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On the Structure and Properties of a Linear Multilevel Programming Problem[1]

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Abstract. Many decision-making situations involve multiple planners with different, and sometimes conflicting, objective functions. One type of model that has been suggested to represent such situations is the linear multilevel programming problem. However, it appears that theoretical and algorithmic results for linear multilevel programming have been limited, to date, to the bounded case or the case of when only two levels exist. In this paper, we investigate the structure and properties of a linear multilevel programming problem that may be unbounded. We study the geometry of the problem and its feasible region. We also give necessary and sufficient conditions for the problem to be unbounded, and we show how the problem is related to a certain parametric concave minimization problem. The algorithmic implications of the results are also discussed.

Key Words. Multilevel programming, multistage optimization, linear multilevel programming, multiple-criteria decision making.

1. Introduction

Many decision-making situations involve multiple planners with different, and sometimes conflicting, objective functions. In some of these problems, each planner independently controls a subset of the decision variables. A hierarchy may exist wherein the planners sequentially choose values for the decision variables. The first planner, in an attempt to optimize his objective function, chooses values for the variables that he controls. These values may partially determine the value of the objective function of the second planner. Furthermore, they may help to restrict the values that

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the second planner can choose for the variables that he controls. Given the values chosen by the first planner for the variables that he controls, the second planner, also in an attempt to optimize his objective function, chooses values for the variables that he controls. His choices, along with those of the first planner, can partially determine the objective function value and the permissible decision variable values of the third planner. Continuing in this way, the third planner, and each subsequent planner, in turn, in an attempt to optimize his objective function, chooses permissible values for the decision variables that he controls. At each stage, the planner's objective function value and permissible values for the variables in his control may be partially determined by the choices made earlier by other planners.

In recent years, several researchers have suggested models to represent such hierarchical decision-making problems. These models have come to be known as multilevel programming problems. In a problem of this type, when, at each level, the objective function and the functions in the constraints are linear, the problem is referred to as a linear multilevel programming problem. Otherwise, it is called a nonlinear multilevel programming problem.

Multilevel programming problems can apply in a variety of situations. For example, detailed applications have been described by Bracken, Falk, and Miercort (Ref. 1) to strategic weapons exchange problems; by Cassidy, Kirby, and Raike (Ref. 2) to the distribution of federal budgets among states; and by Fortuny-Amat and McCarl (Ref. 3) to the pricing and purchasing of fertilizers within an agricultural region. In addition, Candler and Townsley (Ref. 4) have suggested that multilevel programming problems can apply to various governmental problems involving issues such as the setting of penalties for illegal drug importation, the fixing of import quotas, and the determination of the extent to which transportation systems should be developed. Typically, in such problems, higher-level planners set policies, and lower-level planners react to these policies with certain behaviors, actions, or policies of their own. Thus, these problems can also be viewed as multilevel games (Refs. 5 and 6).

Although several researchers, including Candler and Townsley (Ref. 4), Bard (Ref. 5), Bard and Falk (Ref. 7), and Bialas and Karwan (Ref. 8), have defined the general multilevel programming problem, it appears that theoretical and algorithmic results have been limited, to date, to the bounded case or the case of when only two levels exist. In the case when two levels exist, the multilevel programming problem is referred to as the bilevel programming problem.

Most of the research concerning the bilevel programming problem has dealt with the linear case. Candler and Townsley (Ref. 4), Bialas and Karwan (Ref. 8), and Bard (Ref. 9) have studied the geometry of the linear bilevel programming problem. Bard (Refs. 9 and 10) also has stated some relationships between the linear bilevel programming problem and certain multiple-objective mathematical programs with two objective functions. In addition, Candler and Townsley (Ref. 4), Bard and Falk (Ref. 7), Bialas and Karwan (Ref. 8), Bard (Ref. 10), and Bialas and Karwan (Ref. 11) have proposed various algorithms for finding locally optimal and globally optimal solutions to the linear bilevel programming problem.

In the case of nonlinear bilevel programming, Bard (Refs. 9 and 12) has given various necessary and sufficient conditions for a vector to be an optimal solution. In addition, he has proposed a one-dimensional search algorithm that sometimes yields a locally optimal or globally optimal solution to the nonlinear bilevel programming problem (Ref. 12). Algorithms for finding globally optimal solutions to certain special cases of the nonlinear bilevel programming problem have been developed by Cassidy, Kirby, and Raike (Ref. 2), Bracken, Falk, and Miercort (Ref. 1), and Fortuny-Amat and McCarl (Ref. 3).

For the case of multilevel programming with three or more planners, Bard (Ref. 5) has studied the geometry of, and offered an algorithm for, the linear case. However, he assumed that the problem is bounded.

In this paper, the structure and properties of a linear multilevel programming problem that may be unbounded are investigated for the first time. Some of the results generalize those already given for the linear bilevel programming problem. Others are new not only for the multilevel case, but for the bilevel case as well. Throughout this paper, we show how the results relate to the known literature on linear multilevel programming. In addition, we discuss the algorithmic implications, if any, of each result.

The organization of this paper is as follows. Basic definitions, preliminaries, and the linear multilevel programming problem (LMPP) that we shall investigate are presented in Section 2. In Section 3, the geometry of the problem (LMPP) is studied. Necessary and sufficient conditions for the problem to be unbounded are presented in Section 4. In Section 5, we explore how the linear multilevel programming problem (LMPP) is related to a certain parametric concave minimization problem. This relationship suggests that certain one-dimensional search procedures could be used to find an optimal solution for the problem.

2. Basic Definitions and Preliminaries

For any finite-dimensional vectors y and z in Euclidean space, we will let $\langle y, z\rangle$ denote the inner product of y and z. To help to define the linear multilevel programming problem that we shall consider, let $n \geq 2$ be an integer, and let $k_1, k_2, \ldots, k_n$ be positive integers. In the linear multilevel program that we will define, n planners will exist. For each $i \in \{1, 2, \ldots, n\}$, planner i will have control over the vector of variables $x^i \in R^{m_i}$, where $m_1, m_2, \ldots, m_n$ are positive integers whose sum equals $\bar{n}$.

