JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 73, No. 2, MAY 1992

Partitionable Variational Inequalities with Applications to Network and Economic Equilibria

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Communicated by F. A. Potra

Abstract. In this paper, we describe a useful class of finite-dimensional variational inequalities which we call partitionable. These variational inequalities are characterized by state functions which can be thought of as nonlinear separable functions added to antisymmetric linear functions. In the case of partitionable variational inequalities, questions of the monotonicity and coercivity of the state function can be addressed by considering the monotonicity and coercivity of a series of lower-dimensional functions. These functions are generally simpler to investigate than the state function. In the applications, these lower-dimensional functions are usually the natural functions to consider. To demonstrate, we conclude the paper by reviewing several models in the recent literature which give rise to partitionable variational inequalities.

Key Words. Variational inequalities, network equilibrium, existence, uniqueness, sensitivity analysis, economic equilibrium.

1. Introduction

The concept of equilibrium is central to many disciplines, economics, regional science, transportation science, game theory, and operations research for example. Despite the differing contexts in which they arise, equilibrium problems share many common characteristics, namely a number of competing entities each trying to optimize its personal utility within an environment affected by the actions of itself and its competitors. Mathematically, the equilibrium state, if one exists, resulting from such a situation is characterized by a set of mathematical conditions. These equilibrium conditions vary considerably from one model to the next, and yet many may be

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studied in a mathematically unified manner through the theory of variational inequalities. Specifically, the equilibrium conditions may be cast in the form of the variational inequality problem VI(F, K); namely, find an element $\bar{x}$ in K so that

$$F(\bar{x})^T(x-\bar{x})\geq 0, \qquad \forall x\in K,$$

where F is a function defined on a closed convex set $K\subseteq \mathbb{R}^n$ which takes values in $\mathbb{R}^n$ [see, e.g., Kinderlehrer and Stampacchia (Ref. 1)].

In this paper, we introduce a class of variational inequalities, called partitionable variational inequalities, the members of which capture the underlying structure of many equilibrium problems, including the traffic equilibrium problem [see Dafermos and Nagurney (Ref. 2)], the spatial price equilibrium problem [see Florian and Los (Ref. 3), Freisz *et al.* (Ref. 4), Tobin (Ref. 5)], and the general equilibrium problem in economics [Zhao (Ref. 6)]. While partitionable variational inequalities have appeared in the literature, they have not been characterized as such. In this paper, we characterize this class of variational inequalities and carefully examine the properties of its members.

Formulating equilibrium problems as partitionable variational inequalities allows us to address the questions of the existence and uniqueness of equilibria by considering, independently, members of a set of functions each of which is far simpler than the state function F of the variational inequality. We also obtain detailed sensitivity analysis results through the partitionable variational inequality formulation.

We begin this paper by offering a definition of partitionable functions. If a function F is partitionable, then the definition also defines a set of functions called the partitions of F. These partitions are functions defined on domains of lower dimension than the function F. The partitions of F play a key role in the remainder of the paper.

Following the definition of partitionable functions, a structure theorem is proved giving necessary and sufficient conditions for a function F to be partitionable. Essentially, we show that F is partitionable if and only if F is the sum of linear and nonlinear parts, where the linear parts possess a certain antisymmetry.

The properties of partitionable functions are investigated in the remaining sections of the paper. For general variational inequalities, questions concerning the existence and uniqueness of equilibria, as well as sensitivity analysis, are often answered by referring to the state function F of a variational inequality and asking if F is coercive or monotonic. The question of convergence of algorithms for variational inequalities also often involves monotonicity and coercivity. In Section 3 of this paper, we see that the questions of monotonicity, strict monotonicity, strong monotonicity, or coercivity can be answered by analyzing the partitions of F. We show that, for each of these four properties, the state function F possesses the property if and only if all of the partitions possess the same property. Thus, the analysis can shift away from the state function F to the lower-dimensional, and presumably simpler, partitions of F.

Once we have shown how the coercivity and monotonicity of the function F is related to the coercivity and monotonicity of the partitions, we address the issues of existence and uniqueness of solutions to variational inequalities in which F is partitionable. These results are straightforward applications of standard existence and uniqueness results to the special case of partitionable functions. See Kinderlehrer and Stampacchia (Ref. 1) for a concise discussion of existence and uniqueness in variational inequalities.

Next, we consider the important question of sensitivity analysis. We approach this topic from two distinct perspectives. In the first case, following the approach of Dafermos and Nagurney (Ref. 2), we consider global perturbations in the state function F. The necessary conditions for performing this analysis are quite easily satisfied. We require the continuity and strict monotonicity of the partitions of F before the perturbation and the continuity of the state function after the perturbation. Of course, in both instances, we assume the existence of at least one solution to the variational inequality.

Our second consideration in the realm of sensitivity analysis is the case of local or parametric sensitivity analysis. In this case, we impose only local conditions on the partitions and are rewarded with very detailed information for small perturbations. Our work here follows the approach of Dafermos (Ref. 7).

We end the paper with three examples of equilibrium problems from the recent literature which have been formulated using partitionable variational inequalities. These examples show the breadth of application for partitionable variational inequalities.

2. Partitionable Functions and Partitionable Variational Inequalities

Many equilibrium problems found in the social sciences share a common structure. As we shall see in Section 6, examples include the traffic equilibrium problem [Dafermos and Nagurney (Ref. 8)], the spatial price equilibrium problem [Freisz *et al.* (Ref. 4), Tobin (Ref. 5)], and the market equilibrium problem [Dafermos and McKelvey (Ref. 9)]. In this section, we introduce a class of variational inequalities, called partitionable variational inequalities, which captures this common structure.

