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Asymptotic Expansions for Dynamic Programming Recursions with General Nonnegative Matrices[1]

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Abstract. This paper is concerned with the study of the asymptotic behavior of dynamic programming recursions of the form

$$x(n+1) = \max_{P \in \mathscr{K}} Px(n), \qquad n = 0, 1, 2, \ldots,$$

where $\mathscr{K}$ denotes a set of matrices, generated by all possible interchanges of corresponding rows, taken from a fixed finite set of nonnegative square matrices. These recursions arise in a number of well-known and frequently studied problems, e.g. in the theory of controlled Markov chains, Leontief substitution systems, controlled branching processes, etc. Results concerning the asymptotic behavior of $x(n)$, for $n \to \infty$, are established in terms of the maximal spectral radius, the maximal index, and a set of generalized eigenvectors. A key role in the analysis is played by a geometric convergence result for value iteration in undiscounted multichain Markov decision processes. A new proof of this result is also presented.

Key Words. Dynamic programming, nonnegative matrices, asymptotic expansions, generalized eigenvectors, geometric convergence.

1. Introduction

Discrete dynamic programming recursions for probabilistic systems arise already in Bellman's pathbreaking book (Ref. 1). In Chapter 11 of

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this volume, Markov decision processes are studied; and, on p. 328, the question is raised of what can be said about the asymptotic behavior of dynamic programming recursions of the form

$$x(n+1)=\max_{P\in\mathcal{K}} Px(n), \qquad n=0,1,2,\ldots, x(0)>\underline{0}, \tag{1}$$

where $\mathcal{K}$ denotes a finite set of square nonnegative matrices having the product property (definition follows below). Using the Brouwer fixed-point theorem, Bellman showed that, for the case in which all matrices are strictly positive, the following holds:

$$x(n)\sim\hat{\sigma}^n x, \qquad n\to\infty,$$

where $\hat{\sigma}$ is the maximal spectral radius over the set of matrices and x obeys

$$\hat{\sigma}x=\max_{P\in\mathcal{K}} Px.$$

Systems of the form (1) arise in a number of well-known and frequently studied problems in mathematics (Markov decision processes with additive reward structure, risk-sensitive Markov decision chains, controlled branching processes) and mathematical economics (e.g., Leontief substitution systems). For details, the reader is referred to Howard (Ref. 2), Howard and Matheson (Ref. 3), Pliska (Ref. 4), and Burmeister and Dobell (Ref. 5). In general, however, the matrices involved are only nonnegative, rather than strictly positive. Furthermore, they are not necessarily substochastic. The only assumption which will be made throughout this paper is that the set $\mathcal{K}$ of square nonnegative matrices is finite and possesses the *product property*. This concept deserves a formal definition (for convenience, we give it in a slightly more general form).

Definition 1.1. Let $\mathcal{K}$ be a set of nonnegative $k\times m$ matrices, $k\in\mathbb{N}$, $m\in\mathbb{N}\cup\{\infty\}$, and let P_i denote the ith row of a matrix $P\in\mathcal{K}$. Then, $\mathcal{K}$ has the *product property* if, for each subset V of $\{1,2,\ldots,k\}$ and for each pair of matrices $P(1), P(2)\in\mathcal{K}$, the following holds: The matrix $P(3)$, defined by

$$P(3)_i := \begin{cases} P(1)_i, & i\in V,\\ P(2)_i, & i\in\{1,2,\ldots,k\}\setminus V,\end{cases}$$

is also an element of $\mathcal{K}$. □

Hence, for each $i\in\{1,2,\ldots,k\}$, there exists a collection $\mathcal{K}_i$ of nonnegative row vectors of length m (note that m is allowed to be infinite). $\mathcal{K}$ is the set of all $k\times m$ matrices with the property that their ith row is an element of $\mathcal{K}_i$, $i=1,2,\ldots,k$.

Throughout the rest of this paper, $\mathcal{K}$ denotes a finite set of nonnegative $N \times N$ matrices with the product property ($N \in \mathbb{N}$). The objective of this paper is to study the asymptotic behavior of the vector $x(n)$, defined recursively in (1), for $n \to \infty$. Apart from being interesting in itself, this question appears to be important for the determination of matrices which maximize the growth of systems of the form (1). The following examples may be illustrative.

Example 1.1. Consider a system which is observed at discrete time points. At each time point, the system may be in one of a finite number of states, labeled by 1, 2, ..., N. If, at time t, the system is in state i, one may choose an action, a say, from a finite action space A; this results in an immediate reward $r_i(a)$ and a probability $p_{ij}(a)$ of finding the system in state j at time $t+1$. Suppose that

$$r_i(a) \geqq 0, \quad \sum_{j=1}^{N} p_{ij}(a) \leqq 1, \qquad i, j = 1, \ldots, N, \; a \in A.$$

Let $v(0)_i$ denote a terminal reward in state i, and let $v(n)_i$ be the maximal expected return for the n-period problem. Then Bellman's optimality principle (cf. Ref. 1) implies that

$$v(n)_i = \max_{a \in A} \left\{ r_i(a) + \sum_{j=1}^{N} p_{ij}(a) v(n-1)_j \right\}, \qquad i = 1, \ldots, N, \; n = 1, 2, \ldots. \tag{2}$$

Define a policy as a function f from $\{1, \ldots, N\}$ to A. Let F denote the set of all policies, $P(f)$ the substochastic matrix with entries $p_{ij}(f(i))$, and $r(f)$ the vector with components $r_i(f(i))$, $i, j = 1, 2, \ldots, N$. Note that the collection of matrices

$$\{(P(f), r(f)) \mid f \in F\}$$

possesses the product property. Instead of (2), we may write

$$v(n) = \max_{f \in F} \{r(f) + P(f) v(n-1)\}, \qquad n = 1, 2, \ldots, \tag{3}$$

where $v(n)$ is the vector with components $v(n)_i$, $i = 1, 2, \ldots, N$. By introducing a simple dummy variable, we obtain

$$\begin{bmatrix} v(n) \\ 1 \end{bmatrix} = \max_{f \in F} \begin{bmatrix} P(f) & r(f) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v(n-1) \\ 1 \end{bmatrix}, \qquad n = 1, 2, \ldots,$$

which is a special case of the recursion (1). The process considered here is called a *Markov decision process* (with additive reward structure). We return to it when interpreting our final results.

Example 1.2. Bellman (Ref. 6) considered the following multistage decision process. At each stage, a decision maker has the choice of one of a finite number of actions. Let A denote the set of all possible actions. The choice of $a \in A$ results in a probability distribution with the following properties:

(a) there is a probability $p_i(a)$ that the controller receives i units and the process continues, $i = 1, 2, \ldots, N$;

(b) there is a probability $p_0(a)$ that the controller receives nothing and the process terminates.

Let n be a fixed integer, and suppose that the decision maker wants to maximize the probability that he receives at least a total number of n units before the process terminates. Let u_j denote the maximal probability of obtaining at least j units before termination of the process. Then,

$$u_j = \begin{cases} \max\limits_{a \in A} \sum\limits_{i=1}^{N} p_i(a) u_{j-i}, & j > 0, \\ 1, & j \leqq 0. \end{cases}$$

Applying a simple transformation, we may write, for $j = 1, 2, \ldots, n$,

$$\begin{bmatrix} u_j \\ u_{j-1} \\ \vdots \\ u_{j-N+1} \end{bmatrix} = \max_{a \in A} \begin{bmatrix} p_1(a) & & & & p_N(a) \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{j-1} \\ u_{j-2} \\ \vdots \\ u_{j-N} \end{bmatrix},$$

where we start with

$$(u_0, \ldots, u_{1-N})^T = (1, \ldots, 1)^T.$$

It follows that the decision maker has to solve an n-step sequential decision problem of type (1).

A list of further examples, worked out in detail, can be found in Ref. 7.

Let us now return to the recursion (1). As remarked above, we are interested in the asymptotic behavior of $x(n)$, for $n \to \infty$. More precisely, the objective will be to prove the existence of an asymptotic expansion for $x(n)$. If $\mathscr{K}$ contains exactly one matrix, P say (hence no maximization occurs), and if in addition P is aperiodic (which means that its spectral radius is the only eigenvalue on the spectral circle, compare Section 2), then, by applying a familiar Jordan transformation, it is easily proved that

$$x(n) = \begin{bmatrix} n \\ \nu - 1 \end{bmatrix} \sigma^{n-\nu+1} y(1) + \cdots + \begin{bmatrix} n \\ 0 \end{bmatrix} \sigma^n y(\nu) + \mathcal{O}(\rho^n), \qquad n \to \infty, \tag{4}$$

where σ denotes the spectral radius of P, ν its index (i.e., the size of the largest Jordan block, associated with σ), $y(1)$ an eigenvector, and $y(2), \ldots, y(\nu)$ generalized eigenvectors [compare, e.g., Pease (Ref. 8)]. Furthermore, $\rho < \sigma$ (in fact, ρ can be chosen arbitrarily close to the modulus of the subdominant eigenvalue of P).

When dealing with a set of square nonnegative matrices, the approach has to be different (in each step, another matrix may be the maximizing one). In a companion paper (Ref. 9), the author discusses a number of structural properties of sets of nonnegative matrices with the product property or, more precisely, of the nonlinear continuous, convex mapping, defined for each $x \in \mathbb{R}^N$ by

$$x \to \max_{P \in \mathcal{K}} Px. \tag{5}$$

The main result of this paper establishes the existence of an asymptotic expansion, similar to the one given in (4), for the vector $x(n)$, defined in (1). Then, σ denotes the maximal spectral radius, taken over the set of matrices $\mathcal{K}$, while ρ is some nonnegative number, strictly smaller than σ.

We conclude this section with an overview of the organization of the rest of the paper. After introducing some notational conventions, we list, in Section 2, a number of basic definitions and results concerning nonnegative matrices which will be needed in the sequel. Also the main results, obtained in Ref. 9, concerning sets of nonnegative matrices with the product property, will be given here; they serve as a starting point for the analysis in Sections 4 and 5.