In the linear multilevel programming problem, the first planner chooses values for the variables $x^1 \in R^{m_1}$ in an attempt to maximize his objective function, which is given by

$$\langle a^{11}, x^1\rangle + \langle a^{12}, x^2\rangle + \cdots + \langle a^{1n}, x^n\rangle,$$

where $a^{1j} \in R^{m_j}$, $j = 1, 2, \ldots, n$, and $x^2, x^3, \ldots, x^n$ are to be chosen by planners $2, 3, \ldots, n$, respectively. He must choose values for x^1 so that $x = (x^1, x^2, \ldots, x^n)$ lies in

$$X^1 = \{x \in R^{\bar{n}} \mid B^{11}x^1 \leq b^1\},$$

where B^{11} is a $k_1 \times m_1$ matrix and $b^1 \in R^{k_1}$.

Given x^1, the second planner chooses values for the variables $x^2 \in R^{m_2}$ that maximize his objective function over X^2. The second planner's objective function is given by $\langle a^{22}, x^2\rangle$, where $a^{22} \in R^{m_2}$, and

$$X^2 = \{x \in R^{\bar{n}} \mid B^{21}x^1 + B^{22}x^2 \leq b^2\},$$

where B^{21} is a $k_2 \times m_1$ matrix, B^{22} is a $k_2 \times m_2$ matrix, and $b^2 \in R^{k_2}$. Then, in a similar manner, given x^1 and x^2, the third planner chooses values for the variables $x^3 \in R^{m_3}$ that maximize his objective function over X^3. The third planner's objective function is given by $\langle a^{33}, x^3\rangle$, where $a^{33} \in R^{m_3}$, and

$$X^3 = \{x \in R^{\bar{n}} \mid B^{31}x^1 + B^{32}x^2 + B^{33}x^3 \leq b^3\},$$

where B^{31} is a $k_3 \times m_1$ matrix, B^{32} is a $k_3 \times m_2$ matrix, B^{33} is a $k_3 \times m_3$ matrix, and $b^3 \in R^{m_3}$. Continuing in this way, each subsequent planner i, in turn, chooses values for the variables $x^i \in R^{m_i}$ that maximize his objective function, which is given by $\langle a^{ii}, x^i\rangle$, where $a^{ii} \in R^{m_i}$, over the set

$$X^i = \{x \in R^{\bar{n}} \mid B^{i1}x^1 + B^{i2}x^2 + \cdots + B^{ii}x^i \leq b^i\},$$

where, for each $j = 1, 2, \ldots, i$, B^{ij} is a $k_i \times m_j$ matrix, and $b^i \in R^{k_i}$.

Mathematically, this linear multilevel programming problem, which we shall refer to as problem (LMPP), seeks to find the value of ϕ such that

$$\phi = \max_{x \in X^1} \langle a^{11}, x^1\rangle + \langle a^{12}, x^2\rangle + \cdots + \langle a^{1n}, x^n\rangle,$$

where, given x^1, x^2 solves

$$\max_{x \in X^2} \langle a^{22}, x^2\rangle,$$

where, given x^1 and x^2, x^3 solves

$$\max_{x \in X^3} \langle a^{33}, x^3 \rangle,$$

$$\vdots$$

where, given $x^1, x^2, \ldots, x^{n-2}$, and x^{n-1}, x^n solves

$$\max_{x \in X^n} \langle a^{nn}, x^n \rangle,$$

where, for each $i = 1, 2, \ldots, n$,

$$X^i = \{x \in R^{\bar{n}} \mid B^{i1}x^1 + B^{i2}x^2 + \cdots + B^{ii}x^i \leq b^i\}.$$

For each $i = 1, 2, \ldots, n$, we will refer to the problem of planner i as problem (i).

Notice that, for each planning level $i \in \{2, 3, \ldots, n\}$, the values for the variables $x^1, x^2, \ldots, x^{i-1}$ are given. Therefore, for each $i \in \{2, 3, \ldots, n\}$, if $a^{ij} \in R^{m_i}$ for each $j = 1, 2, \ldots, i-1$, a problem equivalent to problem (LMPP) is obtained if $\sum_{j=1}^{i} \langle a^{ij}, x^i \rangle$ is substituted for $\langle a^{ii}, x^i \rangle$. In fact, then, our results will apply to a problem slightly more general than problem (LMPP). For simplicity, however, we will deal with problem (LMPP) directly.

Let $X = X^1 \cap X^2 \cap \ldots \cap X^n$. A vector x is said to be a permissible solution for problem (LMPP) when $x \in X$. A vector x is a feasible solution for problem (LMPP) when $x \in X$ and, for each $i = 2, 3, \ldots, n$, given $x^1, x^2, \ldots, x^{i-1}$, x^i maximizes $\langle a^{ii}, x^i \rangle$ subject to $x \in X^i$. The set of all feasible solutions for problem (LMPP), denoted F, is called the feasible solution set for the problem. An optimal solution for problem (LMPP) is a feasible solution that maximizes $\langle a^{11}, x^1 \rangle + \langle a^{12}, x^2 \rangle + \cdots + \langle a^{1n}, x^n \rangle$ over F.

