 18), is the recognition of the geometrical structure of the optimization problems which can lead to new theoretical and algorithmic results.

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In order to check the g-convexity property of a function on the feasible region, it is necessary and sufficient to state the positive semidefiniteness of a suitable matrix in this domain. Such a matrix is constructed by means of the gradient and the Hessian matrix. The corresponding computational complexity is of the same order as in the convexity and less than in the pseudo-convexity. When the g-convexity has been proved, it is concluded that a stationary point is a global optimum point too; consequently, every algorithm which gives a stationary point gives a global minimum point, too.

The g-convex optimization problems contain the convex ones as a special case. Let the optimization problem be set in the form

$$\min f(x), \tag{1a}$$

$$\text{s.t.} \quad h_j(x)=0, \qquad j=1,\dots,n-k, \tag{1b}$$

$$x\in R^n, \tag{1c}$$

where $k\geq 0$ and $f, h_j \in C^2$, $j=1,\dots,n-k$. Denoting by $h: R^n \to R^{n-k}$ the map whose components are the h_j's, we assume that the following regularity assumption holds: 0 is a regular value of the map h; i.e., the Jacobian $h^T(x)\in \mathscr{L}(R^n, R^{n-k})$ of h at x is of full rank $n-k$ for all $x\in h^{-1}(0)$. Under this assumption, the set $h^{-1}(0)$ is a k-dimensional submanifold of R^n of class C^2 (Ref. 19), which is endowed with the Riemannian metric induced by the Euclidean structure of R^n. It will be shown that, under g-convexity, a stationary point of (1) is also a global minimum point.

2. Some Properties of g-Convex Functions

Let $M\subset R^n$ be a connected Riemannian C^2 k-manifold. As is usual in differential geometry, a curve of M is called a geodesic if its tangent is parallel along the curve (Ref. 19).

Definition 2.1. It is said that a set $A\subset M$ is g-convex if any two points of A are joined by a geodesic belonging to A.

This definition differs slightly from that of differential geometry, because here a geodesic appears instead of a shortest geodesic. The difference between the two definitions is shown by the following example.

Example 2.1. If we consider a sphere and the arc-metric on the sphere, then a part of the sphere which is greater than a hemisphere is not g-convex in the former sense, but is g-convex in the latter sense.

Example 2.2. A connected, complete Riemannian manifold is g-convex (Ref. 19).

Example 2.3. For every point m in M, there is a neighborhood U of m which is g-convex; for any two points in U, there is a unique geodesic which joins the two points and lies in U (Ref. 19).

Definition 2.2. Let $A \subset M$ be a g-convex set. Then, it is said that a function $f: A \to R$ is g-convex if its restrictions to all geodesic arcs belonging to A are convex in the arc length parameter.

By the definition, the following inequalities hold for every geodesic $\gamma(s)$, $s \in [0, b]$, joining the two arbitrary points $m_1, m_2 \in A$:

$$f(\gamma(tb)) \leq (1-t)f(\gamma(0)) + tf(\gamma(b)), \qquad 0 \leq t \leq 1, \tag{2}$$

where $\gamma(0) = m_1$, $\gamma(b) = m_2$, and s is the arc length parameter.

If $M \subset R^n$ is a connected Euclidean manifold, then the g-convex set $A \subset M$ is a convex set and the g-convex function $f: A \to R$ is a convex function on A (Ref. 18), where

$$\gamma(tb) = m_1 + t(m_2 - m_1), \tag{3}$$

$b = |m_2 - m_1|$, and $|\cdot|$ means the Euclidean norm of a vector.

In the case of a g-convex function $f: A \to R$, define the following set in R^{n+1} lying above the graph of the function:

$$M_1 + M_2 = \{(x, x_{n+1}) + (0, x'_{n+1}) | (x, x_{n+1}) \in M_1, (0, x'_{n+1}) \in M_2\}, \tag{4}$$

where

$$M_1 = \{(x, x_{n+1}) | x_{n+1} - f(x) = 0, x \in R^n, x_{n+1} \in R\}, \tag{5}$$

$$M_2 = \{(0, x'_{n+1}) | 0 \in R^n, x'_{n+1} \in R\}. \tag{6}$$

It is clear that, in the case of a convex function, the set $M_1 + M_2$ coincides with the epigraph (Ref. 20). The $\tilde{g}$-convexity of the set $M_1 + M_2$ ($\tilde{g}$-convexity means the geodesic convexity in another Riemannian metric) requires further investigation, because the structure of $M_1 + M_2$ is different from that of M.