Section 3 is almost completely devoted to geometric convergence in undiscounted Markov decision chains (with some minor extensions). The results in this section were initially published by Schweitzer and Federgruen (Ref. 10). The proofs in Ref. 10 are unfortunately extremely complicated; therefore, we present in Section 3 new (and in our view, simpler) proofs of all results. Although interesting in themselves, these results serve especially as an important tool in the analysis of dynamic programming recursions of type (1), studied in Sections 4 and 5. Section 4 treats the case in which all matrices with maximal spectral radius have index equal to one. The results in this section are a substantial generalization of those obtained by Bellman (Ref. 1). In Ref. 11, Sladky proves convergence for the case in which all matrices are irreducible. In Ref. 12, he also obtains convergence results for the case discussed in Section 4, however, without showing that this convergence is geometric. It is precisely this result which proves to be indispensible in the analysis of the asymptotic behavior of $x(n)$, for $n \to \infty$, in the most general case (Section 5). The results in Section 5 are believed to be new. Bounds on discrete dynamic programming recursions of type

(1) were published by Sladky (Ref. 13); they also follow immediately from our Lemma 2.3 (see also Ref. 9). The one-matrix case was studied extensively by Rothblum (see, for example, Refs. 14, 15). In Section 6, finally, we consider briefly several special cases, arising from the theory of Markov decision processes, and discuss the implications of our results. We end with mentioning some extensions.

2. Preliminaries

In this section, some notational conventions are introduced. Next, we summarize a number of basic definitions and results concerning sets of nonnegative matrices, which will be used throughout the rest of the paper.

A *nonnegative matrix* is a matrix with all its entries nonnegative. Unless stated otherwise, all matrices will be square and of a fixed dimension, N say. The set $\{1, 2, \ldots, N\}$ will be called the *state space* and denoted by S.

Matrices will be denoted by capitals $P, Q, \ldots$; column vectors by lower case letters $x, y, u, w, \ldots$. The identity matrix is denoted by I, the vector with all components equal to one by e. The null matrix is denoted by $\underline{\underline{0}}$, the null vector by $\underline{0}$. When writing $\|..\|$, we mean the usual supnorm, i.e.,

$$\|x\| = \sup_{i \in S} |x_i|.$$

The nth power of a matrix P is written as P^n; we define

$$P^0 := I.$$

The ijth entry of P^n will be denoted by $p_{ij}^{(n)}$; if $n = 1$, we write p_{ij} instead of $p_{ij}^{(1)}$. P_i denotes the ith row of P, x_i the ith component of the vector x.

A square matrix P is called *positive* if

$$p_{ij} > 0, \qquad \text{for all } i, j \in S.$$

If P is nonnegative (positive), we write

$$P \geqq \underline{\underline{0}} \;(P > \underline{\underline{0}}).$$

We say that P is *semipositive* and write

$$P \geq \underline{\underline{0}}, \qquad \text{if } P \geqq \underline{\underline{0}} \text{ and } P \neq \underline{\underline{0}}.$$

Furthermore, we write

$$P \geqq Q \;(\geq Q, > Q),$$

if

$$P - Q \geqq \underline{\underline{0}} \;(\geq \underline{\underline{0}}, > \underline{\underline{0}}).$$

Similar definitions apply to vectors. Sometimes, the words "strictly positive" are used instead of "positive."

Subsets of the state space S are denoted by $A, B, C, D, \ldots$. If $C \subset S$, then P^C is the restriction of the square matrix P to $C \times C$. Similarly, x^C is the restriction of the column vector x to C. If $\{D(0), D(1), \ldots, D(n)\}$ denotes a partition of the state space, then we often write $P^{(k,l)}$ for the restriction of P to $D(k) \times D(l)$, and $x^{(k)}$ for the restriction of x to $D(k)$, $k, l = 0, 1, \ldots, n$. Note that

$$P^{(k,k)} = P^{D(k)}, \qquad k = 0, 1, \ldots, n.$$

If P is a square matrix of finite dimension, then the *spectral radius* $\sigma(P)$ of P is defined by

$$\sigma(P) := \max\{|\lambda| \,|\, \lambda \text{ is an eigenvalue of } P\}.$$

It is well known from the Perron–Frobenius theorem that $\sigma(P)$ is an eigenvalue of P if P is nonnegative (cf. Ref. 16). P is called *irreducible* if, for each pair $i, j \in S$, there exists a nonnegative integer $n = n(i, j)$ such that

$$p_{ij}^{(n)} > 0;$$

otherwise, P is said to be *reducible*. An irreducible matrix P is called *periodic*, with period k, if there exist precisely k different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with

$$|\lambda_l| = \sigma(P), \qquad \text{for } l = 1, 2, \ldots, k.$$

If $k = 1$, P is said to be *aperiodic*.

The Perron–Frobenius theorem states that there exist semipositive left-eigenvectors and right-eigenvectors, associated with the spectral radius of a square nonnegative matrix P. If P is irreducible, these eigenvectors can even be chosen strictly positive (cf. Ref. 16 or Ref. 17).

If P is reducible, we may partition the state space S into *classes*. A *class* of P is a subset $C \subset S$ such that P^C is irreducible and such that C cannot be enlarged without destroying the irreducibility. It is well known that classes can be ordered, at least partially, according to certain accessibility relations.

We say that state i has access to state j (or j has access from i), if

$$p_{ij}^{(n)} > 0,$$

for some nonnegative integer n. Since classes are irreducible, we may speak of having access to (from) some (and hence, any) state in that class. Exploiting the accessibility relations between classes, it is easily seen that, after possibly permuting the states, P can be written in block-triangular form, as follows:

$$P=\begin{bmatrix} P^{(1,1)} & P^{(1,2)} & \cdots\cdots & P^{(1,n)} \\ & P^{(2,2)} & \cdots\cdots & P^{(2,n)} \\ & & & \vdots \\ & & & P^{(n,n)} \end{bmatrix}, \tag{6}$$

where $P^{(i,i)}$ is irreducible and

$$P^{(i,j)}=\underline{0}, \qquad \text{for } i \geqq j,\ i, j=1, \ldots, n.$$

In other words, S can be partitioned into classes, $C(1), \ldots, C(n)$ say, such that $C(i)$ has only access to classes $C(j)$ with $j \geqq i$, $i=1, \ldots, n$; $P^{(i,j)}$ denotes the restriction of P to $C(i) \times C(j)$. Note that, since the eigenvalues of P are completely determined by the blocks on the diagonal, the spectral radius of the restriction of P to a class never exceeds the spectral radius of P itself. Formally,

$$\sigma(P^{(i,i)}) \leqq \sigma(P), \qquad i=1, \ldots, n.$$

A class C is called *basic*, with respect to P, if

$$\sigma(P^C)=\sigma(P);$$

otherwise, it is called *nonbasic*. A class C is said to be *final* (*initial*) if C has no access to (from) any other class.

A reducible matrix P is said to be *aperiodic* if the restriction to each of its basic classes is aperiodic; otherwise, P is called *periodic*.

The existence of strictly positive eigenvectors, associated with the spectral radius $\sigma(P)$ of a square nonnegative matrix P, depends heavily on accessibility relations between basic and nonbasic classes. This is expressed by the following lemma which, moreover, lists a number of important properties of nonnegative matrices.

Lemma 2.1. Let P be a square nonnegative matrix with spectral radius σ. Then, the following properties hold:

(a) If P is irreducible and $x \geq \underline{0}$, then $Px \geqq \sigma x$ implies $Px=\sigma x$. Similarly, $Px \leqq \sigma x$ implies $Px=\sigma x$.

(b) P possesses a strictly positive right (left)-eigenvector if and only if its basic classes are precisely its final (initial) classes.

(c) If $x > \underline{0}$ and $Px \leqq \sigma x$, then each basic class C of P is final and $(Px)_i=\sigma x_i$, for $i \in C$.

(d) If $Px \geqq \lambda x$, for some real vector x with at least one positive component, then $\sigma \geqq \lambda$.

(e) If $x > \underset{\sim}{0}$ and $Px \geqq \sigma x$, then each final class C of P is basic and $(Px)_i = \sigma x_i$ for $i \in C$.

Proofs of these properties can be found in Ref. 9.

Matrices with strictly positive eigenvectors possess several nice properties which are used throughout this paper. We have the following lemma.

Lemma 2.2. Let P be nonnegative, with spectral radius $\sigma > 0$; and let there exist a strictly positive right-eigenvector, u say, associated with σ. Then, the following properties hold:

(a) There exists a nonnegative matrix P^*, defined by

$$P^* := \lim_{n\to\infty}[1/(n+1)] \sum_{k=0}^{n} \sigma^{-k} P^k. \tag{7}$$

We have

$$PP^* = P^*P = \sigma P^* \quad \text{and} \quad (P^*)^2 = P^*.$$

Furthermore, $p_{ij}^* > 0$, if and only if j belongs to a basic class of P and i has access to j under P (this implies that the restriction of P^* to a basic class of P is strictly positive). If P is aperiodic, then even

$$P^* = \lim_{n\to\infty} \sigma^{-n} P^n.$$

(b) If $P^* y = \underset{\sim}{0}$, for some $y \geq \underset{\sim}{0}$, then $y_i = 0$ for every state i belonging to a basic class of P.

(c) If $Px \geqq \sigma x$, for some x, then $P^* x \geqq x$.

Proof. For (a), see Ref. 18, p. 480. The proofs of (b) and (c) are left to the reader. □

The results stated above, and especially Lemma 2.1, indicate the importance of the position of basic and nonbasic classes of a square nonnegative matrix P. The definition of the position of a class can be made precise by introducing chains. A *chain* of classes of P is a collection of classes $\{C(1), C(2), \ldots, C(n)\}$ such that

$$P_{i_k j_k} > 0,$$

for some pair of states i_k, j_k with

$$i_k \in C(k), \quad j_k \in C(k+1), \qquad k = 1, 2, \ldots, n-1.$$

Such a chain starts with $C(1)$ and ends with $C(n)$. The *length* of a chain is the number of basic classes that it contains. The *depth* of a class C of P is the length of the longest chain starting with C.