To illustrate these concepts, assume that $n = 3$ planners exist, and that, for each $i = 1, 2, 3$, planner i has control over the vector $x^i = x_i \in R^1$. Consider the linear multilevel programming problem of finding ϕ such that

$$\phi = \max_{x \in X^1} x_1 - x_2 + x_3,$$

where, given x_1, x_2 solves

$$\max_{x \in X^2} x_2,$$

where, given x_1 and x_2, x_3 solves

$$\max_{x \in X^3} x_3,$$

where $x=(x_1, x_2, x_3)$ and

$$X^1=\{x\in R^3 \mid x_1\le 6, -x_1\le 0\},$$

$$X^2=\{x\in R^3 \mid 2x_1+x_2\le 10, -x_1\le 0, -x_2\le 0\},$$

$$X^3=\{x\in R^3 \mid 2x_1+x_2+x_3\le 18, -x_1\le 0, -x_2\le 0, -x_3\le 0\}.$$

Here, the set $X=X^1\cap X^2\cap X^3$ of permissible solutions is given by

$$X=\{x\in R^3 \mid x_1\le 6, 2x_1+x_2\le 10, 2x_1 + x_2+x_3\le 18, x_1\ge 0, x_2\ge 0, x_3\ge 0\}.$$

The vector $\bar{x}=(0,0,18)$ is a permissible solution. However, $\bar{x}$ is not a feasible solution since, given $x_1=0$, $x_2=10$ (rather than $x_2=0$) maximizes x_2 over X^2. The vector $\bar{\bar{x}}=(4,2,8)$ is both permissible and feasible, since $\bar{\bar{x}}\in X$ and, for $i=2$ and $i=3$, given $x_j=\bar{\bar{x}}_j$, $j\le i-1$, $\bar{\bar{x}}_i$ maximizes x_i subject to $x\in X^i$. It is not difficult to show that the feasible solution set F for this problem is given by

$$F=\{x\in R^3 \mid x=\lambda(0,10,8)+(1-\lambda)(5,0,8), \text{ for some } \lambda \text{ such that } 0\le\lambda\le 1\}.$$

Notice that F is a face (indeed, an edge) of X. Given F, it is easy to see that the extreme point $x^*=(5,0,8)$ of X is the unique optimal solution for this problem and that $\phi=13$. In contrast, the simple maximum of $x_1-x_2+x_3$ over X is equal to 18 and is achieved at $\bar{x}=(0,0,18)$, which is also an extreme point of X, but does not lie in F. Notice that, from our earlier comments, for any real numbers a_{21}, a_{31}, a_{32}, a problem equivalent to this multilevel problem is obtained whenever the objective function x_2 of planner two is replaced by $a_{21}x_1+x_2$ and the objective function x_3 of planner three is replaced by $a_{31}x_1+a_{32}x_2+x_3$.

We will assume that at least one permissible solution for problem (LMPP) exists. Notice in problem (LMPP) that, once the first planner has chosen values $\bar{x}^1$ for x^1, there may or may not exist values for $x^2, x^3, \ldots, x^n$ such that $x=(\bar{x}^1, x^2, \ldots, x^n)$ is a permissible solution. If such values exist, however, then, barring multiple optimal solutions, unattained suprema, and unboundedness in problems (2)-(n), exactly one feasible solution x exists for problem (LMPP) with $x^1=\bar{x}^1$. In fact, we will assume, for each $i=2,3,\ldots,n$, given $x^1, x^2, \ldots, x^{i-1}$, if problem ($i$) is feasible, that optimal solution values for x^i in problem (i) exist and are unique. In this way, planner one, once he has chosen values for the variables x^1, does not need to make any other choices.

When $n=2$, problem (LMPP) reduces to the linear bilevel programming problem studied by Candler and Townsley (Ref. 4), Bard and Falk (Ref.

7), and Bard (Refs. 9 and 10), and to one of the two versions of the linear bilevel programming problem studied by Bialas and Karwan (Refs. 8 and 11).

Since we have not assumed that the sets $X^1, X^2, \ldots, X^n$ are bounded, it is possible, in problem (LMPP), that, for any given positive number M, a feasible solution x exists for the problem such that $\sum_{j=1}^{n} \langle a^{1j}, x^j \rangle$ exceeds M. When this is the case, we say that problem (LMPP) is unbounded. If problem (LMPP) is not unbounded, then, as we shall see in the next section, an optimal solution will exist. Therefore, we have defined ϕ as a maximum rather than a supremum.

If at least one of the sets $X^1, X^2, \ldots, X^n$ is bounded, then problem (LMPP) is a special case of the problem studied by Bard in Ref. 5.

To conclude this section, we present the following lemma. We will need to refer to this lemma and its proof in the proofs of some of our main results.

Lemma 2.1. The vector $x = (x^1, x^2, \ldots, x^n)$ is a feasible solution for problem (LMPP) if and only if there exists a vector $u = (u^2, u^3, \ldots, u^n)$, where $u^i \in R^{k_i}$, $i = 2, 3, \ldots, n$, such that $(x, u) = (x^1, x^2, \ldots, x^n, u^2, u^3, \ldots, u^n)$ satisfies the conditions

$$(u^i)^T B^{ii} = (a^{ii})^T, \qquad i = 2, 3, \ldots, n,$$

$$\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij} x^j \rangle = \langle a^{ii}, x^i \rangle, \qquad i = 2, 3, \ldots, n,$$

$$u^2, u^3, \ldots, u^n \geq 0,$$

$$x^i \in X^i, \qquad i = 1, 2, \ldots, n.$$

Proof. A vector $x = (x^1, x^2, \ldots, x^n)$ is a feasible solution for problem (LMPP) if and only if it is a permissible solution for problem (LMPP) and, for each $i = 2, 3, \ldots, n$, given $x^1, x^2, \ldots, x^{i-1}$, x^i is an optimal solution for the linear programming problem (P^i) given by

$$\max \langle a^{ii}, x^i \rangle,$$

$$\text{s.t.} \quad B^{ii} x^i \leq b^i - \sum_{j=1}^{i-1} B^{ij} x^j.$$

By duality theory of linear programming (Ref. 13), for each $i = 2, 3, \ldots, n$, given $x^1, x^2, \ldots, x^{i-1}$, $x^i \in X^i$ is an optimal solution for problem (P^i) if and only if there exists a feasible solution u^i for the linear programming dual

(D^i) of problem (P^i) given by

$$\min\left\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij}x^j \right\rangle,$$

$$\text{s.t.} \quad (u^i)^T B^{ii} = (a^{ii})^T,$$

$$u^i \geq 0,$$

such that

$$\langle a^{ii}, x^i \rangle = \left\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij}x^j \right\rangle.$$

The lemma follows by combining these two statements. □

3. Geometric Properties

In this section, we will develop some geometric properties of problem (LMPP). We will show, among other things, that the feasible solution set F for problem (LMPP) is a union of faces of $X = X^1 \cap X^2 \cap \ldots \cap X^n$ and is connected. We will also show that, if problem (LMPP) has an optimal solution, then it has an optimal solution that is an extreme point of X. Throughout this section, for any convex set Y in a finite-dimensional Euclidean space, (ri Y) will denote the relative interior of Y. In addition, for any set Z in a finite-dimensional Euclidean space, (cl Z) will denote the closure of Z.