From (2), we achieve obviously the following lemma.

Lemma 2.1. Let $A \subset M$ be a g-convex set, and let $f: A \to R$ be a g-convex function. Then, the level sets

$$\{m | f(m) \leq f(m_0);\ m, m_0 \in A\} \tag{7}$$

are g-convex.

Theorem 2.1. Let $A \subset M$ be a g-convex set, and let $f: A \rightarrow R$ be a g-convex function. Then, a local minimum point is also a global minimum point.

The proof of Theorem 2.1 is similar to that for the convex case and is omitted.

Theorem 2.2. Let $A \subset M$ be an open g-convex set. Then, a function $f: A \rightarrow R$ is g-convex if and only if it is g-convex in a g-convex neighborhood of every point of A.

Proof. (i) If f is a g-convex function on A, then the statement follows from Example 2.3.

(ii) Ab absurdo, assume that the thesis is not true, so that there exist two points, $m_1, m_2 \in M$, a geodesic $\gamma(tb)$, $0 \leq t \leq 1$, $\gamma(0) = m_1$, $\gamma(b) = m_2$, and $t_0 \in [0, 1]$ such that

$$f(\gamma(t_0 b)) > f(\gamma(0)) + t_0[f(\gamma(b)) - f(\gamma(0))]. \tag{8}$$

Let

$$l(t) = f(\gamma(tb)) - f(\gamma(0)) - t[f(\gamma(b)) - f(\gamma(0))] \tag{9}$$

and

$$l(t^*) = \max_{0 \leq t \leq 1} l(t). \tag{10}$$

Then, $0 < t^* < 1$. If t^* is not unique, then let t^* be the smallest value. It obviously exists if the number of the maximum points belonging to $[0, 1]$ is finite; in the contrary case, the smallest value is the inferior limit point of the maximum points, since the function value of $l(t)$ is equal to the maximum value at this point, because of the continuity of $l(t)$. Let $\epsilon > 0$, $t_1 = t^* - \epsilon$, $t_2 = t^* + \epsilon$ be such that $\gamma(t_1 b)$ and $\gamma(t_2 b)$ are in a g-convex neighborhood of $\gamma(t^* b)$. Since

$$l(t^*) > l(t_1), \qquad l(t^*) \geq l(t_2), \tag{11}$$

we have

$$2l(t^*) > l(t_1) + l(t_2), \tag{12}$$

that is,

$$f(\gamma(t^* b)) > [f(\gamma(t_1 b)) + f(\gamma(t_2 b))]/2, \tag{13}$$

which is a contradiction. □

As a matter of fact, the question arises: What is the relation between convex functions and g-convex functions? The g-convexity means that the function $f: M \to R$ is convex along the geodesics. Thus, in order to answer the question, those Riemannian geometries must be considered where the geodesics are straight lines. This is a special case of Hilbert's fourth problem (Ref. 21), which is characterized by the Beltrami theorem (Ref. 21) as follows. Let M_1, M_2 be two manifolds.

Definition 2.3. A homeomorphism $\varphi: M_1 \to M_2$ is called a geodesic mapping if, for every geodesic γ of M_1, the composition $\varphi\gamma$ is a reparametrization of a geodesic of M_2.

Theorem 2.3. (Beltrami). If M is a connected Riemannian k-manifold such that every point has a neighborhood that can be mapped geodesically to R^k, then M has a constant curvature.

The above theorem is the basis of the next one.

Theorem 2.4. The g-convexity of a function $f: M \to R$ coincides with the convexity (Ref. 18) in a coordinate neighborhood of every point if and only if the manifold M has a constant curvature, that is, in the cases of Euclidean geometries, Riemannian elliptic geometries and Bolyai-Lobachevsky hyperbolic geometries.