Rothblum (Ref. 19) proved an important relationship between these concepts and the index of a nonnegative square matrix P. Let P have spectral radius σ. The *index* $\nu(P)$ of P with respect to σ is the smallest nonnegative integer k such that

$$N^k(P) = N^{k+1}(P),$$

where $N^k(P)$ denotes the null space of $(P-\sigma I)^k$, $k \in \mathbb{N}$. It is known (cf. Ref. 8) that

$$N^1(P) \subsetneqq N^2(P) \subsetneqq \cdots \subsetneqq N^\nu(P) = N^k(P), \qquad \text{for } k \geqq \nu, \tag{8}$$

where ν is the index of P. The elements of $N^k(P) \backslash N^{k-1}(P)$ are called *generalized eigenvectors* of order k. In Ref. 19, it is shown that the index of P equals the length of its longest chain. Recalling the definitions above, it follows that a square nonnegative matrix with index ν possesses classes with depth k, for $k = 1, 2, \ldots, \nu$, and possibly classes with depth 0 (nonbasic classes which do not have access to any basic class). Rothblum also proved a number of structural properties of generalized eigenvectors, associated with the spectral radius σ of P. His results have been extended in Ref. 9 to sets of nonnegative matrices with the product property. The results, obtained in Ref. 9, are basic for this paper. They are summarized in Lemma 2.3.

Lemma 2.3. Let $\mathscr{K}$ be a set of nonnegative square matrices with the product property; let

$$\hat{\sigma} := \max\{\sigma(P) \mid P \in \mathscr{K}\};$$

and let

$$\nu := \max\{\nu(P) \mid P \in \mathscr{K}, \sigma(P) = \hat{\sigma}\}.$$

Then, there exist a matrix $\hat{P} \in \mathscr{K}$, with $\sigma(\hat{P}) = \hat{\sigma}$ and $\nu(\hat{P}) = \nu$, a unique partition $\{D(0), D(1), \ldots, D(\nu)\}$ of S, with $D(0)$ possibly empty, and a sequence of generalized eigenvectors $w(1)$, $w(2), \ldots, w(\nu)$, such that the following properties hold:

(a) $$\max_{P \in \mathscr{K}} Pw(\nu) = \hat{P}w(\nu) = \hat{\sigma}w(\nu), \tag{9a}$$

$$\max_{P \in \mathscr{K}} Pw(k) = \hat{P}w(k) = \hat{\sigma}w(k) + w(k+1),$$

$$k = \nu - 1, \ldots, 2, 1. \tag{9b}$$

(b) $w(k)_i > 0, \quad \text{for } i \in \bigcup_{l=k}^{\nu} D(l),\ k = 1, \ldots, \nu,$ (10a)

$$w(k)_i = 0 \quad \text{for } i \in \bigcup_{l=0}^{k-1} D(l),\ k = 1, \ldots, \nu. \tag{10b}$$

(c) $P^{(k,l)} = \underset{=}{0} \quad \text{for } k < l;\ k, l = 0, 1, \ldots, \nu,\ P \in \mathscr{K},$

where $P^{(k,l)}$ denotes the restriction of P to $D(k) \times D(l)$.

(d) $\max_{P \in \mathscr{K}} \sigma(P^{(k,k)}) = \sigma(\hat{P}^{(k,k)}) = \hat{\sigma}, \qquad k = 1, 2, \ldots, \nu,$ (11a)

$$\max_{P \in \mathscr{K}} \sigma(P^{(0,0)}) < \hat{\sigma}, \qquad \text{if } D(0) \neq \phi. \tag{11b}$$

A complete proof of Lemma 2.3 can be found in Ref. 9. The partition $\{D(0), D(1), \ldots, D(\nu)\}$ is called the *principal partition* of S with respect to $\mathscr{K}$. In fact, $D(k)$ contains exactly all classes of $\hat{P}$ with depth k, $k = 0, 1, \ldots, \nu$.

For the special case that $\mathscr{K}$ contains exactly one matrix, the above results reduce to those of Rothblum (Ref. 19). If each $P \in \mathscr{K}$ is irreducible, we find that

$$\nu = 1 \quad \text{and} \quad D(0) = \phi,$$

and we arrive at

$$\max_{P \in \mathscr{K}} Pu = \hat{\sigma} u, \tag{12}$$

for some $u > \underset{=}{0}$ (cf. Ref. 11, Ref. 20). Block-triangular decompositions for sets of nonnegative matrices are also discussed by Sladky (cf. Ref. 13). A constructive method to obtain a matrix with maximal spectral radius and maximal index has been published by Rothblum and Whittle (Ref. 21) and Zijm (Ref. 7, Chapter 3).

The fact that $w(1)$ has only zero components on $D(0)$, whereas all other components are positive, is of direct importance for dynamic programming recursions of type (1). It can be shown easily (and it will be done in Sections 4 and 5) that

$$\begin{bmatrix} n \\ \nu - 1 \end{bmatrix}^{-1} \hat{\sigma}^{-n} x(n) \leqq ce, \qquad \text{for all } n \in \mathbb{N}, \tag{13}$$

for some constant c and $\hat{\sigma}$, ν defined as in Lemma 2.3. The following result will be helpful in proving (13).

Lemma 2.4. Let $\mathscr{K}$ be a set of nonnegative square matrices with the product property, and let

$$\hat{\sigma} = \max\{\sigma(P) \mid P \in \mathscr{K}\}.$$

Then, there exists, for each $\lambda > \hat{\sigma}$, a vector $w(\lambda) > \underset{\sim}{0}$ such that

$$\max_{P \in \mathscr{K}} Pw(\lambda) < \lambda w(\lambda).$$

Proof. Take

$$w(\lambda) := \max_{P \in \mathscr{K}} (\lambda I - P)^{-1} e,$$

the maximum being taken componentwise. □

However, instead of (13), stronger results can be obtained. Starting with Lemma 2.3, we will establish the existence of polynomial expansions for $x(n)$, for $n \to \infty$, first in the case $\nu = 1$, and then, in Section 5, for any possible value of ν. Before presenting these results, we first have to discuss geometric convergence in Markov decision chains.

3. Geometric Convergence in Stochastic Models

Our main results concerning the asymptotic behavior of dynamic programming recursions of type (1) will be based heavily on similar results for Markov decision chains. Several authors (e.g., Refs. 22, 23, 24, 33) studied the asymptotic behavior of the n-period reward for Markov decision chains. Geometric convergence in undiscounted multichain MDC was studied initially by Schweitzer and Federgruen (Ref. 10). Unfortunately, the analysis, given in Ref. 10, is extremely complicated and, moreover, not suitable for direct application to the recursions studied in this paper. Therefore, this section is devoted to a new, and relatively simple, proof of geometric convergence in undiscounted Markov decision chains.

Consider a finite set $\mathscr{K}$ of stochastic $N \times N$ matrices. With each $P \in \mathscr{K}$ is associated a sequence of vectors $\{r(P), r(1, P), r(2, P), \ldots\}$, such that the set

$$\{(P, r(P), r(1, P), r(2, P), \ldots) \mid P \in \mathscr{K}\} \tag{14}$$

possesses the product property. Furthermore, assume that

$$\|r(n, P) - r(P)\| \leqq c\delta^n, \qquad \text{for all } P \in \mathscr{K},\ n \in \mathbb{N}, \tag{15}$$

for some δ, $0 \leqq \delta < 1$, and some $c > 0$.

In the next two sections, it turns out that a detailed analysis of the general dynamic programming recursions of type (1) ultimately leads to a study of the following recursion:

$$v(n) = \max_{P \in \mathcal{K}} \{r(n, P) + Pv(n-1)\}, \qquad n = 1, 2, \ldots, \tag{16}$$

with $v(0)$ fixed.

This recursion also arises in the study of value iteration in undiscounted MDC. In order to facilitate the analysis in the next two sections, we present a separate treatment here.

Lemma 3.1. Let $\mathcal{K}$ be a finite set of stochastic, aperiodic $N \times N$ matrices, and let vectors $r(P)$, $r(1, P)$, $r(2, P), \ldots$ be given such that the set of rectangular matrices (14) possesses the product property. Suppose furthermore that $v(n)$, defined recursively by (16), is bounded uniformly in n. Finally, let (15) hold. Then, $\lim_{n \to \infty} v(n)$ exists.

Proof. Since $v(n)$ is bounded, we may define finite-valued vectors a, b such that

$$b_i := \limsup_{n \to \infty} v(n)_i, \qquad i \in S,$$

$$a_i := \liminf_{n \to \infty} v(n)_i, \qquad i \in S.$$

Choose $i \in S$ fixed. Let $n_1, n_2, \ldots$ be a sequence such that

$$\lim_{k \to \infty} v(n_k + 1)_i = b_i,$$

and such that

$$x := \lim_{k \to \infty} v(n_k)$$

exists. Then, $x \leqq b$. Putting $n = n_k$ in (16) and letting $k \to \infty$, we obtain [use (15)]

$$b_i \leqq \max_{P \in \mathcal{K}} \{r(P) + Pb\}_i.$$

Repeating the same procedure for each $i \in S$, we get

$$b \leqq \max_{P \in \mathcal{K}} \{r(P) + Pb\} = r(\hat{P}) + \hat{P}b, \qquad \text{for some } \hat{P} \in \mathcal{K}. \tag{17}$$

Similarly, we get

$$a \geqq \max_{P \in \mathcal{K}} \{r(P) + Pa\} \geqq r(\hat{P}) + \hat{P}a, \tag{18}$$

with $\hat{P}$ chosen as in (17).