The following lemma will assist in deriving some of our main geometric results.

Lemma 3.1. Let $\bar{x}, \bar{\bar{x}} \in X$, with $\bar{x} \notin F$. Then, none of the points on the line segment

$$[\bar{x}, \bar{\bar{x}}) = \{x \in R^{\bar{n}} \mid x = \alpha\bar{x} + (1-\alpha)\bar{\bar{x}}, \text{ for some } \alpha \text{ such that } 0 < \alpha \leq 1\}$$

belongs to F.

Proof. Since $\bar{x} \in X$, but $\bar{x} \notin F$, the set

$$I = \{i \in \{2, 3, \ldots, n\} \mid \text{given } \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{i-1},$$
$$\bar{x}^i \text{ does not maximize } \langle a^{ii}, x^i \rangle \text{ subject to } x \in X^i\}$$

is nonempty. Let $\bar{i}$ be the smallest element in I; and, given $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{\bar{i}-1}$, choose any point

$$y = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{\bar{i}-1}, y^{\bar{i}}, \ldots, y^n) \in X^{\bar{i}}$$

which satisfies

$$\langle a^{\bar{i}\bar{i}}, y^{\bar{i}}\rangle > \langle a^{\bar{i}\bar{i}}, \bar{x}^{\bar{i}}\rangle.$$

Let α be any real number such that $0 < \alpha \leq 1$, and let

$$x = \alpha\bar{x} + (1-\alpha)\bar{\bar{x}}.$$

Let

$$\bar{z} = \alpha(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{\bar{i}-1}, y^{\bar{i}}, \ldots, y^n) + (1-\alpha) \times (\bar{\bar{x}}^1, \bar{\bar{x}}^2, \ldots, \bar{\bar{x}}^{\bar{i}-1}, \bar{\bar{x}}^{\bar{i}}, \ldots, \bar{\bar{x}}^n).$$

Then, since $X^{\bar{i}}$ is a convex set, $\bar{z} \in X^{\bar{i}}$.

Given $x^1, x^2, \ldots, x^{\bar{i}-1}$, consider the problem $(P^{\bar{i}})$ of maximizing $\langle a^{\bar{i}\bar{i}}, x^{\bar{i}}\rangle$ over

$$X^{\bar{i}} = \{x \in R^{\bar{n}} \mid B^{\bar{i}1}x^1 + B^{\bar{i}2}x^2 + \cdots + B^{\bar{i},\bar{i}-1}x^{\bar{i}-1} + B^{\bar{i}\bar{i}}x^{\bar{i}} \leq b^{\bar{i}}\}.$$

Since $x = \alpha\bar{x} + (1-\alpha)\bar{\bar{x}}$,

$$\langle a^{\bar{i}\bar{i}}, x^{\bar{i}}\rangle = \alpha\langle a^{\bar{i}\bar{i}}, \bar{x}^{\bar{i}}\rangle + (1-\alpha)\langle a^{\bar{i}\bar{i}}, \bar{\bar{x}}^{\bar{i}}\rangle.$$

Also, since $\alpha > 0$ and $\langle a^{\bar{i}\bar{i}}, \bar{x}^{\bar{i}}\rangle < \langle a^{\bar{i}\bar{i}}, |y^{\bar{i}}|\rangle$,

$$\alpha\langle a^{\bar{i}\bar{i}}\bar{x}^{\bar{i}}\rangle < \alpha\langle a^{\bar{i}\bar{i}}, y^{\bar{i}}\rangle.$$

From the last two sentences, we see that

$$\langle a^{\bar{i}\bar{i}}, x^{\bar{i}}\rangle < \alpha\langle a^{\bar{i}\bar{i}}, y^{\bar{i}}\rangle + (1-\alpha)\langle a^{\bar{i}\bar{i}}, \bar{\bar{x}}^{\bar{i}}\rangle.$$

By definition of $\bar{z}$, this means that

$$\langle a^{\bar{i}\bar{i}}, x^{\bar{i}}\rangle < \langle a^{\bar{i}\bar{i}}, \bar{z}^{\bar{i}}\rangle.$$

Since $\bar{z} \in X^{\bar{i}}$, this implies that, given $x^1, x^2, \ldots, x^{\bar{i}-1}, x^{\bar{i}}$ is not an optimal solution for problem $(P^{\bar{i}})$. Therefore, $x \notin F$, and the proof is complete. □

Using Lemma 3.1, we obtain the following theorem concerning F.

Theorem 3.1. Let K be any nonempty closed, convex subset of X. Suppose that $x \in (\text{ri } K)$ and $x \in F$. Then, $K \subseteq F$.

Proof. Let $\bar{x} \in K$. Since $K \subseteq X$, $\bar{x} \in X$. Suppose that $\bar{x} \notin F$. From Rockafellar (Ref. 14, Theorem 6.4), since $x \in (\text{ri } K)$ and K is a nonempty convex set, we can pick a $\lambda > 1$ such that

$$x^\lambda = (1-\lambda)\bar{x} + \lambda x \in K.$$

Since $K \subseteq X$, $x^\lambda \in X$. From Lemma 3.1, since $\bar{x}, x^\lambda \in X$ and $\bar{x} \notin F$, none of the points on the line segment $[\bar{x}, x^\lambda)$ lies in F. If we set $\alpha = (\lambda - 1)/\lambda$, then $0 < \alpha < 1$ and

$$x = \alpha\bar{x} + (1 - \alpha)x^\lambda.$$

Therefore, $x \in [\bar{x}, x^\lambda)$, so that $x \notin F$. But this contradicts the hypothesis of the theorem that $x \in F$. Therefore, our assumption that $\bar{x} \notin F$ is untenable. Since $\bar{x}$ was an arbitrarily chosen element of K, the proof is complete. □

Theorem 3.1 is new not only for the multilevel case, but for the bilevel case as well. For the bilevel case, Bialas and Karwan (Ref. 8, Theorem 3.1) have presented a result related to Theorem 3.1, but their result is a special case of Theorem 3.1 as applied to the bilevel problem.