Proof. It is sufficient to prove the "only if" part. Assume that the g-convexity of $f: M \to R$ coincides with the convexity in a coordinate neighborhood of every point. This means that every point has a convex coordinate neighborhood $U \subset R^k$ in which n functions $x_i(u) \in C^2$, $i = 1, \ldots, n$, $u \in U$, are determined by the inclusion map from M to R^n such that the composition function $f(x(u))$, $u \in U$ is convex and the function $f(x)$ is g-convex on the g-convex set $x(u)$, $u \in U$ of M. So, every point of M has a neighborhood that can be mapped geodesically to R^k; consequently, M has a constant curvature by the Beltrami theorem. □

It is possible to introduce some other kinds of generalized g-convexity property, as in the case of nonlinear optimization. Only the g-quasiconvex functions are defined here.

Definition 2.4. Let $A \subset M$ be a g-convex set. Then, it is said that a function $f: A \to R$ is g-quasiconvex if all its level sets $\{m \mid f(m) \leq f(m_0);\ m, m_0 \in A\}$ are g-convex.

Theorem 2.5. The g-quasiconvexity of a function $f: M \to R$ coincides with the quasiconvexity (Ref. 18) in a coordinate neighborhood of every point if and only if the manifold M has a constant curvature.

The proof of Theorem 2.5 is similar to that for the g-convex case.

3. First-Order and Second-Order Characterizations

Lemma 3.1. Let $A \subset M$ be an open g-convex set, and let $f: A \to R$ be a continuously differentiable function. Then, f is g-convex on A if and only if, for every pair of points $m_1 \in A$, $m_2 \in A$, and a connecting geodesic $\gamma(tb)$, $0 \leq t \leq 1$, $\gamma(0) = m_1$, $\gamma(b) = m_2$,

$$f(m_2) - f(m_1) \geq \nabla f(m_1) \dot{\gamma}(0), \tag{14}$$

where $\nabla f(m_1)$ and $\dot{\gamma}(0)$ mean, respectively, the gradient of f at the point m_1 and the derivative of $\gamma(tb)$ w.r.t. t at the point 0.

The proof of Lemma 3.1 is similar to that for the convex case.

Definition 3.1. The point $m \in M$ is a stationary point of the continuously differentiable function $f: A \to R$ if the gradient $\nabla f(m)$ is orthogonal to the tangent space, say TM, of M at m.

Corollary 3.1. Let $A \subset M$ be an open g-convex set, and let $f: A \to R$ be a continuously differentiable g-convex function. Then, every stationary point of f is a global minimum point. Moreover, the set of global minimum points is g-convex.

Proof. As the right side of Ineq. (14) is equal to zero at a stationary point $m \in A$ by Definition 3.1, it follows that the first part of the statement is proved. The second part is obtained from Lemma 2.1. □

Theorem 3.1. Let $A \subset M$ be an open g-convex set, and let $f: A \to R$ be a twice continuously differentiable function. Then, f is g-convex on A if and only if the following matrix is positive semidefinite at every point:

$$H^g f = Hf_{|TM} + |\nabla f_N| B_{\nabla f_N}. \tag{15}$$

Here, $Hf_{|TM}$ is the Hessian matrix of the function f restricted to the tangent space TM of M, and $B_{\nabla f_N}$ is the second fundamental form of M in the normal direction of the vector ∇f.

Proof. As the statement is valid in a g-convex neighborhood (Ref. 14), the theorem is a simple consequence of Theorem 2.2. □

Now, it will be shown that $H^g f$ determines a second-order, symmetrical tensor field on A. The most important consequence of this fact is that the quadratic form $w^T H^g w$, $w \in R^k$, is invariant under nonlinear coordinate transformations.

Let $u \in U \subset R^k$ be coordinates for a g-convex region of the C^2 k-manifold $M \subset R^n$. Then, the inclusion map from M to R^n determines n functions $x_i(u) \in C^2$, $i=1,\dots,n$.