Repeated substitution in (17) leads to

$$b \leqq \sum_{k=0}^{m-1} \hat{P}^k r(\hat{P}) + \hat{P}^m b, \qquad \text{for all } m \in \mathbb{N}.$$

On the other hand, (16) yields

$$v(n+m) \geqq \sum_{k=0}^{m-1} \hat{P}^k r(n+m-k, \hat{P}) + \hat{P}^m v(n), \qquad \text{for all } n, m \in \mathbb{N}.$$

Combining these two results with (15), we obtain

$$b - \hat{P}^m b \leqq v(n+m) - \hat{P}^m v(n) + c\delta^n \sum_{k=0}^{m-1} \delta^{m-k} e. \tag{19}$$

Take $n \in \mathbb{N}$ fixed. As before, we may define, for each $i \in S$, a sequence $\{m_{k,i};\ k = 1, 2, \ldots\}$, such that

$$\lim_{k \to \infty} v(n + m_{k,i})_i = a_i,$$

which, together with (19), leads to (apply Lemma 2.2)

$$b - \hat{P}^* b \leqq a - \hat{P}^* v(n) + c\delta^{n+1}(1-\delta)^{-1} e. \tag{20}$$

Clearly, (20) can be derived for any $n \in \mathbb{N}$. Let now x be an arbitrary limit point of $v(n)$. Then, $a \leqq x$. By choosing an appropriate subsequence, we find that

$$b - \hat{P}^* b \leqq a - \hat{P}^* x. \tag{21}$$

Multiplication with $\hat{P}^*$ yields

$$\underset{\sim}{0} = \hat{P}^*(b - \hat{P}^* b) \leqq \hat{P}^*(a - \hat{P}^* x) = \hat{P}^*(a - x).$$

However, since

$$a - x \leqq \underset{\sim}{0},$$

we have

$$\hat{P}^*(a - x) = \underset{\sim}{0},$$

and hence $a_i = x_i$ for every state i belonging to a basic class of $\hat{P}$. Since x was chosen arbitrarily, this implies that $b_i = a_i$ for every state i belonging to a basic class of $\hat{P}$.

Finally, let E be the set of all states which are not contained in any basic class of $\hat{P}$. From (17) and (18), we deduce that

$$(b - a) \leqq \hat{P}(b - a); \tag{22}$$

and, since $a_i = b_i$ for $i \in S \backslash E$, this reduces to

$$b^E - a^E \leqq \hat{P}^E (b^E - a^E), \tag{23}$$

which implies

$$b^E - a^E \leqq \underset{\sim}{0},$$

since $\sigma(\hat{P}^E) < 1$ (use Lemma 2.1). Hence, $b = a$, which proves the theorem. □

Remark 3.1. Taking limits in (16) yields

$$v = \max_{P \in \mathscr{K}} \{r(P) + Pv\},$$

from which we easily deduce that

$$\max_{P \in \mathscr{K}} P^* r(P) = \underset{\sim}{0}.$$

In general, it can be shown that, if we omit the boundedness assumption in the formulation of Lemma 3.1, there exists a vector w^* such that

$$\lim_{n \to \infty} (v(n) - ng^*) = w^*,$$

where $v(n)$ obeys (16) and g^* is defined by

$$g^* := \max_{P \in \mathscr{K}} P^* r(P). \tag{24}$$

This result is well known if $r(n, P) = r(P)$ for all n, P; see, e.g., Derman (Ref. 31). A proof of the more general result is contained in Ref. 7, Appendix 4A.

The next lemma provides the basis for the proof of the geometric convergence of $v(n)$, for $n \to \infty$.

Lemma 3.2. Let $\mathscr{K}$ be a finite set of stochastic, aperiodic $N \times N$ matrices with the product property. Choose $z(0)$ fixed, and define $z(n)$ recursively by

$$z(n) = \max_{P \in \mathscr{K}} Pz(n-1), \qquad n = 1, 2, \ldots. \tag{25}$$

Then, $\lim_{n \to \infty} z(n)$ exists. Denote this limit by z. Then, the convergence of $z(n)$ to z, for $n \to \infty$, is geometric.

Proof. Clearly,

$$\|z(n)\| \leqq \|z(0)\|, \qquad \text{for all } n;$$

hence, we may apply Lemma 3.1, with

$$r(n, P) = \underline{0} \qquad \text{for all } n, P.$$

Let

$$z := \lim_{n\to\infty} z(n).$$

Then, obviously,

$$\max_{P\in\mathcal{K}} Pz = z,$$

by taking limits in (25). Let

$$\mathcal{K}(1) := \{P \in \mathcal{K} \mid Pz = z\}.$$

Since $\mathcal{K}$ is finite, there exists a positive integer n_0 such that

$$z(n) = \max_{P\in\mathcal{K}(1)} Pz(n-1), \qquad \text{for } n \geqq n_0.$$

Let

$$w(n) := z(n) - z, \qquad n \in \mathbb{N}.$$

Then,

$$w(n) = \max_{P\in\mathcal{K}(1)} Pw(n-1), \qquad n \geqq n_0, \tag{26}$$

$$\lim_{n\to\infty} w(n) = \underline{0}. \tag{27}$$

Hence, it suffices to show that the convergence in (27) is geometric. With respect to $w(n)$, we will prove that, for some fixed integer $m > 0$ and for some ϵ, $0 \leqq \epsilon < 1$, the following two assertions hold:

(i) $0 \leqq \max_{i\in S} w(n+m)_i \leqq (1-\epsilon) \max_{i\in S} w(n)_i, \qquad \text{for } n \geqq n_0;$

(ii) $0 \geqq \min_{i\in S} w(n+m)_i \geqq (1-\epsilon) \min_{i\in S} w(n)_i, \qquad \text{for } n \geqq n_0.$

We will use only (26) and (27) to prove assertions (i) and (ii). Together, these two assertions imply that the convergence in (27) is geometric.

Proof of (i). If

$$\max_{i\in S} w(n)_i = \alpha < 0,$$

then, for all i,

$$w(n+1)_i \leqq \alpha \max_{P \in \mathcal{K}} (Pe)_i = \alpha;$$

therefore, for all m,

$$w(n+m)_i \leqq \alpha < 0,$$

contradicting (27). This proves the first inequality in (i). Now, define

$$C(n) := \{i \in S \mid w(n)_i > 0\},$$

and suppose that $C(n_0) \neq \Phi$, otherwise the result holds trivially with $\epsilon = 1$. Define $R(n_0) = C(n_0)$ and, for $n \geqq n_0$, recursively,

$$R(n) := \left\{ i \in S \mid \exists P \in \mathcal{K}(1): \sum_{j \in R(n-1)} p_{ij} = 1 \right\}.$$

Choose $i \in R(n_0+1)$, and let P be such that

$$\sum_{j \in R(n_0)} p_{ij} = 1.$$

Then,

$$w(n_0+1)_i \geqq \sum_{j \in S} p_{ij} w(n_0)_j = \sum_{j \in R(n_0)} p_{ij} w(n_0)_j > 0;$$

therefore,

$$R(n_0+1) \subset C(n_0+1).$$

By induction, we obtain

$$R(n) \subset C(n), \qquad \text{for all } n \geqq n_0.$$

Let $m := 2^N$ (recall that N is the dimension of the state space S). If

$$R(n) \neq \Phi, \qquad \text{for } n = n_0 + m,$$

then

$$R(k) = R(l), \qquad \text{for some } k, l \in \mathbb{N}, \text{ with } n_0 \leqq k < l \leqq n_0 + m,$$

since there exist at most $m-1$ nonempty subsets of S. Define

$$R := R(k) = R(l).$$

By definition of $R(n)$, there exists a finite sequence of matrices $P(k+1)$, $P(k+2), \ldots, P(l)$ such that, for Q defined by

$$Q := P(l)P(l-1) \cdots P(k+1),$$

we have

$$\sum_{j\in R} q_{ij} = 1, \qquad \text{for } i \in R.$$

Let

$$\delta := \min_{i\in R} w(k)_i.$$

Then, as

$$R = R(k) \subset C(k),$$

we have that $\delta > 0$. From (26) we derive

$$w(l)_i \geqq \sum_{j\in S} q_{ij} w(k)_j = \sum_{j\in R} q_{ij} w(k)_j \geqq \delta, \qquad \text{for all } i \in R;$$

and, by induction,

$$w(k+n(l-k))_i \geqq \delta > 0, \qquad \text{for all } i \in R,\ n \in \mathbb{N},$$

contradicting (27). Hence,

$$R(n) = \Phi, \qquad \text{for } n = n_0 + m.$$

Let now $\epsilon(P(1)\cdots P(m))$ be defined as the smallest positive entry of the matrix $P(1)\cdots P(m)$; and let

$$\epsilon := \min\{\epsilon(P(1)P(2)\cdots P(m)) \mid P(1), P(2), \ldots, P(m) \in \mathcal{K}(1)\}. \tag{28}$$

Then, $0 < \epsilon \leqq 1$. For $n = n_0 + 1, \ldots, n_0 + m$, choose $P(n)$ such that

$$w(n) = P(n) w(n-1),$$

and let

$$T := P(n_0 + m) \cdots P(n_0 + 1).$$

Since

$$R(n_0 + m) = \Phi,$$

we have, for all $i \in S$,

$$\begin{aligned} w(n_0+m)_i &= \sum_{j\in S} t_{ij} w(n_0)_j \leqq \sum_{j\in R(n_0)} t_{ij} w(n_0)_j \\ &\leqq \sum_{j\in R(n_0)} t_{ij} \max_{h\in R(n_0)} w(n_0)_h \leqq (1-\epsilon) \max_{h\in R(n_0)} w(n_0)_h \\ &= (1-\epsilon) \max_{h\in S} w(n_0)_h. \end{aligned}$$

Hence,

$$\max_{i\in S} w(n_0+m)_i \leqq (1-\epsilon) \max_{i\in S} w(n_0)_i.$$

Similar results are obtained when starting with $C(n_0+1)$, $C(n_0+2), \ldots$. Since m and ϵ do not depend on n_0, assertion (i) follows.