Recall that a face Y' of a convex set Y is a convex subset of Y such that every closed line segment in Y with a relative interior point in Y' has both endpoints in Y'. From Theorem 3.1, we have the following key result.

Corollary 3.1. Let X' be a nonempty face of X. If $x \in (\text{ri } X')$ and $x \in F$, then $X' \subseteq F$.

Proof. Since X is a polyhedral set, its faces are also polyhedral (Ref. 14) and are therefore closed. Hence, the corollary is an immediate consequence of Theorem 3.1. □

Since X is a polyhedral set, it has a finite number of faces (Ref. 14). Using Corollary 3.1, we can now show that F is a union of some or all of these faces.

Theorem 3.2. Let $X(1), X(2), \ldots, X(w)$ be the nonempty faces of X. If $F \neq \emptyset$, then

$$F = \bigcup_{i \in \bar{W}} X(i),$$

where $\bar{W}$ is some subset of $W = \{1, 2, \ldots, w\}$.

Proof. Let $x \in F$. Then, since $F \subseteq X$, from Rockafellar (Ref. 14), $x \in [\text{ri } X(i)]$, for some $i \in W$. From Corollary 3.1, since $x \in F$, $X(i) \subseteq F$. Therefore,

$$F \subseteq \bigcup_{i \in \bar{W}} X(i),$$

for some set $\bar{W} \subseteq W$ such that $X(i) \subseteq F$ for all $i \in \bar{W}$.

Choose any such set $\bar{W}$. Since

$$F \supseteq \bigcup_{i \in \bar{W}} X(i),$$

the proof is complete. □

In the bilevel case, Bard (Ref. 9) and Bialas and Karwan (Ref. 11) have stated that F consists of faces of X. Theorem 3.2 shows that this result holds for our multilevel case as well. Notice that Theorem 3.2 applies even if X is an unbounded set. If X is bounded, Theorem 3.2 would follow from Bard (Ref. 5). Also, the set $\bar{W}$ in Theorem 3.2 need not be unique.

Theorem 3.2 implies that the feasible solution set F for problem (LMPP) may be a nonconvex set. This implies that problem (LMPP) may have locally optimal solutions that are not globally optimal. Therefore, any potential algorithm for finding an optimal solution for problem (LMPP) must have the capability of distinguishing between locally optimal solutions and globally optimal solutions.

As another consequence of Theorem 3.2, we have the following result. The proof of this result is a simple exercise and is therefore omitted.

Corollary 3.2. Suppose that ϕ in problem (LMPP) is finite. Then, there exists a point $x^* \in F$ such that

$$\phi = \langle a^{11}, x^{*1} \rangle + \langle a^{12}, x^{*2} \rangle + \cdots + \langle a^{1n}, x^{*n} \rangle.$$

Corollary 3.2 justifies our definition of ϕ in Section 2 as a maximum, rather than a supremum. In previous studies, researchers have simply defined linear bilevel and multilevel programming problems as maximization problems, rather than as problems of finding a supremum. Corollary 3.2 justifies this definition for the first time not only for the multilevel case considered here, but also for the bilevel case studied in Refs. 8–11, even if X is unbounded.

Recall that a connected set Z is one that cannot be written as $Z = A \cup B$, where A and B are nonempty, open sets such that

$$A \cap (\mathrm{cl}\, B) = \varnothing \quad \text{and} \quad (\mathrm{cl}\, A) \cap B = \varnothing.$$

Since the feasible solution set F for problem (LMPP) may be a nonconvex set, there is the possibility, at least in theory, that F is not connected. However, by the next result, F is in fact a connected set.

Theorem 3.3. F is a connected set.

Proof. From Lemma 2.1, $x = (x^1, x^2, \ldots, x^n) \in F$ if and only if $x^i \in X^i$, $i = 1, 2, \ldots, n$, and there exist vectors $u^2, u^3, \ldots, u^n \geq 0$ that satisfy

$$(u^i)^T B^{ii} = (a^{ii})^T, \qquad i = 2, 3, \ldots, n,$$

such that

$$\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij}x^j\rangle - \langle a^{ii}, x^i\rangle = 0, \qquad i = 2, 3, \ldots, n.$$

Therefore, if we let

$$U = \{(u^2, u^3, \ldots, u^n) \geq 0 \,|\, (u^i)^T B^{ii} = (a^{ii})^T, i = 2, 3, \ldots, n\},$$

and, for any $u \in U$, if we let

$$M(u) = \left\{(x^1, x^2, \ldots, x^n) \,|\, x^i \in X^i, i = 1, 2, \ldots, n, \text{ and } \left\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij}x^j\right\rangle - \langle a^{ii}, x^i\rangle = 0, i = 2, 3, \ldots, n\right\},$$

then

$$F = M(U), \qquad \text{where } M(U) = \{M(u) \,|\, u \in U\}.$$

From Hogan (Ref. 15, Theorem 10), M (considered as a point-to-set mapping from U into $R^{\bar{n}}$) is closed at each $u \in U$. (See Ref. 15 for a definition of when a point-to-set mapping is closed at a point in its domain. A point-to-set mapping that is closed at a point is also sometimes called upper semicontinuous at the point.) In addition, U and $M(u)$, for each $u \in U$, are convex sets. Therefore, they are connected sets. From Naccache (Ref. 16), this implies that $M(U) = F$ is also a connected set. □

It is well known that, if an optimal solution exists for a linear programming problem (P) whose feasible solution set contains no lines, then an optimal solution exists for (P) that is an extreme point of its feasible solution set. This property also holds for certain other problems, such as problems involving the minimization of a concave function over a polyhedron (Ref. 17). To close this section, we show that this extreme point property holds for problem (LMPP) as well.