Let us introduce the following notations and operations:

$$Hx(u) = \begin{bmatrix} Hx_1(u) \\ \vdots \\ Hx_n(u) \end{bmatrix}, \qquad J = \partial x/\partial u,$$

where $Hx_i(u)$, $i=1,\dots,n$, are $k \times k$ Hessian matrices,

$$y^T = (y_1,\dots,y_n) \in R^n, \qquad w^T = (w_1,\dots,w_k) \in R^k,$$

$$y^T Hx(u) = \sum_{i=1}^n y_i Hx_i(u), \qquad w^T Hx(u) w = \begin{bmatrix} w^T Hx_1(u) w \\ \vdots \\ w^T Hx_n(u) w \end{bmatrix},$$

$$\lambda w^T Hx(u) w = w^T \lambda Hx(u) w, \qquad \lambda \in R.$$

Theorem 3.2. $H_x f + J(J^TJ)^{-1}\nabla_x f_N Hx(u)(J^TJ)^{-1}J^T : TM \times TM \to R$ is a second-order symmetrical tensor field on A, where $H_x f$ denotes the Hessian matrix of $f(x)$ by x.

Proof. First, the following identity will be proved:

$$v^T(H_x f + J(J^TJ)^{-1}\nabla_x f_N Hx(u)(J^TJ)^{-1}J^T)v$$
$$= w^T H_u^g f w, \qquad v \in TM,\ w \in R^k. \tag{16}$$

Since

$$Jw = v, \qquad w \in R^k,\ v \in TM, \tag{17}$$

we have

$$v^T(H_x f + J(J^TJ)^{-1}\nabla_x f_N Hx(u)(J^TJ)^{-1}J^T)v$$
$$= v^T H_x f v + v^T J(J^TJ)^{-1}\nabla_x f_N Hx(u)(J^TJ)^{-1}J^T)v$$
$$= w^T J^T H_x f J w + w^T J^T J(J^TJ)^{-1}\nabla_x f_N Hx(u)(J^TJ)^{-1}J^T J w$$
$$= w^T(J^T H_x f J + \nabla_x f_N Hx(u))w. \tag{18}$$

But

$$J^T H_x fJ = Hf_{|TM} \quad \text{and} \quad |\nabla f_N| B_{\nabla f_N} = \nabla_x f_N Hx(u);$$

so, the identity is true. Consider a nonlinear coordinate transformation $u(z)$, $\det(\partial u/\partial z) \neq 0$ of R^k. Then,

$$\begin{aligned} H_z^g f &= (\partial u^T/\partial z) J^T H_x fJ(\partial u/\partial z) + \nabla_x f_N H_z x(u(z)) \\ &= (\partial u^T/\partial z) J^T H_x fJ(\partial u/\partial z) + \nabla_x f_N (\partial u^T/\partial z) H_u x(u)(\partial u/\partial z) \\ &= (\partial u^T/\partial z)(J^T H_x fJ + \nabla_x f_N Hx(u))(\partial u/\partial z) \\ &= (\partial u^T/\partial z) H_u^g f(\partial u/\partial z), \end{aligned} \tag{19}$$

which justifies the statement. □

Remark 3.1. We observe that

$$H_u f = J^T H_x fJ + \nabla_x fHx(u)$$

does not determine a tensor field on A.

4. Optimality Conditions and g-Convexity

We consider the problem

$$\min f(x), \qquad x \in A \subset M, \tag{20}$$

where $f \in C^2$, $M \subset R^n$ is a Riemannian C^2 k-manifold, and A is a subset of M. In order to characterize the local optimality, it is sufficient to investigate the manifold in a neighborhood, so that instead of (20) we are faced with the following problem:

$$\min f(x), \tag{21a}$$

$$\text{s.t.} \quad x = x(u) \in R^n, \tag{21b}$$

$$u \in U \subset R^k, \tag{21c}$$

where $f(x)$, $x_i(u) \in C^2$, $i = 1, \ldots, n$, and U is an open set. The optimality conditions are obtained by direct computation, elaborated in detail in Ref. 22.

Theorem 4.1. If u_0 is a local minimum of (20), then

$$\nabla f_N(x(u_0)) = \nabla f(x(u_0)) \tag{22}$$

and

$$H^g_u f(x(u_0)) \text{ is positive semidefinite.} \tag{23}$$

If (22) holds at u_0 and

$$H^g_u f(x(u_0)) \text{ is positive definite,} \tag{24}$$

then u_0 is a strict local minimum of (20).

Corollary 4.1. It follows from Theorem 4.1 and Example 2.3 that the function $f(x)$ is g-convex in a neighborhood of u_0, if (22) and (24) hold at u_0.

The optimality conditions (22), (23), and (24) as well as the relation between the nonlinear optimization problems and (20) were investigated in Refs. 22–24.