Proof of (ii). If

$$\min_{i\in S} w(n+m)_i > 0,$$

then (27) would be violated (by arguments similar to those given above); hence, the first inequality in (ii) must hold. Define

$$D(n) := \{i \in S \mid w(n)_i < 0\}.$$

If $D(n_0) = \Phi$, nothing is left to prove. Hence, suppose that $D(n_0) \neq \Phi$. Define $U(n_0) = D(n_0)$ and, for $n > n_0$, recursively,

$$U(n) := \left\{ i \in S \mid \sum_{j \in U(n-1)} p_{ij} = 1, \text{ for all } P \in \mathcal{K}(1) \right\};$$

note the difference in the definitions of $R(n)$ and $U(n)$. Then, $U(n) \subset D(n)$ and

$$U(n) = \Phi, \qquad \text{for } n = n_0 + 2^N,$$

by arguments similar to those given in the proof of (i). Again, let

$$m := 2^N.$$

Since

$$U(n_0 + m) = \Phi,$$

there exists a sequence $P(1), P(2), \ldots, P(m)$ such that the matrix $\hat{Q}$, defined by

$$\hat{Q} := P(m)P(m-1) \cdots P(1),$$

obeys the relation

$$\sum_{j \notin U(n_0)} \hat{q}_{ij} > 0, \qquad \text{for all } i \in S.$$

Defining ϵ as in (28), it follows that

$$\min_{i \in S} w(n_0 + m)_i \geqq (1-\epsilon) \min_{i \in S} w(n_0)_i.$$

Since m and ϵ do not depend on n_0, assertion (ii) follows.

Combination of (i) and (ii) yields

$$\left\{ \max_{i \in S} w(n_0 + mk)_i - \min_{i \in S} w(n_0 + mk)_i \right\}$$

$$\leqq (1-\epsilon)^k \left\{ \max_{i \in S} w(n_0)_i - \min_{i \in S} w(n_0)_i \right\}, \qquad k \in \mathbb{N}.$$

This completes the proof of Lemma 3.2, since $0 < \epsilon \leqq 1$. □

Once having proved Lemma 3.2, it is relatively easy to show the geometric convergence of the recursion (16).

Lemma 3.3. Let the assumptions of Lemma 3.1 hold. Then, the convergence established in Lemma 3.1 is geometric.

Proof. Let

$$v := \lim_{n\to\infty} v(n).$$

By taking limits in (16), we obtain

$$v = \max_{P\in\mathcal{K}} \{r(P) + Pv\}.$$

Let

$$\mathcal{K}(1) := \{P \in \mathcal{K} \mid r(P) + Pv = v\}.$$

There exists some $n_0 \in \mathbb{N}$ such that [use (15)]

$$v(n) = \max_{P\in\mathcal{K}(1)} \{r(n, P) + Pv(n-1)\}, \qquad \text{for } n \geqq n_0.$$

Define

$$w(n) := v(n) - v.$$

By definition of $\mathcal{K}(1)$, we find that

$$w(n) = \max_{P\in\mathcal{K}(1)} \{(r(n, P) - r(P)) + Pw(n-1)\}, \qquad \text{for } n \geqq n_0,$$

$$\lim_{n\to\infty} w(n) = \underline{0}.$$

Choose $k \geqq n_0$, and define $w(k, n)$ by

$$w(k, n) = \max_{P\in\mathcal{K}(1)} Pw(k, n-1), \qquad n = 1, 2, \ldots,$$

with $w(k, 0) = w(k)$. By Lemma 3.2, $w(k, n)$ converges geometrically to some vector $\hat{w}(k)$, as $k \to \infty$. Hence,

$$\|w(k, n) - \hat{w}(k)\| \leqq \alpha\rho^n, \qquad n = 1, 2, \ldots, \tag{29}$$

where $\alpha > 0$, $0 \leqq \rho < 1$. Note that α and ρ can be chosen independent of k (compare the proof of Lemma 3.2). From (15), we derive

$$\|w(k+n) - w(k, n)\| \leqq c\delta^k \sum_{l=1}^{n} \delta^l \leqq c\delta^{k+1}(1-\delta)^{-1}, \qquad n = 1, 2, \ldots; \tag{30}$$

hence, for $n \to \infty$,

$$\|\hat{w}(k)\| \leqq c\delta^{k+1}(1-\delta)^{-1}. \tag{31}$$

This argument can be repeated for any $k \geqq n_0$. Combination of (29), (30), (31) yields

$$\|w(k+n)\| \leqq \|w(k+n) - w(k, n)\| + \|w(k, n) - \hat{w}(k)\| + \|\hat{w}(k)\|$$
$$\leqq c\delta^{k+1}(1-\delta)^{-1} + \alpha\rho^n + c\delta^{k+1}(1-\delta)^{-1};$$

in particular,

$$\|w(2n)\| \leqq 2c\delta^{n+1}(1-\delta)^{-1} + \alpha\rho^n.$$

Since

$$|w(2n+1)_i| \leqq \max_{P \in \mathscr{K}(1)} \sum_j p_{ij} |w(2n)_j| \leqq \max_{P \in \mathscr{K}(1)} \sum_j p_{ij} \|w(2n)\| = \|w(2n)\|,$$

and therefore

$$\|w(2n+1)\| \leqq \|w(2n)\|,$$

the geometric convergence of $w(n)$, for $n \to \infty$, is proved. □

The next lemma relaxes the stochasticity assumption.

Lemma 3.4. The conclusions of the preceding three lemmas remain valid if the stochasticity assumption is relaxed to

$$\max_{P \in \mathscr{K}} Pe = e,$$

and aperiodicity is assumed only for those matrices P with $\sigma(P) = 1$.

Proof. The proof follows easily by extending all matrices to be stochastic by adding a single absorbing state. Consider, for example, Lemma 3.1. Define

$$\tilde{S} := S \cup \{N+1\} = \{1, 2, \ldots, N, N+1\};$$

and define $\tilde{P}$ by

$$\tilde{p}_{ij} = p_{ij}, \qquad j = 1, 2, \ldots, N,$$

$$\tilde{p}_{i,N+1} = 1 - \sum_{j=1}^{N} p_{ij}, \qquad i = 1, 2, \ldots, N,$$

$$\tilde{p}_{N+1,j} = 0, \qquad j = 1, 2, \ldots, N,$$

$$\tilde{p}_{N+1,N+1} = 1.$$

Let, furthermore,

$$\tilde{r}(n, \tilde{P})_i = r(n, P)_i, \qquad \text{for all } n \text{ and } P,$$

$$\tilde{v}(0)_i = v(0)_i, \qquad i = 1, 2, \ldots, N,$$

$$\tilde{r}(n, \tilde{P})_{N+1} = 0, \qquad \text{for all } n \text{ and } P,$$

$$\tilde{v}(0)_{N+1} = 0.$$

Then, applying the dynamic programming recursion (16) to the extended model, it follows easily that

$$\tilde{v}(n)_i = v(n)_i, \qquad \text{for all } n,\ i = 1, 2, \ldots, N,$$

and

$$\tilde{v}(n)_{N+1} = 0, \qquad \text{for all } n.$$

Application of Lemma 3.1 to the extended model now yields the desired result. A similar argument applies to Lemma 3.2 and Lemma 3.3. □

The main result of this section, Theorem 3.1, is in fact a further generalization of the preceding results. However, it is this theorem that will be exploited extensively in the next two sections, when analyzing dynamic programming recursions of type (1).

Theorem 3.1. Let $\mathcal{K}$ be a finite set of nonnegative $N \times N$ matrices, and let vectors $r(P)$, $r(1, P)$, $r(2, P), \ldots$ be given such that the set of rectangular matrices (14) possesses the product property. Suppose that

$$\max_{P \in \mathcal{K}} Pw = \hat{\sigma} w,$$

for some strictly positive vector w, where

$$\hat{\sigma} = \max\{\sigma(P) \mid P \in \mathcal{K}\} > 0.$$

Let all matrices P with $\sigma(P) = \hat{\sigma}$ be aperiodic. Suppose also that

$$\|\hat{\sigma}^{-n} r(n, P) - r(P)\| \leqq c\delta^n, \qquad \text{for all } P \in \mathcal{K},\ n \in \mathbb{N},$$

where $0 \leqq \delta < 1$ and $c > 0$. Let finally $v(n)$ be defined by (16) again and suppose that

$$\max_n \|\hat{\sigma}^{-n} v(n)\| < \infty.$$

Then, $\hat{\sigma}^{-n} v(n)$ converges geometrically to some vector v, for $n \to \infty$.

Proof. Define $\bar{v}(n)$, $\bar{P}$, $\bar{r}(n, P)$, $\bar{r}(P)$ by

$$\bar{v}(n)_i = \hat{\sigma}^{-n} w_i^{-1} v(n)_i, \qquad i \in S,\ n \in \mathbb{N},$$

$$\bar{r}(P)_i = w_i^{-1} r(P)_i, \qquad i \in S,\ P \in \mathcal{K},$$

$$\bar{r}(n, P)_i = \hat{\sigma}^{-n} w_i^{-1} r(n, P)_i, \qquad i \in S,\ P \in \mathcal{K},\ n \in \mathbb{N},$$

$$\bar{p}_{ij} = \hat{\sigma}^{-1} w_i^{-1} p_{ij} w_j, \qquad i, j \in S,\ p \in \mathcal{K}.$$

The result now follows easily by applying Lemma 3.3 and Lemma 3.4.

4. Asymptotic Expansions of $x(n)$: Case $\nu = 1$

Throughout the rest of this paper, $\mathscr{K}$ denotes a finite set of nonnegative square matrices with the product property. By $\hat{\sigma}$ we denote the maximal spectral radius, i.e.,

$$\hat{\sigma} = \max\{\sigma(P) \mid P \in \mathscr{K}\};$$

and by ν we denote the maximal index among the indices of those matrices with maximal spectral radius, i.e.,

$$\nu = \max\{\nu(P) \mid P \in \mathscr{K}, \hat{\sigma}(P) = \hat{\sigma}\}.$$

In order to avoid complicating technical difficulties, we assume $\hat{\sigma} > 0$ for the rest of this paper. The case $\hat{\sigma} = 0$ is treated separately at the end of Section 5. A sensitive analysis of the case $\hat{\sigma} = 0$ is contained in Rothblum (Ref. 32).