Theorem 3.4. Suppose that X contains no lines, and that problem (LMPP) has an optimal solution. Then, problem (LMPP) has an optimal solution that is an extreme point of X.

Proof. Let $x = (x^1, x^2, \ldots, x^n)$ be an optimal solution for problem (LMPP) that is not an extreme point of X. From Theorem 3.2, $x \in X(1)$, where $X(1)$ is some nonempty face of X and $X(1) \subseteq F$. Since $X(1)$ is a face of the polyhedral set X, $X(1)$ is itself polyhedral (Ref. 14). Also, since X contains no lines, $X(1)$ contains no lines. Therefore, from Ref. 14, there exist positive integers t and u, with $1 \leq t \leq u$, extreme points $x_1, x_2, \ldots, x_t$ of $X(1)$, and directions $x_{t+1}, x_{t+2}, \ldots, x_u$ such that $x(1) \in X(1)$ if and only if $x(1)=\sum_{s=1}^{u}\lambda_s x_s$ for some nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ satisfying $\sum_{s=1}^{t}\lambda_s=1$. If we let $a=(a^{11}, a^{12}, \ldots, a^{1n})$, then, for any nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ satisfying $\sum_{s=1}^{t}\lambda_s=1$,

$$\left\langle a, \sum_{s=1}^{u}\lambda_s x_s\right\rangle=\sum_{s=1}^{t}\lambda_s\langle a, x_s\rangle+\sum_{s=t+1}^{u}\lambda_s\langle a, x_s\rangle.$$

Since $X(1)\subseteq F$ and problem (LMPP) is not unbounded, the last two statements imply that

$$\langle a, x_s\rangle\leq 0, \qquad \text{for each } s=t+1, t+2, \ldots, u.$$

Also, since $x\in X(1)$, there exist nonnegative numbers $\lambda_1^*, \lambda_2^*, \ldots, \lambda_u^*$ with $\sum_{s=1}^{t}\lambda_s^*=1$ such that

$$\langle a, x\rangle=\sum_{s=1}^{t}\lambda_s^*\langle a, x_s\rangle+\sum_{s=t+1}^{u}\lambda_s^*\langle a, x_s\rangle.$$

Since

$$\sum_{s=t+1}^{u}\lambda_s^*\langle a, x_s\rangle\leq 0,$$

this implies that

$$\langle a, x\rangle\leq\sum_{s=1}^{t}\lambda_s^*\langle a, x_s\rangle.$$

From this inequality and the fact that x is an optimal solution for problem (LMPP), it is easy to see that

$$\langle a, x\rangle=\langle a, x_s\rangle, \qquad \text{for each } s\in\{1, 2, \ldots, t\} \text{ such that } \lambda_s^*>0.$$

Since $X(1)$ is a face of X, each extreme point of $X(1)$ is also an extreme point of X, so that the proof is complete. □

Notice that both Theorems 3.3 and 3.4 hold even if X is unbounded. If X is bounded, they would follow from Bard (Ref. 5). Theorem 3.4 suggests that one approach for finding an optimal solution for problem (LMPP) might be to search among the extreme points of X. Indeed, approaches of this sort have been suggested for the bilevel case (Refs. 4, 10, and 11).

4. Unbounded Case

As explained in Section 2, since we have not assumed that the sets $X^1, X^2, \ldots, X^n$ are bounded, it is possible for problem (LMPP) to be unbounded. In this section, we present some necessary and sufficient conditions for problem (LMPP) to be unbounded. Since the unbounded case has not been explored even for the linear bilevel programming problem, all of the results in this section are new for both the linear bilevel and the linear multilevel programming problems. Throughout this section, to indicate that a maximization problem is unbounded, we may write sometimes that its optimal objective function value equals $+\infty$.

Our first result gives a simple necessary condition for problem (LMPP) to be unbounded. This result is immediate; therefore, its proof is omitted.

Theorem 4.1. If $\phi = +\infty$, then

$$+\infty = \max \sum_{j=1}^{n} \langle a^{1j}, x^j \rangle,$$
$$\text{s.t.} \quad x^j \in X^j, \qquad j = 1, 2, \ldots, n.$$

The necessary condition given in Theorem 4.1 for problem (LMPP) to be unbounded is not a sufficient one, even in the bilevel case. Simple examples can be constructed to illustrate this. In the next result, however, we present a sufficient condition for problem (LMPP) to be unbounded.

Theorem 4.2. Suppose that

$$+\infty = \max \langle a^{11}, x^1 \rangle,$$
$$\text{s.t.} \quad x = (x^1, x^2, \cdot \cdot, x^n) \in X,$$

and that, for each $j = 2, 3, \ldots, n$, the optimal objective function value of the problem

$$\min \langle a^{1j}, x^j \rangle,$$
$$\text{s.t.} \quad x = (x^1, x^2, \ldots, x^n) \in X,$$

is finite. Then, $\phi = +\infty$.

Proof. From Epelman (Ref. 18), since the problem of maximizing $\langle a^{11}, x^1 \rangle$ over X is unbounded, there exists $d = (d^1, d^2, \ldots, d^n) \in R^{\bar{n}}$ such that

$$B^{i1} d^1 + B^{i2} d^2 + \cdots + B^{ii} d^i \leq 0, \qquad i = 1, 2, \ldots, n,$$
$$\langle a^{11}, d^1 \rangle > 0.$$

Let $x \in X$ and, for any $t \geq 0$, let $x_t = x + td$. Then $x_t \in X$, for all $t \geq 0$. For each $t \geq 0$, given $x^1 = x_t^1$, let $\bar{x}_t^2$ maximize $\langle a^{22}, x^2 \rangle$ over X^2. For each $t \geq 0$, given $x^1 = x_t^1$ and $x^2 = \bar{x}_t^2$, let $\bar{x}_t^3$ maximize $\langle a^{33}, x^3 \rangle$ over X^3. Continuing in this manner, we can construct, for each $t \geq 0$, a feasible solution $\hat{x}_t = (x_t^1, \bar{x}_t^2, \bar{x}_t^3, \ldots, \bar{x}_t^n)$ for problem (LMPP).