Definition 2.1. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous, where K is a convex set. The function F is said to be partitionable of order m over K if

$$[F(x)-F(y)]^T(x-y)=\sum_{i=1}^{m}[f_i(x_i)-f_i(y_i)]^T(x_i-y_i), \tag{1}$$

for some continuous functions

$$f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}, \qquad i=1,\ldots,m, \tag{2}$$

with convex domains $K_i \subseteq \mathbb{R}^{n_i}$, each of which contains an open neighborhood of $\mathbb{R}^{n_i}$, such that

$$\prod_{i=1}^{m} K_i = K \subseteq \mathbb{R}^n, \tag{3}$$

and for any x_i, y_i, $i=1,\ldots,m$, where we let

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

The functions f_i are the partitions of F.

The condition that each K_i contains an open neighborhood may seem a bit restrictive. However, it is easy to see that this is really no restriction at all. Suppose that some K_i fails to contain an open ball in $\mathbb{R}^{n_i}$; then, the convexity of K_i implies that K_i lies in a hyperplane of $\mathbb{R}^{n_i}$ with dimension strictly less than n_i. If we let n_i' be the dimension of the smallest hyperplane which contains K_i, then there exists an invertible linear change of variables which places the image of K_i into $\mathbb{R}^{n_i'}$. This image does contain an open neighborhood of $\mathbb{R}^{n_i'}$, and the conditions of the definition are met.

In applications, we seldom need to invoke this transformation. As we shall see, all the applications developed in this paper lead naturally to formulations with suitably large K_i.

The partitions of a partitionable function are not, in general, unique. From the definition of partitionable function, it is clear that, if $f_i(x_i)$ is a partition of F, then any function which differs from $f_i(x_i)$ by a constant vector is also a partition of F.

It is also possible that the feasible set K can be written as a Cartesian product in many ways. In this case, the domains of the partitions will differ, so the partitions are not identical. For example, let $K=K_1\times K_2\times K_3 \subseteq \mathbb{R}^n$, and let F be partitionable into three partitions,

$$f_1: K_1 \subseteq \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}, \qquad f_2: K_2 \subseteq \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}, \qquad f_3: K_3 \subseteq \mathbb{R}^{n_3} \to \mathbb{R}^{n_3}.$$

Then, it is easy to see that F can be partitioned into two partitions, namely,

$$g_1=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}: K_1\times K_2 \subseteq \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2}, \qquad g_2=f_3: K_3 \subseteq \mathbb{R}^{n_3} \to \mathbb{R}^{n_3}.$$

We see from this example that if a function F is partitionable of order m, then it is also partitionable of any order less than m. In particular, any function defined on a convex set is partitionable of order one, where the function's partition is itself. Of course the interesting cases are those where the order of partitionability is greater than one.

Partitionable functions and separable functions over Cartesian products share many important properties. In fact, it is clear from the definition that separable functions defined over Cartesian products are a subclass of the class of partitionable functions. However, the class of partitionable functions is much larger than the class of separable functions.

There is, nonetheless, a strong relationship between all partitionable functions and functions which are separable and defined over Cartesian products. Our first theorem shows that partitionable functions can be considered separable functions plus linear terms, where the linear terms have a special form.

Theorem 2.1. A function

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \qquad F: \prod_{i=1}^{m} K_i \to \mathbb{R}^{\sum_{i=1}^{m} n_i},$$

is partitionable of order m if and only if there exist constant real matrices M_{ij}, $j = i+1, \ldots, m$, $i = 1, \ldots, m-1$, of dimension $n_1 \times n_j$ such that

$$F_i(x) = f_i(x_i) + \sum_{j>i} M_{ij} x_j - \sum_{j<i} M_{ji}^T x_j, \qquad i = 1, \ldots, m. \tag{4}$$

Furthermore, the $f_i(x_i)$ of Eq. (4) are partitions of $F(x)$. Partitions which satisfy Eq. (4) are called principal partitions of F.

Proof. We begin with the proof of sufficiency. Suppose that $F(x)$ has the form (4). Then,

$$\begin{aligned}
&[F(x) - F(y)]^T (x - y) \\
&= \sum_{i=1}^{m} [F_i(x) - F_i(y)]^T (x_i - y_i) \\
&= \sum_{i=1}^{m} \left[f_i(x_i) - f_i(y_i) + \sum_{j>i} M_{ij} x_j - \sum_{j<i} M_{ji}^T x_j - \sum_{j>i} M_{ij} y_j + \sum_{j<i} M_{ji}^T y_j \right]^T (x_i - y_i) \\
&= \sum_{i=1}^{m} [f_i(x_i) - f_i(y_i)]^T (x_i - y_i) \\
&+ \sum_{i=1}^{m} \left[\sum_{j>i} M_{ij}(x_j - y_j) - \sum_{j<i} M_{ji}^T (x_j - y_j) \right]^T (x_i - y_i).
\end{aligned} \tag{5}$$

After a simple yet tedious computation, it can be shown that the second term of (5) is zero. Therefore, we have shown that

$$[F(x)-F(y)]^T(x-y)=\sum_{i=1}^{m}[f_i(x_i)-f_i(y_i)]^T(x_i-y_i),$$

and we see that $F(x)$ is partitionable of order m.

We continue with the proof of necessity. We induct on m, the order of partitionability. It is clear that necessity holds when $m=1$. Simply note that condition (4) becomes

$$F(x)=f(x).$$

Fix $m>1$, and assume that all partitionable functions of order less than m are of the form (4). $F(x)$ is then also partitionable of order two, where the first partition $f_1(x)$ is itself partitionable of order $m-1$. The second partition is denoted $f_2(x)$.

Given these functions, we have

$$\begin{aligned}&[F(x)-F(y)]^T(x-y)\\&=[f_1(x_1)-f_1(y_1)]^T(x_1-y_1)+[f_2(x_2)-f_2(y_2)]^T(x_2-y_2),\end{aligned} \tag{6}$$

for every $x, y\in K_1\times K_2$. Letting

$$x=\begin{bmatrix}x_1\\x_2\end{bmatrix},\qquad y=\begin{bmatrix}y_1\\x_2\end{bmatrix},$$

we see that

$$\begin{aligned}[F(x)-F(y)]^T(x-y)&=[F_1(x)-F_1(y)]^T(x_1-y_1)\\&=[f_1(x_1)-f_1(y_1)]^T(x_1-y_1),\end{aligned}$$

for all $x_2\in K_2$ and all $x_1, y_1\in K_1$. This in turn implies that $F_1(x)$ is separable into two components.