5. g-Convexity in Nonlinear Optimization

Next, we deal with the problem

$$\min f(x), \tag{25a}$$

$$\text{s.t.} \quad h_j(x)=0, \qquad j=1,\ldots,n-k, \tag{25b}$$

$$x \in R^n, \tag{25c}$$

where $f, h_j \in C^2$, $j=1,\ldots,n-k$. Let

$$M=\{x \mid h_j(x)=0, j=1,\ldots,n-k\}. \tag{26}$$

If $\nabla h_j(x)$, $j=1,\ldots,n-k$, $x\in M$, are linearly independent, then M is a Riemannian C^2 k-manifold, where the Riemannian metric is induced by the Euclidean metric. Assume that M is connected.

It is convenient to introduce the Lagrangian associated with (25), defined as

$$L(x, \mu(x)) = f(x) - \sum_{j=1}^{n-k} \mu_j(x) h_j(x), \tag{27}$$

where

$$\mu(x)^T = \nabla f(x) \nabla h(x)^T [\nabla h(x) \nabla h(x)^T]^{-1}, \tag{28}$$

$$\nabla h(x) = \begin{bmatrix} \nabla h_1(x) \\ \vdots \\ \nabla h_{n-k}(x) \end{bmatrix}. \tag{29}$$

It is emphasized that the multiplier $\mu(x)$ depends on x and is no longer a constant term.

Lemma 5.1. If $\nabla h_j(x)$, $j=1,\ldots,n-k$, $x \in M$, are linearly independent, then

$$|\nabla f_N| B_{\nabla f_N} = -\left[\sum_{j=1}^{n-k} \mu_j(x) H h_j(x)\right]_{|TM}, \qquad x \in M. \tag{30}$$

Proof. Since

$$\nabla h_j(x) \neq 0, \qquad j=1,\ldots,n-k, \qquad x \in M, \tag{31}$$

the level surfaces $h_j(x)=0$, $j=1,\ldots,n-k$, can be given in a small neighborhood of every point as elementary surfaces of $(n-1)$-dimensions $y_j(z)$, $z \in Z \subset R^{n-1}$. We can assume that the $(n-1)$-dimensional parameter set Z is the same for all level surfaces and that $U \subset Z$ is a coordinate domain of M. These assumptions do not mean loss of generality, for this is always obtainable with linear transformations.

Thus, in an arbitrary coordinate domain U of M given in the form $x(u)$, $u \in U$ is contained in every level surface; that is,

$$y_j(u) = x(u), \qquad j=1,\ldots,n-k, \qquad u \in U \subset R^k. \tag{32}$$

It turns out that

$$h_j(x(u)) = 0, \qquad j=1,\ldots,n-k; \tag{33}$$

and differentiating (33) twice w.r.t. u, we have

$$\nabla_x h_j(x(u)) H x(u) = -J^T H_x h_j(x(u)) J, \qquad j=1,\ldots,n-k. \tag{34}$$

Multiplying both sides of Eqs. (34), respectively, by $\mu_1(x),\ldots,\mu_{n-k}(x)$ and adding term by term leads to an equation equivalent to (30). □

Theorem 5.1. Let

$$M = \{x \mid h_j(x)=0, j=1,\ldots,n-k, x \in R^n\}$$

be connected; let $f: M \to R$ be a twice continuously differentiable function; and let $\nabla h_j(x)$, $j=1,\ldots,n-k$, $x \in M$, be linearly independent. Then, f is g-convex on M if and only if the matrix $H_x^g L(x, \mu(x))_{|TM}$ is positive semidefinite at every point $x \in M$.

Proof. Because of the assumptions, M is a complete Riemannian manifold. From the Hopf–Rinow theorem (Ref. 19), it turns out that any two points in M can be joined by a geodesic segment, i.e., M is g-convex.

Since

$$H_x^g L(x, \mu(x))_{|TM} = \left[H_x f(x) - \sum_{j=1}^{n-k} \mu_j(x) H_x h_j(x) \right]_{|TM}, \qquad x \in M, \tag{35}$$

the thesis follows from Lemma 5.1 and Theorem 3.1. □

Corollary 5.1. Let

$$M = \{x \mid h_j(x) = 0, j = 1, \ldots, n-k, x \in R^n\}$$

be connected; let $f: M \to R$ be a twice continuously differentiable function; and let $\nabla h_j(x)$, $j = 1, \ldots, n-k$, $x \in M$, be linearly independent. If $Hf(x)$, $-\mu_j(x) Hh_j(x)$, $j = 1, \ldots, n-k$, $x \in M$, are positive semidefinite, then $f(x)$ is g-convex on M.