In this section, we suppose $\nu = 1$; hence, the principal partition of S with respect to $\mathscr{K}$ can be denoted by $\{D(0), D(1)\}$. The aim of this section is to show the convergence of $\hat{\sigma}^{-n}x(n)$, for $n \to \infty$, where $x(n)$ is defined by (1). First, we need boundedness.

Lemma 4.1. Let $\{D(0), D(1)\}$ be the principal partition of S with respect to $\mathscr{K}$, and let $x(n)$ be given by (1), for $n = 0, 1, 2, \ldots$, where $x(0) > \underline{0}$. Then,

$$\lim_{n\to\infty} \hat{\sigma}^{-n}x(n)_i = 0, \qquad \text{for } i \in D(0), \tag{32}$$

$$\beta w(1)_i \leqq \hat{\sigma}^{-n}x(n)_i \leqq \gamma w(1)_i, \qquad \text{for } n \in \mathbb{N},\ i \in D(1), \tag{33}$$

with $w(1)$ as defined in Lemma 2.3 and β, γ some positive constants.

Proof. Let $P^{(k,l)}$ denote the restriction of P to $D(k) \times D(l)$, for k, $l = 0, 1$. Then,

$$P^{(0,1)} = \underline{0}, \qquad \text{for all } P \in \mathscr{K}.$$

By Lemma 2.3(d) and Lemma 2.4, there exist a positive real number λ, with $\lambda < \hat{\sigma}$, and some vector $w^{(0)} > \underline{0}$, defined on $D(0)$, such that

$$\max_{P\in\mathscr{K}} P^{(0,0)}w^{(0)} \leqq \lambda w^{(0)}.$$

We note in passing that, by appropriate scaling, it is possible to choose $w^{(0)}$ such that

$$\max_{P\in\mathscr{K}} (P^{(1,0)}w^{(0)})_i \leqq w(1)_i, \qquad \text{for } i \in D(1), \tag{34}$$

where $w(1)$ is defined as in Lemma 2.3; note that

$$w(1)_i > 0, \qquad \text{for } i \in D(1).$$

Choose $\alpha > 0$, such that

$$x(0)_i \leqq \alpha w_i^{(0)}, \qquad \text{for } i \in D(0),$$

$$x(0)_i \leqq \alpha w(1)_i, \qquad \text{for } i \in D(1).$$

By induction, we find that

$$x(n)_i \leqq \alpha \lambda^n w_i^{(0)}, \qquad \text{for } n \in \mathbb{N},\ i \in D(0), \tag{35}$$

which establishes (32) immediately, since $\lambda < \hat{\sigma}$. In fact, (35) implies that the convergence in (32) is geometric.

Let $\beta > 0$ be such that

$$\beta w(1) \leqq x(0),$$

and recall that

$$\max_{P \in \mathscr{K}} P w(1) = \hat{\sigma} w(1).$$

Then, the first inequality of (33) follows trivially.

Furthermore, the choice of α, together with (34), implies that

$$x(1)_i = \max_{P \in \mathscr{K}} \left\{ \sum_{j \in D(1)} p_{ij} x(0)_j + \sum_{j \in D(0)} p_{ij} x(0)_j \right\}$$

$$\leqq \alpha \hat{\sigma} w(1)_i + \alpha w(1)_i = \alpha(\hat{\sigma} + 1) w(1)_i, \qquad \text{for } i \in D(1);$$

and, by induction [use (34) and (35)],

$$x(n)_i \leqq \alpha \left\{ \hat{\sigma}^n + \sum_{k=0}^{n-1} \hat{\sigma}^{n-1-k} \lambda^k \right\} w(1)_i$$

$$= \alpha \hat{\sigma}^n \left\{ 1 + \hat{\sigma}^{-1} \sum_{k=0}^{n-1} (\lambda \hat{\sigma}^{-1})^k \right\} w(1)_i \leqq \gamma \hat{\sigma}^n w(1)_i, \qquad i \in D(1),$$

where we choose

$$\gamma = \alpha(1 + (\hat{\sigma} - \lambda)^{-1}). \qquad \square$$

Once having boundedness, the convergence of $\{\hat{\sigma}^{-n} x(n); n = 0, 1, 2, \ldots\}$ follows almost immediately from the results of the preceding section. We have the following theorem.

Theorem 4.1. Let $\{D(0), D(1)\}$ be the principal partition of S with respect to $\mathscr{K}$, and let $x(n)$ be given by (1), for $n = 0, 1, 2, \ldots$, where $x(0) > \underline{0}$.

Suppose that each $P \in \mathscr{K}$, for which $\sigma(P) = \hat{\sigma}$, is aperiodic. Then, there exists a vector x, with

$$x_i = 0, \qquad \text{for } i \in D(0),$$

$$x_i > 0, \qquad \text{for } i \in D(1),$$

such that

$$\lim_{n \to \infty} \hat{\sigma}^{-n} x(n) = x, \tag{36}$$

and this convergence is geometric. Furthermore,

$$\max_{P \in \mathscr{K}} Px = \hat{\sigma}x. \tag{37}$$

Proof. The geometric convergence of $\hat{\sigma}^{-n}x(n)^{(0)}$ to $\underline{0}$, for $n \to \infty$, was already established in the proof of Lemma 4.1 [compare (35)]. Furthermore, we may write

$$x(n)^{(1)} = \max_{P \in \mathscr{K}} \{P^{(1,1)}x(n-1)^{(1)} + P^{(1,0)}x(n-1)^{(0)}\}, \qquad n \in \mathbb{N}. \tag{38}$$

Note that, since

$$w(1)_i = 0, \qquad \text{for } i \in D(0),$$

we have

$$\max_{P \in \mathscr{K}} P^{(1,1)} w(1)^{(1)} = \hat{\sigma} w(1)^{(1)},$$

where

$$w(1)^{(1)} > \underline{0}.$$

The geometric convergence of $\hat{\sigma}^{-n}x(n)^{(1)}$, for $n \to \infty$, to some vector $x^{(1)}$ now follows immediately from Lemma 4.1 and Theorem 3.1. Note that (33) implies that

$$x^{(1)} > \underline{0}.$$

Multiplying (1) with $\hat{\sigma}^{-n}$ and taking limits yields (37), since

$$x^{(0)} = \underline{0}.$$ □

Partial results in the case $\nu = 1$ have also been published by Sladky (Refs. 10, 11). In fact, he proved convergence of $\hat{\sigma}^{-n}x(n)$, for $n \to \infty$, without mentioning the fact that this convergence appears to be geometric.

Without proof, we state a periodic analogue of Theorem 4.1. Define

$$\tilde{\mathscr{K}} := \{P \in \mathscr{K} \mid \sigma(P) = \hat{\sigma}\}.$$

Now, let some $P \in \tilde{\mathscr{K}}$ have k basic classes, $B(1), \ldots, B(k)$ say, and let $d_l(P)$ denote the period of $P^{B(l)}$, for $l = 1, 2, \ldots, k$. Define $d(P)$ as the least common multiple of $\{d_1(P), \ldots, d_k(P)\}$, for each $P \in \tilde{\mathscr{K}}$. Finally, let

$$d := \text{g.c.d.}\{d(P) \mid P \in \tilde{\mathscr{K}}\}.$$

Then, the following result can be formulated.

Theorem 4.2. Let $\{D(0), D(1)\}$ be the principal partition of S with respect to $\mathscr{K}$, and let $\tilde{\mathscr{K}}$, d be defined as above. Then, there exist vectors $w(l) \geq \underline{0}$, $l = 0, 1, \ldots, d-1$, such that, for $x(n)$ defined by (1), the following holds:

$$\lim_{k\to\infty} \hat{\sigma}^{-(l+kd)} x(l+kd) = w(l), \qquad \text{for } l = 0, 1, \ldots, d-1. \tag{39}$$

Furthermore, we have

$$\max_{P\in\mathscr{K}} Pw(l) = \hat{\sigma} w(l+1), \qquad \text{for } l = 0, 1, \ldots, d-1,$$

where $w(d) := w(0)$. Finally, the convergence, established in (39), is geometric.

In Section 5, more general results concerning the asymptotic behavior of $x(n)$ will be proved. The results of this section (in particular Theorem 4.1) will serve as a first step in an inductive argument.

5. Asymptotic Expansions of $x(n)$: General Case

In this section, it is shown that an asymptotic expansion, similar to (4), can be obtained for $x(n)$, when ν is arbitrary. As mentioned already in Section 4, we assume $\hat{\sigma} > 0$ (at the end of this section, the case $\hat{\sigma} = 0$ is discussed briefly). The main result of this paper reads as follows.

Theorem 5.1. Let $\{D(0), D(1), \ldots, D(\nu)\}$ be the principal partition of S with respect to $\mathscr{K}$. Suppose that all matrices $P \in \mathscr{K}$, with $\sigma(P) = \hat{\sigma}$ and $\nu(P) = \nu$, are aperiodic. Then, there exist unique vectors $y(1), y(2), \ldots, y(\nu)$ and constants $c > 0$, $\rho < \hat{\sigma}$, such that $x(n)$, defined by (1), obeys

$$\left\| x(n) - \left\{ \binom{n}{\nu-1} \hat{\sigma}^{n-\nu+1} y(\nu) + \cdots + \binom{n}{1} \hat{\sigma}^{n-1} y(2) + \hat{\sigma}^n y(1) \right\} \right\| \leqq c\rho^n, \qquad \text{for all } n \in \mathbb{N}. \tag{40}$$

These vectors $y(k)$ satisfy

$$y(k)_i > 0, \qquad i \in D(k), \qquad k = 1, \ldots, \nu, \tag{41a}$$

$$y(k)_i = 0, \qquad i \in \bigcup_{l=0}^{k-1} D(l), \qquad k = 1, \ldots, \nu. \tag{41b}$$

Furthermore, the following relationships hold:

$$\max_{P\in\mathscr{K}} Py(\nu)=\hat{\sigma}y(\nu), \tag{42a}$$

$$\max_{P\in\mathscr{K}(l+1)} Py(l)=\hat{\sigma}y(l)+y(l+1), \qquad l=\nu-1,\ldots,2,1, \tag{42b}$$

with

$$\mathscr{K}(\nu):=\{P\in\mathscr{K}\mid Py(\nu)=\hat{\sigma}y(\nu)\}, \tag{43a}$$

$$\mathscr{K}(l):=\{P\in\mathscr{K}(l+1)\mid Py(l)=\hat{\sigma}y(l)+y(l+1)\},$$
$$l=\nu-1,\ldots,2,1. \tag{43b}$$

The proof of Theorem 5.1 will be given by an inductive argument. A key role in the analysis is played by the following technical result, which will be proved before we verify Theorem 5.1.