To see this, note that the equation above implies that the difference $F_1(x)-F_1(y)$ is a function of only x_1 and y_1 and does not depend on the choice of x_2. We can therefore define the function

$$\tilde{F}(x_1, y_1)=F_1(x)-F_1(y).$$

Recalling the definitions of the vectors x and y, we can rewrite the definition of $\tilde{F}$ as

$$F_1(x_1, x_2)=\tilde{F}(x_1, y_1)+F_1(y_1, x_2),\qquad \forall y_1\in K_1.$$

Since this equation holds for any choice of y_1, we are free to fix y_1 and we see $F_1(x_1, x_2)$ is written as the sum of a function of x_1 alone and a function of x_2 alone. Hence, we can write F_1 as

$$F_1(x) = f_{11}(x_1) + f_{12}(x_2).$$

By symmetry, we know that $F_2(x)$ is separable, and we write

$$F_2(x) = f_{22}(x_2) + f_{21}(x_1).$$

We will see that the functions f_{11} and f_{22} are partitions of the function F. For now, we seek to show that f_{12} and f_{21} are affine. Using the separability just established, we see that

$$\begin{aligned}
&[F(x) - F(y)]^T(x-y)\\
&= [F_1(x) - F_1(y)]^T(x_1 - y_1) + [F_2(x) - F_2(y)]^T(x_2 - y_2)\\
&= [f_{11}(x_1) - f_{11}(y_1)]^T(x_1 - y_1) + [f_{22}(x_2) - f_{22}(y_2)]^T(x_2 - y_2)\\
&+ [f_{12}(x_2) - f_{12}(y_2)]^T(x_1 - y_1) + [f_{21}(x_1) - f_{21}(y_1)]^T(x_2 - y_2)\\
&= [f_1(x_1) - f_1(y_1)]^T(x_1 - y_1) + [f_2(x_2) - f_2(y_2)]^T(x_2 - y_2),
\end{aligned}$$

the last equality coming from (6). By setting $x_1 = y_1$ and letting x_2 and y_2 be arbitrary, we see that

$$[f_2(x_2) - f_2(y_2)]^T(x_2 - y_2) = [f_{22}(x_2) - f_{22}(y_2)]^T(x_2 - y_2).$$

Similarly, we have

$$[f_1(x_1) - f_1(y_1)]^T(x_1 - y_1) = [f_{11}(x_1) - f_{11}(y_1)]^T(x_1 - y_1),$$

for all x_1, y_1. These equations give us

$$[f_{12}(x_2) - f_{12}(y_2)]^T(x_1 - y_1) = -[f_{21}(x_1) - f_{21}(y_1)]^T(x_2 - y_2). \tag{7}$$

This expression implies that f_{12} and f_{21} are affine in their arguments. To see this, rewrite (7) as

$$\begin{aligned}
f_{12}(x_2)^T(x_1 - y_1) = [&f_{12}(y_2)^T(x_1 - y_1) + [f_{21}(x_1) - f_{21}(y_1)]^T y_2]\\
&- [f_{21}(x_1) - f_{21}(y_1)]^T x_2.
\end{aligned}$$

If we hold x_1, y_1, y_2 constant, we see that $f_{12}(x_2)$ has been written as a constant vector plus the product of a constant vector and x_2. This implies that f_{12} is affine. The same argument implies that f_{21} is also affine. We write

$$f_{12}(x_2) = Mx_2 + C_1, \qquad f_{21}(x_1) = Nx_1 + C_2,$$

for appropriate matrices M and N and constant vectors C_1 and C_2.

We note that we can assume without loss of generality that $C_1=0$ and $C_2=0$. If not, we simply modify f_{11} and f_{22} to include C_1 and C_2, respectively.

Letting

$$f_{12}(x_2)=Mx_2, \qquad f_{21}(x_1)=Nx_1,$$

we see that

$$[M(x_2-y_2)]^T(x_1-y_1)=-[N(x_1-y_1)]^T(x_2-y_2),$$

which implies that

$$N=-M^T.$$

Therefore,

$$F_1(x)=f_1(x_1)+Mx_2, \tag{8a}$$

$$F_2(x)=f_2(x_2)-M^Tx_1. \tag{8b}$$

Using the proof of sufficiency, we see that (8) implies that f_1 and f_2 are principal partitions of F.

Recalling that f_1 is itself partitionable of order $m-1$, rewriting the matrix M as

$$M^T=[M_{1,m}|M_{2,m}|\cdots|M_{m-1,m}],$$

and relabelling the vectors x and F appropriately, Eq. (8) can be rewritten as

$$F_i(x)=f_i(x_i)+\sum_{j>i} M_{ij}x_j-\sum_{j<i} M_{ji}^T x_j,$$

and the theorem is shown. □

Having defined the notion of a partitionable function, we are ready to define partitionable variational inequalities.

Definition 2.2. Let $F: K\subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous on the closed convex set K. If F is partitionable of order m over K, then the variational inequality VI(F, K), i.e., find $\tilde{x}\in K$ such that

$$F(\tilde{x})^T(x-\tilde{x})\geq 0, \qquad \text{for all } x\in K,$$

is a partitionable variational inequality of order m.

It should be noted that all partitionable variational inequalities are defined over product sets. Iterative schemes for solving variational inequalities defined over product sets and applications of such variational inequalities are discussed in Pang (Ref. 10).

Partitionability will enable us to establish existence, uniqueness, and sensitivity analysis results for variational inequalities by imposing conditions only on the partitions. As we shall see in the examples, the partitions tend to be the interesting functions in those problems amenable to analysis through partitionable variational inequalities.