Example 5.1. Let

$$x(u)^T = [x_1 = u_1, x_2 = u_2, x_3 = (1 - x_1^2 - x_2^2)^{1/2}] \in R^3, \tag{36}$$

$$U = \{(u_1, u_2) \mid -1 < u_1 < 1, -1 < u_2 < 1\} \subset R^2, \tag{37}$$

$$f(x) = [x_1, x_2, x_3] \begin{bmatrix} e & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = e x_1^2 + l x_2^2 + q x_3^2. \tag{38}$$

Then, we can compute the Hessian matrix of the composition function $f(x(u))$, $u \in U$,

$$H_u f = \begin{bmatrix} e - q & 0 \\ 0 & l - q \end{bmatrix}, \tag{39}$$

and the geodesic Hessian matrix of $f(x(u))$, $u \in U$,

$$H_u^g f = ((e-q)x_1^2 + (l-q)x_2^2) \begin{bmatrix} 1 + x_1^2/x_3^2 & x_1 x_2/x_3^2 \\ x_1 x_2/x_3^2 & 1 + x_2^2/x_3^2 \end{bmatrix} + \begin{bmatrix} e + q x_1^2/x_3^2 & q x_1 x_2/x_3^2 \\ q x_1 x_2/x_3^2 & l + q x_2^2/x_3^2 \end{bmatrix}. \tag{40}$$

If $e > q$, $l > q$, $q > 0$, then $f(x(u))$, $u \in U$, is convex and f is g-convex on $x(u)$, $u \in U$. If $0 < e < q < l$, then $f(x(u))$, $u \in U$, is not convex, but is g-convex on $x(u)$,

$$u \in U \cap \{(u_1, u_2) \mid u_2 > [(q-e)/(l-q)]^{1/2} u_1\}; \tag{41}$$

that is, a local optimum is also a global optimum.

Remark 5.1. The matrix $H_x^g L(x, \mu(x))_{|TM}$ formulated in (35) and (28), (29) can be given explicitly at every point via the gradient vector and the Hessian matrix. After this, the computational work to check the g-convexity consists of the study of the positive semidefiniteness of $H_x^g L(x, \mu(x))_{|TM}$ similarly to the convex case.

The determination of the matrix $H_x^g L(x, \mu(x))_{|TM}$ is presented in the quadratic case.

Example 5.2. Consider the problem

$$\min[(1/2)x^T Cx + p^T x], \tag{42a}$$

$$\text{s.t.} \quad (1/2)x^T C_j x + p_j^T x = 0, \qquad j = 1, \ldots, n-k, \tag{42b}$$

$$x \in R^n. \tag{42c}$$

Let

$$M = \{x \mid (1/2)x^T C_j x + p_j^T x = 0, j = 1, \ldots, n-k, x \in R^n\}$$

be connected; and let the vectors $C_j x + p_j, j = 1, \ldots, n-k, x \in M$, be linearly independent. Then, by Theorem 5.1, the necessary and sufficient condition of the g-convexity of the objective function on M is the positive semidefiniteness of $H_x^g L(x, \mu(x))_{|TM}$ at every point $x \in M$, where

$$H_x^g L(x, \mu(x))_{|TM} = \left[C - \sum_{j=1}^{n-k} \mu_j(x) C_j \right]_{|TM}, \tag{43}$$

$$\mu(x)^T = (Cx + p)^T \nabla h(x)^T [\nabla h(x) \nabla h(x)^T]^{-1}, \tag{44}$$

$$\nabla h(x) = \begin{bmatrix} (C_1 x + p_1)^T \\ \vdots \\ (C_{n-k} x + p_{n-k})^T \end{bmatrix}. \tag{45}$$

The restriction of the matrix $H_x^g L(x, \mu(x))$ to the tangent space TM is obtained via the projection matrix (Ref. 18)

$$P = I - \nabla h(x)^T [\nabla h(x) \nabla h(x)^T]^{-1} \nabla h(x), \tag{46}$$

so that

$$H_x^g L(x, \mu(x))_{|TM} = P^T \left[C - \sum_{j=1}^{n-k} \mu_j(x) C_j \right] P. \tag{47}$$

If the matrices $-C$, C_j, $j = 1, \ldots, n-k$, are negative semidefinite and if $\mu(x) \geq 0$, $x \in M$, $\mu(x)$ given by (44), then the function $(1/2)x^T Cx + p^T x$ is g-convex on M by Corollary 5.1.