Let M be any positive number. For each $j = 2, 3, \ldots, n$, and for any $t \geq 0$, since the problem of minimizing $\langle a^{1j}, x^j \rangle$ over X has a finite optimal objective function value, there exists a real number α_j such that $\langle a^{1j}, \bar{x}_t^j\rangle > \alpha_j$. Choose t to be any positive number that satisfies

$$t > \left(M - \langle a^{11}, x^1\rangle - \sum_{j=2}^{n} \alpha_j\right) \Big/ \langle a^{11}, d^1\rangle.$$

Let

$$a = (a^{11}, a^{12}, \ldots, a^{1n}) \in R^{\bar{n}}.$$

Then,

$$\begin{aligned}\langle a, \hat{x}_t\rangle &= \langle a^{11}, x_t^1\rangle + \sum_{j=2}^{n} \langle a^{1j}, \bar{x}_t^j\rangle \\ &= \langle a^{11}, x^1\rangle + t\langle a^{11}, d^1\rangle + \sum_{j=2}^{n} \langle a^{1j}, \bar{x}_t^j\rangle \\ &> \langle a^{11}, x^1\rangle + t\langle a^{11}, d^1\rangle + \sum_{j=2}^{n} \alpha_j \\ &> M,\end{aligned}$$

so that problem (LMPP) is unbounded. □

Notice that, to test the sufficient condition for problem (LMPP) to be unbounded given in Theorem 4.2, only linear programming problems need to be solved. It can be shown that this condition, although sufficient, is not necessary for problem (LMPP) to be unbounded.

It appears that any condition that is both necessary and sufficient for problem (LMPP) to be unbounded will not be as simple to state or to test as the two conditions given in Theorems 4.1 and 4.2. For instance, the necessary and sufficient condition given in the next result is not as simple to test as either of these two conditions. However, this condition gives a complete characterization of the unbounded case.

Theorem 4.3. Let

$$U = \{(u^2, u^3, \ldots, u^n) \geq 0 \,|\, (u^i)^T B^{ii} = (a^{ii})^T, i = 2, 3, \ldots, n\}.$$

Then, $\phi = +\infty$ if and only if, for some $u = (u^2, u^3, \ldots, u^n) \in U$, the linear programming problem (LP) given by

$$\begin{aligned}&\max \langle a^{11}, x^1\rangle + \langle a^{12}, x^2\rangle + \cdots + \langle a^{1n}, x^n\rangle, \\ &\text{s.t.} \quad \sum_{j=1}^{i-1} \langle (u^i)^T B^{ij}, x^j\rangle + \langle a^{ii}, x^i\rangle = \langle u^i, b^i\rangle, \qquad i = 2, 3, \ldots, n, \\ &\qquad\; x^i \in X^i, \qquad i = 1, 2, \ldots, n,\end{aligned}$$

is unbounded.

Proof. To prove the "if" portion of the theorem, let $u \in U$ be a point such that problem (LP) is unbounded. Then, for any positive integer M, there exists a vector $x = (x^1, x^2, \ldots, x^n)$ such that x satisfies the constraints of problem (LP) and

$$\sum_{j=1}^{n} \langle a^{1j}, x^j \rangle > M.$$

From Lemma 2.1, this implies that problem (LMPP) is unbounded.

To prove the "only if" portion of the theorem, let $\phi = +\infty$, and let $\{M_t\}_{t=1}^{\infty}$ be a strictly increasing sequence of positive real numbers with $\lim_{t\to\infty} M_t = +\infty$. Then, for each t, there exists a point $x(t) \in F$ such that

$$\sum_{j=1}^{n} \langle a^{1j}, x(t)^j \rangle > M_t.$$

For any $x = (x^1, x^2, \ldots, x^n) \in R^{\bar{n}}$ and any $i = 2, 3, \ldots, n$, define a function $q_i : R^{\bar{n}} \to R$ by

$$q_i(x) = \min \left\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij} x^j \right\rangle,$$

$$\text{s.t.} \quad (u^i)^T B^{ii} = (a^{ii})^T,$$

$$u^i \geq 0.$$

Also, for each $i = 2, 3, \ldots, n$, let

$$U_i = \{u^i \geq 0 \mid (u^i)^T B^{ii} = (a^{ii})^T\}.$$

Then, for each t, since $x(t) \in F$, using reasoning similar to that used in the proof of Lemma 2.1, it can be shown that $x(t) \in X$ and

$$q_i(x(t)) = \langle a^{ii}, x(t)^i \rangle, \qquad \text{for each } i = 2, 3, \ldots, n.$$

For each $i = 2, 3, \ldots, n$, by definition of q_i, this implies that, for each t, there exists an extreme point $u(t)^i$ of U_i such that

$$\left\langle u(t)^i, b^i - \sum_{j=1}^{i-1} B^{ij} x(t)^j \right\rangle = \langle a^{ii}, x(t)^i \rangle.$$

Since the number of points in $\{x(t)\}_{t=1}^{\infty}$ is infinite and, for each $i = 2, 3, \ldots, n$, the number of extreme points in U_i is finite, there exists a subsequence $\{x(t)\}_{t \in I}$ of $\{x(t)\}_{t=1}^{\infty}$ and extreme points $\bar{u}^i$ of U^i, $i = 2, 3, \ldots, n$, such that, for each $t \in I$ and each $i = 2, 3, \ldots, n$,

$$\left\langle \bar{u}^i, b^i - \sum_{j=1}^{i-1} B^{ij} x(t)^j \right\rangle = \langle a^{ii}, x(t)^i \rangle.$$

Therefore, with $u = \bar{u} \in U$, $x(t)$ is a feasible solution for problem (LP) for each $t \in I$. Since

$$\sum_{j=1}^{n} \langle a^{1j}, x(t)^j \rangle > M_t,$$

this completes the proof. □

Remark 4.1. Notice, from the "only if" portion of the proof of Theorem 4.3, that, if problem (LMPP) is unbounded, then there exists a point $\bar{u} \in U$ that is actually an extreme point of U such that, with $u = \bar{u}$, the linear program (LP) is unbounded.