3. Properties of Partitionable Functions and Partitionable Variational Inequalities

Coercivity and monotonicity are central properties in the study of the existence and uniqueness of solutions to variational inequalities. Monotonicity is also often required to show the convergence of iterative schemes for the solution of variational inequalities. In this section, we show that a partitionable function $F(x)$ possesses one of these important properties if and only if each of the partitions of $F(x)$ possesses the same property. The implication of these results is that partitionable functions can be studied through the examination of each partition separately, rather than the typically more complicated partitionable function $F(x)$. A more detailed discussion of the importance of monotonicity and coercivity can be found in the excellent survey of variational inequality problems by Harker and Pang (Ref. 11).

Definition 3.1. Let K be a closed convex subset of $\mathbb{R}^n$, and let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a function defined on K. We say that:

(i) F is monotone if

$$[F(x) - F(y)]^T (x - y) \geq 0, \qquad \forall x, y \in K;$$

(ii) F is strictly monotone if

$$[F(x) - F(y)]^T (x - y) > 0, \qquad \forall \text{ distinct } x, y \in K;$$

(iii) F is strongly monotone if there exists an $\alpha > 0$ so that

$$[F(x) - F(y)]^T (x - y) > \alpha \|x - y\|^2, \qquad \forall x, y \in K;$$

(iv) F is coercive if there exists an $x_0 \in K$ so that, for any unbounded sequence in K, we have

$$\lim_{\|x\| \to \infty} [F(x) - F(x_0)]^T (x - x_0) / \|x - x_0\| = +\infty.$$

The next theorem relates the presence of these properties in the function F to the presence of these properties in the partitions f_i of F.

Theorem 3.1. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$. The function F is monotone if and only if all the f_i are monotone.

Proof. Suppose that F is monotone. Let x_1 and y_1 be arbitrary elements of K_1, and let x_i be an arbitrary element of K_i for $i = 1, \ldots, m$. To show that f_1 is monotone, let x and y denote the vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

Then, note that

$$\begin{aligned} &[f_1(x_1) - f_1(y_1)]^T (x_1 - y_1) \\ &= [f_1(x_1) - f_1(y_1)]^T (x_1 - y_1) + \sum_{i=2}^{m} [f_i(x_i) - f_i(x_i)]^T (x_i - x_i) \\ &= [F(x) - F(y)]^T (x - y) \geq 0. \end{aligned}$$

Therefore, f_1 is monotone. By symmetry, the same proof implies that all the f_i, $i = 1, 2, \ldots, m$, are monotone.

To show the converse, assume that f_i, $i = 1, 2, \ldots, m$, are monotone, and pick x and y arbitrarily from K. The components of x and y can each be partitioned into two vectors so that

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

where $x_i, y_i \in K_i$. To show that F is monotone, we note that

$$\begin{aligned} [F(x) - F(y)]^T (x - y) &= \sum_{i=1}^{m} [f_i(x_i) - f_i(y_i)]^T (x_i - y_i) \\ &\geq \sum_{i=1}^{m} 0 = 0. \end{aligned}$$

Therefore, F is monotone. □

Similar arguments lead to the following theorem.

Theorem 3.2. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$. The function F is strictly or strongly monotone if and only if all the f_i are strictly or strongly monotone, respectively.

We now turn to the coercivity of F, which can also be determined by the coercivity of the partitions f_i.

Theorem 3.3. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$. The function F is coercive if and only if all the f_i are coercive.

Proof. We prove this theorem by induction on m, the number of partitions. If $m = 1$, the theorem follows trivially. If $m = 2$, suppose that F is coercive. Choose $y \in K$ so that

$$\lim_{\|x\|_n \to \infty} [F(x) - F(y)]^T (x - y)/\|x - y\|_n = +\infty, \qquad x \in K,$$

where $\|\cdot\|_n$ is the standard Euclidean norm in $\mathbb{R}^n$. Partition this y into $y_1 \in K_1$ and $y_2 \in K_2$ so that

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

For any element x_1 of K_1, let

$$x = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

Since y_2 is fixed, we have the following two conditions:

(i) $\|x_1\|_{n_1} \to \infty$, if and only if $\|x\|_n \to \infty$;
(ii) $\|x_1 - y_1\|_{n_1} = \|x - y\|_n$.

To show that f_1 is coercive, we consider

$$\lim_{\|x_1\|_{n_1} \to \infty} [f_1(x_1) - f_1(y_1)]^T (x_1 - y_1)/\|x_1 - y_1\|_{n_1}, \qquad x_1 \in K_1,$$

$$= \lim_{\|x_1\|_{n_1} \to \infty} \{[f_1(x_1) - f_1(y_1)]^T (x_1 - y_1)$$

$$+ [f_2(y_2) - f_2(y_2)]^T (y_2 - y_2)\}/\|x_1 - y_1\|_{n_1}, \qquad x_1 \in K_1,$$

$$= \lim_{\|x\|_n \to \infty} [F(x) - F(y)]^T (x - y)/\|x - y\|_n = +\infty.$$

Therefore, f_1 is coercive. Symmetry shows that f_2 is also coercive.

To show the converse, we assume f_1 and f_2 are coercive. Let $y_1 \in K_1$ and $y_2 \in K_2$ be fixed points which satisfy the definition of coercivity for f_1 and f_2, respectively. Let

$$x^i = \begin{bmatrix} x_1^i \\ x_2^i \end{bmatrix}$$

be any unbounded sequence in K, each element of which is partitioned into $x_1^i \in K_1$ and $x_2^i \in K_2$. By the coercivity of f_1 and f_2, there is a bound $B > 0$ so that

$$-B \leq [f_1(x_1^i) - f_1(y_1)]^T (x_1^i - y_1)/\|x_1^i - y_1\|_{n_1}, \qquad \text{for all } i, \tag{9}$$

$$-B \leq [f_2(x_2^i) - f_2(y_2)]^T (x_2^i - y_2)/\|x_2^i - y_2\|_{n_2}, \qquad \text{for all } i. \tag{10}$$