6. g-Convex Optimization Problem

Introduce the g-convex programming problem as follows:

$$\min f(x), \tag{48a}$$

$$\text{s.t.} \quad g_i(x) \leq 0, \qquad i = 1, \ldots, l, \tag{48b}$$

$$x \in A \subset M \subset R^n, \tag{48c}$$

where M is a Riemannian manifold, A is a g-convex set, and $f, g_i, i=1, \ldots, l$, are g-convex functions on A. The constraint set of (48) is g-convex, and the problem (48) contains the so-called convex programming problem as a special case.

7. Concluding Remarks

In this paper, the g-convexity (geodesic convexity) of functions is developed and characterized in nonlinear optimization. It is likely that this is the appropriate generalization of the classical convexity notion for the case of nonlinear constraints defining a subset of a Riemannian manifold. This characterization can be useful for solving nonconvex problems.

After checking the g-convexity property of a problem, it is possible to use any algorithm to find the global optimum point, because in this case every stationary point is a global optimum point. In order to check the g-convexity property of a function on the feasible domain, it is necessary and sufficient to state the positive semidefiniteness of the geodesic Hessian matrix in this domain, where this matrix is given by the gradient vector and by the Hessian matrix. To show the positive semidefiniteness of a matrix, there are efficient computer codes (e.g., to state the nonnegativity of the smallest eigenvalue).

By using the property of the g-convexity, it is possible to obtain sufficient conditions for the connectedness of the feasible region of complementarity systems (Ref. 25). In case of a linear complementarity system (Ref. 25), these conditions can be formulated in explicit form.

Consider the linear complementarity problem as follows:

$$x^T M x + q^T x = 0, \qquad Mx + q \geq 0, \qquad x \geq 0, \qquad x, q \in R^n, \tag{49}$$

where M is a symmetrical $n \times n$ matrix. Assume that $2Mx + q \neq 0$ and $n \geq 2$. The equality in (49) determines a Riemannian manifold, so the g-convexity of the feasible region is a consequence of the g-convexity of the linear functions appearing in the inequalities of (49).

The necessary and sufficient conditions of the g-convexity of the functions $-x_i$, $i=1, \ldots, n$, and $-(m_j x + q_j)$, $j=1, \ldots, n$, are the positive semidefiniteness of the following matrices at every point satisfying the equality in (49):

$$H^g x_i = (2Mx + q)_i PMP, \qquad i=1, \ldots, n, \tag{50a}$$

$$H^g(m_j x + q_j) = m_j(2Mx + q)PMP, \qquad j=1, \ldots, m. \tag{50b}$$

Here, $(2Mx + q)_i$, $i=1, \ldots, n$, is the ith component of the vector $2Mx + q$;

m_j, $j=1,\ldots,n$, is the jth row vector of the matrix M; and

$$P=I-(2Mx+q)(2Mx+q)^T/(2Mx+q)^2. \tag{51}$$

The following example indicates that the class of complementarity systems which satisfies the g-convexity property is not empty.

Example 7.1. Let

$$M=\begin{bmatrix}-1 & 0 & -1 & -1\\ 0 & 1 & 0 & 0\\ -1 & 0 & -1 & -1\\ -1 & 0 & -1 & -1\end{bmatrix}, \qquad q=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}. \tag{52}$$

Then the complementarity region determined by (49) and (52) is equal to the g-convex set

$$\{x_1+x_3+x_4=1,\ x_2=0,\ x_1, x_3, x_4 \geq 0\}.$$

In this domain, the conditions (50) hold because $x^T PMPx=0$.

Some questions are open, for example: How wide is the class of complementarity systems which satisfies the g-convexity property? Is it possible to generalize the concept of epigraph to the g-convex functions?

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