Theorem 5.2. Let $\{D(0), D(1),\ldots, D(\nu)\}$ be the principal partition of S with respect to $\mathscr{K}$, and suppose that all $P\in\mathscr{K}$, with $\sigma(P)=\hat{\sigma}$ and $\nu(P)=\nu$, are aperiodic. Let $y(1), y(2),\ldots, y(\nu)$ be vectors satisfying (41) and (42), and moreover

$$-c_1\delta^n \leqq x(n)_i-\sum_{l=1}^{\nu-1}\binom{n}{l-1}\hat{\sigma}^{n-l+1}y(l)_i$$
$$\leqq c_1\delta^n, \qquad \text{for } i\in S\backslash D(\nu),\ n\in\mathbb{N}, \tag{44}$$

for some $c_1>0$ and some δ, $0\leqq\delta<\hat{\sigma}$. Then, there exists a vector $\hat{y}(1)$, with

$$\hat{y}(1)_i=y(1)_i, \qquad \text{for } i\in S\backslash D(\nu),$$

such that

$$-c_2\rho^n\leqq x(n)_i-\left\{\sum_{l=2}^{\nu}\binom{n}{l-1}\hat{\sigma}^{n-l+1}y(l)_i+\hat{\sigma}^n\hat{y}(1)_i\right\}\leqq c_2\rho^n,$$
$$\text{for } i\in S\backslash D(\nu),\ n\in\mathbb{N}, \tag{45}$$

for some $c_2>c_1$ and some ρ, $\delta\leqq\rho\leqq\hat{\sigma}$. Moreover, the set of vectors $\hat{y}(1)$, $y(2),\ldots, y(\nu)$ also satisfies (41) and (42).

Proof. In order to simplify notations, we assume $\hat{\sigma}=1$. Define

$$z(n):=\begin{bmatrix} n \\ \nu-1\end{bmatrix}y(\nu)+\cdots+\begin{bmatrix} n\\ 1\end{bmatrix}y(2)+y(1), \qquad n\in\mathbb{N}.$$

The conditions of the theorem state that $(x(n)-z(n))$ converges geometrically to zero on $S\backslash D(\nu)$, for $n\to\infty$; note that $y(\nu)_i=0$, for $i\in S\backslash D(\nu)$. We are concerned with the behavior of $(x(n)-z(n))$ on $D(\nu)$.

First, we establish boundedness on $D(\nu)$. Note that (42) implies that

$$\max_{P\in\mathcal{K}(2)} Pz(n) = z(n+1), \qquad \text{for all } n\in\mathbb{N}. \tag{46}$$

Since

$$\lim_{n\to\infty} \begin{bmatrix} n \\ k \end{bmatrix} \Big/ \begin{bmatrix} n \\ k-1 \end{bmatrix} = \infty,$$

there exists an integer n_0 such that

$$\max_{P\in\mathcal{K}} Pz(n) = \max_{P\in\mathcal{K}(2)} Pz(n), \qquad \text{for } n \geqq n_0. \tag{47}$$

Since $y(\nu)^{(\nu)} > \underline{0}$, it is possible to choose a constant $\alpha > 0$ such that

$$x(n_0)^{(\nu)} \leqq z(n_0)^{(\nu)} + y(\nu)^{(\nu)}. \tag{48}$$

Obviously, we also have for $n \geqq n_0$ [since $y(\nu)_i = 0$, for $i \in S\backslash D(\nu)$],

$$\max_{P\in\mathcal{K}} P(z(n) + \tau y(\nu)) = z(n+1) + \tau y(\nu), \qquad \text{for any } \tau \geqq \alpha. \tag{49}$$

Finally, choose β such that

$$\max_{P\in\mathcal{K}} \sum_{l=0}^{\nu-1} P^{(\nu,l)} e^{(l)} \leqq \beta y(\nu)^{(\nu)}. \tag{50}$$

By induction, it is now easily shown that [combine (43), (47), (48), (49)]

$$x(n)^{(\nu)} \leqq z(n)^{(\nu)} + \left(\alpha + \beta c_1 \sum_{k=n_0}^{n-1} \delta^k\right) y(\nu)^{(\nu)}, \qquad \text{for } n \geqq n_0. \tag{51}$$

Hence, $(x(n)^{(\nu)} - z(n)^{(\nu)})$ is bounded from above.

On the other hand, we may choose $\gamma > 0$ such that

$$x(0)^{(\nu)} \geqq y(1)^{(\nu)} - \gamma y(\nu)^{(\nu)} = z(0)^{(\nu)} - \gamma y(\nu)^{(\nu)}; \tag{52}$$

and since, for $n \geqq n_0$,

$$\max_{P\in\mathcal{K}} P(z(n) - \gamma y(\nu)) \geqq \max_{P\in\mathcal{K}(2)} P(z(n) - \gamma y(\nu)) = z(n+1) - \gamma y(\nu), \tag{53}$$

it follows easily that [combine (43), (49), (51), (52)]

$$x(n)^{(\nu)} \geqq z(n)^{(\nu)} - \left(\gamma + \beta c_1 \sum_{k=0}^{n-1} \delta^k\right) y(\nu)^{(\nu)}, \qquad \text{for } n \geqq 0. \tag{54}$$

Together, (50) and (53) imply the boundedness of $(x(n)^{(\nu)} - z(n)^{(\nu)})$, since $\delta < 1$.

Our next step will be to prove that $x(n)^{(\nu)} - z(n)^{(\nu)}$ converges geometrically to some vector, for $n \to \infty$. To this end, notice that, for n large enough, $n \geqq n_1$ say,

$$\max_{P\in\mathcal{K}} Px(n) = \max_{P\in\mathcal{K}(2)} Px(n), \tag{55}$$

by (45) and the boundedness shown above. Also note that, for all $P \in \mathcal{K}(2)$,

$$P(z(n) - y(1)) + y(2) = z(n+1) - y(1). \tag{56}$$

Hence, if we define

$$v(n) := x(n) - z(n) + y(1) = z(n) - \sum_{l=2}^{\nu} \begin{bmatrix} n \\ l-1 \end{bmatrix} y(l),$$

then $v(n)$ is bounded uniformly in n, and by (1), (55), (56),

$$v(n+1) = \max_{P\in\mathcal{K}(2)} (Pv(n) - y(2)), \qquad \text{for } n \geqq n_1. \tag{57}$$

In particular,

$$v(n+1)^{(\nu)} = \max_{P\in\mathcal{K}(2)} \left(P^{(\nu,\nu)} v(n)^{(\nu)} + \left\{ \sum_{l=0}^{\nu-1} P^{(\nu,l)} v(n)^{(l)} - y(2)^{(\nu)} \right\} \right),$$

$$n \geqq n_1. \tag{58}$$

Note that, for each $P \in \mathcal{K}(2)$,

$$\sum_{l=0}^{\nu-1} P^{(\nu,l)} v(n)^{(l)} - y(2)^{(\nu)}$$

converges geometrically to

$$\sum_{l=0}^{\nu-1} P^{(\nu,l)} y(1)^{(l)} - y(2)^{(\nu)}, \qquad n \to \infty.$$

Furthermore,

$$P^{(\nu,\nu)} y(\nu)^{(\nu)} = y(\nu)^{(\nu)} > \underline{0}, \qquad \text{for each } P \in \mathcal{K}(2).$$

Since $v(n)^{(\nu)}$ is bounded, we may apply Theorem 3.1. As a result, we find that $v(n)^{(\nu)}$ converges geometrically to some vector, $v^{(\nu)}$, say, for $n \to \infty$.

From (43) and the definition of $v(n)$, we know that $v(n)_i$ converges geometrically to $y(1)_i$, for $i \in S \backslash D(\nu)$, $n \to \infty$. Hence, if we define $\hat{y}(1)$ by

$$\hat{y}(1)_i := \begin{cases} v_i^{(\nu)}, & i \in D(\nu), \\ y(1)_i, & i \in S \backslash D(\nu), \end{cases}$$

then (45) follows immediately. Taking limits in (57) yields

$$y(2) + \hat{y}(1) = \max_{P\in\mathcal{K}(2)} P\hat{y}(1);$$

hence, (42) is satisfied also by $\hat{y}(1), y(2), \ldots, y(\nu)$. Finally, (41) is trivially verified. □

Before we proceed with the proof of Theorem 5.1, one additional remark must be made.

Remark 5.1. It follows immediately from (43) that the sets $\mathscr{K}(l)$ also possess the product property. Hence, for each $i \in S$, we may write

$$\mathscr{K}_i(\nu) := \left\{ P_i \in \mathscr{K}_i \;\middle|\; \sum_{j \in S} p_{ij} y(\nu)_j = \hat{\sigma} y(\nu)_i \right\},$$

$$\mathscr{K}_i(l) := \left\{ P_i \in \mathscr{K}_i(l+1) \;\middle|\; \sum_{j \in S} p_{ij} y(l)_j = \hat{\sigma} y(l)_i + y(l+1)_i \right\}, \qquad l = \nu - 1, \ldots, 2, 1.$$

Then, obviously,

$$\mathscr{K}(l) = \{ P \in \mathscr{K} \mid P_i \in \mathscr{K}_i(l),\ i \in S \}.$$

Hence, instead of $\mathscr{K}(l)$ we may also consider sets $\mathscr{K}_i(l)$, $i \in S$. This relationship will sometimes be used in the next proof.