Pick the positive number M to be arbitrarily large. Given M, choose M' large enough to ensure the following four conditions:

(i) $\|x_1^i\|_{n_1} > M'$

$$\Rightarrow [f_1(x_1^i) - f_1(y_1)]^T (x_1^i - y_1)/\|x_1^i - y_1\|_{n_1} > 2\sqrt{2}M + 2B; \tag{11}$$

(ii) $\|x_2^i\|_{n_2} > M'$

$$\Rightarrow [f_2(x_2^i) - f_2(y_2)]^T (x_2^i - y_2)/\|x_2^i - y_2\|_{n_2} > 2\sqrt{2}M + 2B; \tag{12}$$

(iii) $\|x_1^i\|_{n_1} > M'$ and $\|x_1^i\|_{n_1} \geq \|x_2^i\|_{n_2}$

$$\Rightarrow \|x_2^i - y_2\|_{n_2} \leq 2\|x_1^i - y_1\|_{n_1}; \tag{13}$$

(iv) $\|x_2^i\|_{n_2} > M'$ and $\|x_2^i\|_{n_2} \geq \|x_1^i\|_{n_1}$

$$\Rightarrow \|x_1^i - y_1\|_{n_1} \leq 2\|x_2^i - y_2\|_{n_2}. \tag{14}$$

We will be done with the proof if we can show the existence of an N such that

$$i > N \Rightarrow [F(x^i) - F(y)]^T (x^i - y)/\|x^i - y\|_n \geq M. \tag{15}$$

We claim that an N which suffices is any N with the property that

$$i > N \Rightarrow \|x^i\|_n > \sqrt{2}M'. \tag{16}$$

Such an N exists by the fact that

$$\|x^i\|_n \to \infty, \qquad \text{as } i \to \infty.$$

To prove our claim, we note that (16) implies

$$\|x_1^i\|_{n_1} > M' \quad \text{or} \quad \|x_2^i\|_{n_2} > M'.$$

By the symmetry of x_1^i and x_2^i, we can assume without loss of generality that

$$\|x_1^i\|_{n_1} \geq \|x_2^i\|_{n_2}, \qquad \|x_1^i\|_{n_1} > M'. \tag{17}$$

Consider now the inner product

$$
\begin{aligned}
&[F(x^i)-F(y)]^T(x^i-y)/\|x^i-y\|_n \\
&=[f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)+[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\|x^i-y\|_n \\
&\geq [f_1(x_1^i)-f_1(t_1)]^T(x_1^i-y_1) \\
&+[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\sqrt{2}\max\{\|x_1^i-y_1\|_{n_1}, \|x_2^i-y_2\|_{n_2}\}.
\end{aligned}
$$

Assumption (17) prevents us from using symmetry here, so we must consider separately each of two possible cases.

Case 1. $\|x_1^i-y_1\|_{n_1}\geq\|x_2^i-y_2\|_{n_2}$. In this case, we have

$$
\begin{aligned}
&[F(x^i)-F(y)]^T(x^i-y)/\|x^i-y\|_n \\
&\geq [f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)/\sqrt{2}\|x_1^i-y_1\|_{n_1} \\
&+[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\sqrt{2}\|x_1^i-y_1\|_{n_1}.
\end{aligned} \tag{18}
$$

From (11), we have

$$[f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)/\|x_1^i-y_1\|_{n_1}>2\sqrt{2}M+2B. \tag{19}$$

We can also minorize the second half of the right-hand side of (18) by noting that, if

$$[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)\geq 0, \tag{20}$$

then clearly

$$[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\|x_1^i-y_1\|_{n_1}\geq -B. \tag{21}$$

On the other hand, if the left-hand side of (20) is negative, then we can use the case assumption that

$$\|x_1^i-y_1\|_{n_1}\geq\|x_2^i-y_2\|_{n_2}$$

to show that

$$
\begin{aligned}
&[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\|x_1^i-y_1\|_{n_1} \\
&\geq [f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\|x_2^i-y_2\|_{n_2}\geq -B,
\end{aligned} \tag{22}
$$

and (21) once again holds. Inequality (18), taken together with (19) and (21), implies that

$$
\begin{aligned}
&[F(x^i)-F(y)]^T(x^i-y)/\|x^i-y\|_n \\
&\geq (1/\sqrt{2})[2\sqrt{2}M+2B-B]\geq M.
\end{aligned} \tag{23}
$$

From (23), it is clear that (15) holds for this case.

Case 2. $\|x_1^i - y_1\|_{n_1} \leq \|x_2^i - y_2\|_{n_2}$. Then, we have

$$\begin{aligned}
&[F(x^i)-F(y)]^T(x^i-y)/\|x^i-y\|_n\\
&\geq (1/\sqrt{2})\{[f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)/\|x_2^i-y_2\|_{n_2}\\
&\qquad +[f_2(x_2^i)-f_2(y_2)]^T(x_2^i-y_2)/\|x_2^i-y_2\|_{n_2}\}\\
&\geq (1/\sqrt{2})\{[f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)/\|x_2^i-y_2\|_{n_2}-B\}\\
&\geq (1/\sqrt{2})\{[f_1(x_1^i)-f_1(y_1)]^T(x_1^i-y_1)/2\|x_1^i-y_1\|_{n_1}-B\}\\
&\geq (1/\sqrt{2})\{(2\sqrt{2}M+2B)/2-B\}=M.
\end{aligned}$$

Thus, (15) holds in the second case as well, so F is coercive.