5. Relationship to a Parametric Concave Minimization Problem

In the bilevel case, various researchers have explored how problem (LMPP) is related to various other mathematical programming problems, including multiple-objective (Ref. 10) and nonconvex (Ref. 7) programming problems. In this section, we will show how problem (LMPP) is related to a certain parametric concave minimization problem. To do so, for the purposes of this section only, we will assume that X is a compact set.

To define this parametric problem, let

$$U = \{(u^2, u^3, \ldots, u^n) \geq 0 \,|\, (u^i)^T B^{ii} = (a^{ii})^T, i = 2, 3, \ldots, n\};$$

and, for any $x = (x^1, x^2, \ldots, x^n) \in R^{\bar{n}}$, let $q(x)$ be given by

$$q(x) = \min_{u \in U} \sum_{i=2}^{n} \left\langle u^i, b^i - \sum_{j=1}^{i-1} B^{ij} x^j \right\rangle.$$

From Rockafellar (Ref. 14), $q: R^{\bar{n}} \to R$ is a concave, and therefore continuous, function on $R^{\bar{n}}$. Let $t \in R$ be a parameter, and define the parametric concave minimization problem (C_t) as the problem that seeks to find ϕ_t, where

$$\phi_t = \min - \sum_{i=2}^{n} \langle a^{ii}, x^i \rangle + q(x),$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \langle a^{ij}, x^j \rangle \geq t,$$

$$x \in X.$$

The essentials of the relationship between problem (LMPP) and problem (C_t) are summarized in the following theorem.

Theorem 5.1. Assume that X is a compact set and that problem (LMPP) has an optimal solution. Let t^* denote the largest value of t in problem (C_t) for which ϕ_t equals zero. Then, these results hold:

(a) $\phi = t^*$.

(b) x^* is an optimal solution for problem (LMPP) if and only if x^* is an optimal solution for problem (C_{t^*}).

Proof. To show part (a), we will first show that, if $t \leq \phi$, then $\phi_t = 0$. Choose t so that $t \leq \phi$. Let x^* be an optimal solution for problem (LMPP). Then,

$$\phi = \sum_{j=1}^{n} \langle a^{1j}, x^{*j} \rangle \geq t.$$

Furthermore, using reasoning similar to that used in the proof of Lemma 2.1, since $x^* \in F$,

$$x^* \in X \quad \text{and} \quad -\sum_{i=2}^{n} \langle a^{ii}, x^{*i} \rangle + q(x^*) = 0.$$

Therefore, x^* is a feasible solution for problem (C_t) with an objective function value of zero. But for any $x \in X$, from the weak duality theorem of linear programming,

$$-\sum_{i=2}^{n} \langle a^{ii}, x^{i} \rangle + q(x) \geq 0.$$

Therefore, x^* is an optimal solution for problem (C_t) and $\phi_t = 0$.

To complete the proof of part (a), we will show that, if $t > \phi$, then $\phi_t = +\infty$ or $\phi_t > 0$. To show this, choose t so $t > \phi$. If no $x \in X$ exists for which

$$\sum_{j=1}^{n} \langle a^{1j}, x^{j} \rangle \geq t,$$

then problem (C_t) has no feasible solution, so that $\phi_t = +\infty$. Otherwise, since $t > \phi$, any $x \in X$ that satisfies

$$\sum_{j=1}^{n} \langle a^{1j}, x^{j} \rangle \geq t$$

must not belong to F. From the previous paragraph, for any $x \in X$ that does not belong to F,

$$-\sum_{i=2}^{n} \langle a^{ii}, x^{i} \rangle + q(x) > 0.$$

Taken together, the last two statements imply that $\phi_t > 0$, so that the proof of part (a) is complete.

From the first part of the proof of part (a), for any $t \leq \phi$, if x^* is an optimal solution for problem (LMPP), then x^* is an optimal solution for problem (C_t). Since, by part (a), $\phi = t^*$, this proves the "only if" portion of part (b).

Finally, to prove the "if" portion of part (b), let x^* be an optimal solution for problem (C_{t^*}). This implies that

$$\sum_{j=1}^{n} \langle a^{1j}, x^{*j} \rangle \geq t^*,$$

that $x^* \in X$, and, from part (a), that

$$-\sum_{i=1}^{n} \langle a^{ii}, x^{*i} \rangle + q(x^*) = 0.$$

Using reasoning similar to that used in the proof of Lemma 2.1, since $x^* \in X$ and

$$-\sum_{i=2}^{n} \langle a^{ii}, x^{*i} \rangle + q(x^*) = 0,$$

then $x^* \in F$. Therefore, x^* is a feasible solution for problem (LMPP) with

$$\sum_{j=1}^{n} \langle a^{1j}, x^{*j} \rangle \geq t^*.$$

Since, by part (a), $t^* = \phi$, this implies that x^* is an optimal solution for problem (LMPP), and the proof is complete. □

Theorem 5.1 is a new result not only for the multilevel case, but for the bilevel case as well.

Let X be a compact set, and suppose that problem (LMPP) has at least one optimal solution. Theorem 5.1 then suggests that, to find an optimal solution for problem (LMPP), a one-dimensional search can be undertaken for the largest value t^* of t for which $\phi_t = 0$ in problem (C_t). Various schemes for one-dimensional search procedures could be employed in such a procedure. To implement many of these schemes, it would be necessary to be able to find an optimal solution for problem (C_t) for various values of t. For each value of t, problem (C_t) involves the minimization of a continuous, nonseparable concave function over a polyhedral set. Various algorithms for minimizing a nonseparable concave function over a polyhedron have been proposed, including the algorithms of Benson (Ref. 19), Cabot (Ref. 20), Carrillo (Ref. 21), Falk and Hoffman (Ref. 22), Majthay and Whinston (Ref. 23), Thoai and Tuy (Ref. 24), and Zwart (Ref. 25). See Ref. 19 for a brief survey of algorithms for minimizing a concave function over a polyhedron. Several of these algorithms, for any given value of t, could be applied to problem (C_t).

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