Proof of Theorem 5.1. Again, $\hat{\sigma} = 1$ is assumed, in order to simplify notations. The proof will be given by induction with respect to ν. If $\nu = 1$, the results follow immediately from Theorem 4.1. Hence, let $\nu \geqq 2$, and suppose that the theorem has been proved for all sets of matrices with maximal index $\nu - 1$.

Consider now the recursion (1), where $\mathscr{K}$ is a set with maximal index ν. Since

$$p_{ij} = 0, \qquad \text{for } i \in S \backslash D(\nu),\ j \in D(\nu), \text{ for all } P \in \mathscr{K},$$

it follows that the asymptotic behavior or $x(n)_i$, for $i \in S \backslash D(\nu)$, is completely determined by a set of matrices with maximal index $\nu - 1$. Hence, according to the induction hypothesis, there exist vectors $w(1), w(2), \ldots, w(\nu - 1)$, defined on $S \backslash D(\nu)$, such that

$$-c_1 \delta^n \leqq x(n)_i - \sum_{l=1}^{\nu-1} \binom{n}{l-1} w(l)_i \leqq c_1 \delta^n, \qquad i \in S \backslash D(\nu),\ n \in \mathbb{N}, \tag{59}$$

for some $\delta < 1$, $c_1 > 0$. Furthermore, we have, for all $i \in S \backslash D(\nu)$,

$$\max_{P_i \in \mathscr{K}_i} \sum_{j \in S \backslash D(\nu)} p_{ij} w(\nu - 1)_j = w(\nu - 1)_i, \tag{60a}$$

$$\max_{P_i \in \mathscr{K}_i(l+1)} \sum_{j \in S \backslash D(\nu)} p_{ij} w(l)_j = w(l)_i + w(l+1)_i, \qquad l = \nu - 2, \ldots, 2, 1, \tag{60b}$$

where

$$\mathcal{H}_i(\nu-1) := \left\{ P_i \in \mathcal{K}_i \,\middle|\, \sum_{j \in S \setminus D(\nu)} p_{ij} w(\nu-1)_j = w(\nu-1)_i \right\},$$

$$\mathcal{H}_i(l) := \left\{ P_i \in \mathcal{H}_i(l+1) \,\middle|\, \sum_{j \in S \setminus D(\nu)} p_{ij} w(l)_j = w(l)_i + w(l+1)_i \right\},$$

$$l = \nu-2, \ldots, 2, 1.$$

Finally,

$$w(l)_i > 0, \qquad \text{for } i \in D(l),$$

$$w(l)_i = 0, \qquad \text{for } i \in \bigcup_{k=0}^{l-1} D(k),\ l = \nu-1, \ldots, 2, 1.$$

So far we have only used the induction hypothesis. It is now easily verified that, if there exist vectors $y(1), y(2), \ldots, y(\nu)$ such that (40), (41), (42) hold, such vectors must satisfy the identities

$$y(\nu)_i = 0, \qquad i \in S \setminus D(\nu), \tag{61a}$$

$$y(l)_i = w(l)_i, \qquad i \in S \setminus D(\nu),\ l = 1, 2, \ldots, \nu-1. \tag{61b}$$

Furthermore, again since

$$p_{ij} = 0, \text{ for } i \in S \setminus D(\nu),\ j \in D(\nu), \text{ for all } P \in \mathcal{K},$$

we choose

$$\mathcal{K}_i(\nu) = \mathcal{K}_i, \qquad i \in S \setminus D(\nu), \tag{62a}$$

$$\mathcal{K}_i(l) = \mathcal{H}_i(l), \qquad i \in S \setminus D(\nu),\ l = 1, 2, \ldots, \nu-1. \tag{62b}$$

Therefore, we may concentrate on $y(l)_i$, for $i \in D(\nu)$, $l = 1, 2, \ldots, \nu$. Substitution of (61) in (42) yields

$$\max_{P_i \in \mathcal{K}_i} \sum_{j \in D(\nu)} p_{ij} y(\nu)_j = y(\nu)_i, \qquad i \in D(\nu), \tag{63a}$$

$$\max_{P_i \in \mathcal{K}_i(l+1)} \left\{ \sum_{j \in D(\nu)} p_{ij} y(l)_j + \sum_{j \in S \setminus D(\nu)} p_{ij} w(l)_j \right\} = y(l)_i + y(l+1)_i,$$

$$i \in D(\nu),\ l = \nu-1, \ldots, 2, 1. \tag{63b}$$

The set of equations (63) has been analyzed in detail in Ref. 9 (a special case was studied by Miller and Veinott, compare Ref. 25). Here, we only remark that a solution $\{\tilde{y}(\nu)^{(\nu)}, \tilde{y}(\nu-1)^{(\nu)}, \ldots, \tilde{y}(1)^{(\nu)}\}$ to (60) indeed exists.

Except for $\tilde{y}(1)^{(\nu)}$, these vectors are even uniquely determined; moreover, $\tilde{y}(\nu)^{(\nu)}$ is strictly positive (compare Ref. 9, Lemma A1). Hence, define

$$y(l)_i := \tilde{y}(l)_i^{(\nu)}, \qquad i \in D(\nu),\ l = 1, 2, \ldots, \nu. \tag{64}$$

Recall that (63) was obtained by substituting (61) in (42). It follows that we have found a solution to (42) which also satisfies (41) and, by the induction hypothesis, (44) with $\hat{\sigma} = 1$. Application of Theorem 5.2 yields the final result. □

Theorem 5.1 yields rather strong results concerning the asymptotic behavior of dynamic programming recursions of type (1), at least under suitable aperiodicity assumptions. The results for arbitrary $\hat{\sigma} > 0$ are obtained immediately from the corresponding results for $\hat{\sigma} = 1$.

However, the proof given above cannot be copied when $\hat{\sigma} = 0$. If P is a nonnegative $N \times N$ matrix with $\sigma(P) = 0$, then it is easily seen that $\{i\}$, the set containing state i only, is a basic class of P, for all $i \in S$. Hence, $p_{ij} > 0$ can only occur for $i < j$ (P is in fact a *nilpotent* matrix). It follows immediately that, for $\hat{\sigma} = 0$ and for $n \in N$, each product $\prod_{k=0}^{n} P(k)$, with $P(k) \in \mathcal{K}$, is equal to the zero matrix. Hence, the conclusions of Theorem 5.1 remain valid for the case $\hat{\sigma} = 0$.

Finally, we remark that results similar to those of Theorem 5.1 can be obtained without assuming that certain matrices have to be aperiodic. Instead of (40), we get results for certain subsequences. Details will not be given here.

In Section 6, some implications of our results with respect to the theory of controlled Markov chains are discussed briefly.

6. Extensions and Implications for Controlled Markov Chains

Consider the Markov decision process introduced in Section 1. Let $v(0) > \underset{\sim}{0}$, and define $v(n)$ recursively by

$$v(n) = \max_{f \in F} \left\{ \binom{n}{k} r(f) + P(f) v(n-1) \right\}, \qquad n = 1, 2, \ldots, \tag{65}$$

where $k \in \mathbb{N}_0$ is fixed [for $k = 0$, we obtain (3) again]. Recall that each $P(f)$ is stochastic and that the set of $N \times (N+1)$ matrices

$$\{(P(f), r(f)) \mid f \in F\}$$

possesses the product property. In addition, assume that each $P(f)$ is aperiodic.

Recursions of type (65) play an important role in the study of k-average optimality criteria in Markov decision processes (cf. Ref. 26). Van der Wal (Ref. 27) showed that there exist vectors $y(1), (2), \ldots, y(k+1)$ and a positive constant $\rho < 1$, such that

$$v(n) = \binom{n}{k+1} y(k+2) + \cdots + \binom{n}{1} y(2) + y(1) + \mathcal{O}(\rho^n), \qquad n \to \infty.$$

However, by a simple trick, (65) can be reformulated as

$$\begin{bmatrix} v(n) \\ \binom{n}{k} \\ \vdots \\ \vdots \\ \binom{n}{1} \\ 1 \end{bmatrix} = \max_{f \in F} \begin{bmatrix} P(f) & r(f) & 0 & & 0 \\ 0 & 1 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} v(n-1) \\ \binom{n-1}{k} \\ \vdots \\ \vdots \\ \binom{n-1}{1} \\ 1 \end{bmatrix}, \qquad n = 1, 2, \ldots.$$

Using this translation, the asymptotic behavior of $v(n)$ for $n \to \infty$ is obtained immediately from Theorem 5.1. For $k = 0$, the geometric convergence result of Schweitzer and Federgruen (Ref. 10) is obtained again.

The reader may also verify that, for this example, Eqs. (42) turn into the policy-iteration equations for k-average optimal policies. In particular, for $k = 0$, Howard's optimality equations with respect to the average reward criterion are obtained.

Results similar to those obtained in this paper exist for continuous-time models. In Ref. 28, the following nonlinear differential equation is studied:

$$dz(t)/dt = \max_{Q \in \mathcal{M}} Qz(t), \qquad z(0) > \underset{\sim}{0},\ t \in [0, \infty), \tag{66}$$

where $\mathcal{M}$ is a set of M-matrices with the product property (an M-matrix is a square matrix with all its nondiagonal entries nonnegative). Without going into details, we remark that a solution $z(t)$ of (65) exists which obeys

$$z(t) = \{\exp(\hat{\lambda} t)\}\{t^{\nu-1} y(\nu) + \cdots + ty(2) + y(1)\} + \mathcal{O}(\exp(\mu t)), \qquad t \to \infty,$$

where $\hat{\lambda}$ and μ are certain real numbers, completely determined by $\mathcal{M}$, with $\mu < \hat{\lambda}$, while furthermore $\nu \in \mathbb{N}$ and $y(1), \ldots, y(\nu)$ is a set of vectors. Specialization of these results to continuous-time Markov decision chains is possible again. Exponential convergence in undiscounted continuous-time Markov decision chains is one of the new results which are obtained in this way (compare also Ref. 29).

Finally, we remark that extensions to models with a denumerable state space are discussed in Ref. 7 and Ref. 30.

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