Now fix $m > 2$, and suppose that the theorem has been shown for functions partitionable into fewer than m partitions. Let F be partitionable with partitions $f_1(x_1), \ldots, f_m(x_m)$. Then, F can be partitioned into two partitions, namely,

$$g_1(X_1)=\begin{bmatrix} f_1(x_1)\\ \vdots \\ f_{m-1}(x_{m-1})\end{bmatrix} : \prod_{i=1}^{m-1} K_i \subseteq \mathbb{R}^{\sum_{i=1}^{m-1} n_i} \to \mathbb{R}^{\sum_{i=1}^{m-1} n_i}, \qquad X_1=\begin{bmatrix} x_1\\ \vdots \\ x_{m-1}\end{bmatrix},$$

$$g_2(X_2)=f_m(x_m): K_m \subseteq \mathbb{R}^{n_m} \to \mathbb{R}^{n_m}, \qquad X_2 = x_m.$$

We have shown that F is coercive if and only if g_1 and g_2 are coercive. Using the induction hypothesis, we see that F is coercive if and only if all the f_i are coercive, and the theorem is proved. □

4. Existence and Uniqueness of Solutions to Partitionable Variational Inequalities

Here, we use the properties of partitionable functions to derive sufficient conditions for the existence and uniqueness of solutions to partitionable variational inequalities, and we perform a sensitivity analysis. The results have conditions stated in terms of the partitions, rather than the entire state function F.

It is well known that the coercivity of the state function F is sufficient to guarantee the existence of a solution to any variational inequality with a convex (not necessarily compact) feasible set and state function F. Recalling Theorem 3.3, we immediately find the following theorem.

Theorem 4.1. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous and partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, where each set K_i is closed. If each partition f_i is coercive, then VI(F, K) has at least one solution.

The coercivity of F is by no means necessary to show existence. It is a rather strong condition to place on a function, and in many applications it is unrealistic to impose such a condition. In many cases, however, the special structure of the problem under study will allow us to prove the existence of solutions to partitionable variational inequalities under much weaker conditions. See, for example, McKelvey (Ref. 12).

To guarantee the uniqueness of a solution to a variational inequality, the standard condition to place on the state function F is to require it to be strictly monotone. In the case of partitionable variational inequalities, this result becomes the following theorem.

Theorem 4.2. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous and partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, where each set K_i is closed. If each partition f_i is strictly monotone, then $\mathrm{VI}(F, K)$ has at most one solution.

Using the well-known fact that strong monotonicity implies coercivity and strict monotonicity, we see that the following corollary holds.

Corollary 4.1. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous and partitionable with partitions $f_i: K_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, where each set K_i is closed. If each partition f_i is strongly monotone, then $\mathrm{VI}(F, K)$ has at least one solution.

5. Sensitivity Analysis

In problems which lead to variational inequalities, it is often important to know how the perturbations in the state function F are reflected in the solution of a variational inequality. Two approaches are commonly used to address this issue. The first, nonparametric or global sensitivity analysis, allows for a very general perturbation in the state function F. This approach is discussed for general variational inequalities in Dafermos and Nagurney (Ref. 2). The second approach, local or parametric sensitivity analysis, is useful when the perturbation in F is governed by a parameter. Under certain conditions, for small enough changes in the parameter, the continuity or differentiability of the solution of a variational inequality as a function of the parameter can be established. In the case of differentiability, it is possible to establish an analytical expression for the gradient. This parametric sensitivity analysis is undertaken for general variational inequalities by Dafermos (Ref. 7) and, under slightly more restrictive conditions, Kyparisis (Ref. 13) and Tobin (Ref. 14). A second approach to parametric sensitivity analysis for general variational inequalities is given in Harker and Pang

(Ref. 11). A comparison of all these approaches can be found in Kyparisis (Ref. 15).

We begin with a discussion of nonparametric sensitivity analysis in the case of partitionable variational inequalities. Specifically, we consider the case where the state function

$$F: \prod_{i=1}^{m} K_i \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

is partitionable of order m and can be written as

$$F_i(x) = f_i(x_i) + \sum_{j>i} M_{ij} x_j - \sum_{j<i} M_{ji}^T x_j, \qquad i = 1, 2, \ldots, m, \tag{24}$$

for appropriate matrices M_{ij}. Given the perturbations $f_i^*(x_i)$ of the partitions $f_i(x_i)$, we define the perturbed state function $F^*(x)$ by

$$F_i^*(x) = f_i^*(x_i) + \sum_{j>i} M_{ij} x_j - \sum_{j<i} M_{ji}^T x_j, \qquad i = 1, 2, \ldots, m. \tag{25}$$

The key to our nonparametric sensitivity analysis is the following result due to Dafermos and Nagurney (Ref. 2).

Theorem 5.1. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be any continuous, strictly monotone function over the convex set K. Let $F^*: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be any other continuous (although not necessarily monotone) function on K. Assume furthermore that x and x^* are solutions to $\mathrm{VI}(F, K)$ and $\mathrm{VI}(F^*, K)$, respectively. If $x \neq x^*$, we have

$$[F^*(x^*) - F(x^*)]^T (x^* - x) < 0.$$

It is important to note that there are no restrictions on the size or nature of the perturbations introduced to create F^* from F. The only assumption is the existence of at least one solution to $\mathrm{VI}(F^*, K)$.

Important consequences follow when this result is applied to the special case of partitionable variational inequalities.

Theorem 5.2. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be partitionable of order m with strictly monotone principal partitions $f_i(x_i)$. Define the matrices M_{ij} by Eq. (24). Consider perturbations $f_i^*(x_i)$ of the partitions $f_i(x_i)$ and define $F^*(x)$ by Eq. (25). If x is a solution to $\mathrm{VI}(F, K)$, x^* is a solution to $\mathrm{VI}(F^*, K)$, and $x \neq x^*$, then,

$$\sum_{i=1}^{m} [f_i^*(x_i^*) - f_i(x_i^*)]^T (x_i^* - x_i) < 0.$$

Proof. The first step of the proof is to note that, for any two elements

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

in the feasible set K, we have the equality

$$[F^*(x) - F(x)]^T(x-y) = \sum_{i=1}^{m} [f_i^*(x_i) - f_i(x_i)]^T(x_i - y_i).$$

This relationship can be verified in a tedious but straightforward manner using Eqs. (24) and (25). Using this observation, we see that

$$[F^*(x^*) - F(x^*)]^T(x^* - x)$$
$$= \sum_{i=1}^{m} [f_i^*(x_i^*) - f_i(x_i^*)]^T(x_i^* - x_i). \tag{26}$$

The strict monotonicity of the partitions f_i implies the strict monotonicity of the state function $F(x)$. Theorem 5.1 gives us

$$[F^*(x^*) - F(x^*)]^T(x^* - x) < 0. \tag{27}$$

Combining (26) with (27), we see that

$$\sum_{i=1}^{m} [f_i^*(x_i^*) - f_i(x_i^*)]^T(x_i^* - x_i) < 0. \qquad \square$$

Theorem 5.2 permits the isolation of a perturbation effect on the solution of a partitionable variational inequality. To illustrate this, suppose that only the first partition is perturbed; then, the theorem guarantees that

$$[f_1^*(x_1^*) - f_1(x_1^*)]^T(x_1^* - x_1) < 0.$$

Since f_1 and f_1^* are both evaluated at the same point, knowledge of the perturbation directly yields information on the shift in the vector x_1.

In addition to changing the value of the solution of a variational inequality, perturbing the state function F also changes the values of the partitions at the new solution. Since, in practice, the partitions are themselves of importance, it is interesting to examine how their values shift as a result of a perturbation.

Dafermos and Nagurney also consider this question in the realm of general variational inequalities; see Ineq. (3.5) in Ref. 2. Their result is the following theorem.

Theorem 5.3. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be any strictly monotone function over the convex set K. Let $F^*: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be any other continuous function on K. Assume furthermore that x and x^* are solutions to VI(F, K) and VI(F^*, K), respectively. Then,

$$[F^*(x^*) - F(x)]^T (x^* - x) \leq 0.$$

In the case of partitionable variational inequalities, this result becomes the following theorem.

Theorem 5.4. Let $F: K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be partitionable of order m with strictly monotonic principal partitions $f_i(x_i)$. Define the matrices M_{ij} by Eq. (24). Consider the perturbations $f_i^*(x_i)$ of the partitions $f_i(x_i)$, and define $F^*(x)$ by Eq. (25). If x is a solution of VI(F, K), x^* is a solution of VI(F^*, K), and $x \neq x^*$, then

$$\sum_{i=1}^{m} [f_i^*(x_i^*) - f_i(x_i)]^T (x_i^* - x_i) \leq 0.$$

Proof. The first step of the proof is to note that, for any two elements

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

in the feasible set K, we have the equality

$$[F^*(y) - F(x)]^T (y - x) = \sum_{i=1}^{m} [f_i^*(y_i) - f_i(x_i)]^T (y_i - x_i).$$

This relationship can be verified in a tedious but straightforward manner using Eqs. (24) and (25). Letting x and x^* play the roles of x and y above, we have

$$[F^*(x^*) - F(x)]^T (x^* - x) = \sum_{i=1}^{m} [f_i^*(x_i^*) - f_i(x_i)]^T (x_i^* - x_i).$$

From this statement, Theorem 5.3 tells us

$$\sum_{i=1}^{m} [f_i^*(y_i) - f_i(x_i)]^T (y_i - x_i) \leq 0. \qquad \square$$

This theorem demonstrates how the values of the decision variables shift with respect to the changes in the values of the partitions. As an example, if some partition $f_j(x_j)$ were a scalar function of the scalar x_j, then any perturbation in f_j designed to increase the value at the solution of $f_j(x_j)$ cannot simultaneously increase the value at the solution of x_j.

Next, we consider parametric perturbations of the partitions of a partitionable function F. In particular, we consider parameters $\lambda \in P \subseteq \mathbb{R}^k$, where P is an open set of parameter values. We are interested in the case where the function F is partitionable of order m and the dependence of F on the parameter λ is found only in the partitions, i.e.,

$$F(x; \lambda) = \begin{bmatrix} F_1(x; \lambda) \\ \vdots \\ F_m(x; \lambda) \end{bmatrix},$$

where

$$F_i(x; \lambda) = f_i(x_i; \lambda) + \sum_{j>i} M_{ij} x_j - \sum_{j<i} M_{ji}^T x_j, \qquad i = 1, 2, \ldots, m, \tag{28}$$

for appropriate matrices M_{ij}.

Given a partitionable function of the form (28), we are interested in the following problem: Suppose that, for some $\tilde{\lambda} \in P$, the variational inequality

$$F(\tilde{x}; \tilde{\lambda})^T (x - \tilde{x}) \geq 0, \qquad \forall x \in K,$$

has a solution $\tilde{x}$. We want to determine how the solution changes, given small changes in the parameter λ.

Dafermos (Ref. 7) considers the very general case where the state function F need not be partitionable and the feasible set $K = K(\lambda)$ is itself a function of the parameter λ. Most of the equilibrium problems which are amenable to partitionable variational inequality formulations have fixed feasible sets, so we consider here the special case where the feasible set is fixed.

In the case of fixed feasible sets, Dafermos' result is the following theorem.

Theorem 5.5. Consider the parametric variational inequality

$$F(\tilde{x}; \lambda)^T (x - \tilde{x}) \geq 0, \qquad \forall x \in K, \tag{29}$$

where $F(x; \lambda)$: $K \times P \subseteq \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. Further, suppose that VI(F, K) admits a solution $\tilde{x}$ for some $\tilde{\lambda} \in P$. If the function F satisfies

$$\begin{aligned} &[F(x; \lambda) - F(y; \lambda)]^T (x - y) \\ &\geq \alpha \|x - y\|^2, \qquad \forall \lambda \in P, x, y \in X \cap K, \end{aligned} \tag{30}$$

for some $\alpha > 0$, and

$$\|F(x; \lambda) - F(y; \lambda)\| \leq L\|x - y\|, \qquad \forall \lambda \in P, x, y \in X \cap K, \tag{31}$$

for some $L > 0$, where X is some open neighborhood about $\tilde{x}$, and if $F(x; \lambda)$ is Lipschitz continuous with respect to λ, then there exists a neighborhood Γ of $\tilde{\lambda}$ such that VI(F, K) admits a unique solution $x(\lambda)$ in $X \cap K$, $x(\tilde{\lambda}) = \tilde{x}$, and $x(\lambda)$ is Lipschitz continuous at $\lambda = \tilde{\lambda}$.

In the case where the state function F is partitionable, we are able to state all the conditions needed to demonstrate the Lipschitz continuity of $x(\lambda)$ in terms of the partitions.

Theorem 5.6. Consider the parametric variational inequality (29), and suppose that it admits a solution $\tilde{x}$ for some $\tilde{\lambda} \in P$. If F is partitionable of order m with partitions $f_i(x_i; \lambda)$ which satisfy

$$[f_i(x_i; \lambda) - f_i(y_i; \lambda)]^T (x_i - y_i) \geq \alpha_i \|x_i - y_i\|^2, \qquad \forall \lambda \in P, x_i, y_i \in X_i \cap K_i, \tag{32}$$

for some $\alpha_i > 0$, and

$$\|f_i(x_i; \lambda) - f_i(y_i; \lambda)\| \leq L_i \|x_i - y_i\|, \qquad \forall \lambda \in P, x_i, y_i \in X_i \cap K_i, \tag{33}$$

for some $L_i > 0$, where $X_i \subseteq \mathbb{R}^{n_i}$ is an open neighborhood about $\tilde{x}_i$, and if $f_i(x_i; \lambda)$ is Lipschitz continuous with respect to λ, then there exists an open neighborhood Γ of $\tilde{\lambda}$ such that variational inequality (29) admits a unique solution $x(\lambda)$ in $X \cap K$, where $X = \prod_{i=1}^m X_i$, $x(\tilde{\lambda}) = \tilde{x}$, and $x(\lambda)$ is Lipschitz continuous at $\lambda = \tilde{\lambda}$.

Proof. From Theorem 3.2, we see that (32) implies (30). From the form of the state function F given by (28), it is clear that (31) is implied from (33) and the Lipschitz continuity of F with respect to λ is equivalent to the Lipschitz continuity of the partitions f_i. Thus, the conditions of Theorem 5.5 are satisfied, and Theorem 5.6 is proved. □

Next, we move to the study of differentiable dependence on the parameter λ. We begin this study with a result for general variational inequalities due to Dafermos (Ref. 7). We consider again the parametric variational inequality (29), and we assume that the variational inequality has a solution $\tilde{x}$ when $\lambda = \tilde{\lambda}$.

For this analysis, the feasible set K is defined locally by means of equality-inequality constraints of the form

$$K \cap X = \{x \in X \| g_i(x) = 0, i = 1, \ldots, s, g_i(x) \geq 0, i = s+1, \ldots, l\},$$

where the g_i are twice continuously differentiable functions defined on X, a neighborhood of $\tilde{x}$, and satisfy

$$g_i(\tilde{x}) = 0, \qquad i = 1, \ldots, l.$$

To ensure that $K \cap X$ is locally convex, we add the condition that the $g_i(x)$ are locally affine about $\tilde{x}$ for $i = 1, \ldots, s$ and the $g_i(x)$ are locally concave about $\tilde{x}$ for $i = s, \ldots, l$. Furthermore, we assume that the l vectors

$$\{\nabla_x^T g_1(\tilde{x}), \ldots, \nabla_x^T g_l(\tilde{x})\} \tag{34}$$

are linearly independent. We also assume that the state function $F(x; \lambda)$ is continuously differentiable on $X \times \Gamma$, where Γ is a neighborhood of $\tilde{\lambda}$.

The vector $F(\tilde{x}; \tilde{\lambda})$ is orthogonal to K at $\tilde{x}$ and is directed toward the interior of K, so it lies inside the positive cone spanned by the vectors of (34); i.e., there exist nonnegative constants a_i such that

$$F(\tilde{x}; \tilde{\lambda}) = \sum_{i=1}^{l} a_i \nabla_x^T g_i(\tilde{x}). \tag{35}$$

For future reference, we define the matrices

$$A = \sum_{i=1}^{l} a_i \nabla_x \nabla_x^T g_i(\tilde{x}), \qquad G = \begin{bmatrix} \nabla_x g_1(\tilde{x}) \\ \vdots \\ \nabla_x g_l(\tilde{x}) \end{bmatrix}.$$

The rows of G span an l-dimensional subspace H which is orthogonal to the boundary of K at $\tilde{x}$. We let Q denote the orthogonal projection onto H.

We are now ready to state Dafermos' result for general variational inequalities. See Dafermos (Ref. 7).

Theorem 5.7. Consider the l-subspaces of dimension $l-1$, denoted by $H_1, \ldots, H_l$, where H_k is spanned by

$$\{\nabla_x^T g_1(\tilde{x}), \ldots, \nabla_x^T g_{k-1}(\tilde{x}), \nabla_x^T g_{k+1}(\tilde{x}), \ldots, \nabla_x^T g_l(\tilde{x})\}.$$

If

$$F(\tilde{x}; \tilde{\lambda}) \notin H_{s+1} \cup \cdots \cup H_l,$$

and if the linear transformation

$$(I - Q)(A - \nabla_x F(\tilde{x}; \tilde{\lambda})) : H^\perp \to H^\perp$$

is nonsingular, then the parametric variational inequality (29) admits a locally unique solution $x(\lambda)$ which is continuously differentiable on some neighborhood Λ of $\tilde{\lambda}$, and $\nabla_\lambda x(\tilde{\lambda})$ is given by

$$\nabla_\lambda x(\tilde{\lambda}) = -\rho D_\rho^{-1}(I - Q)\nabla_\lambda F, \tag{36}$$

where ρ is a sufficiently small positive number and D_ρ is the nonsingular matrix

$$D_\rho = Q - \rho(I - Q)(A - \nabla_x F).$$

In the case of partitionable variational inequalities, this result simplifies somewhat. As is the case with most results arising from partitionable variational inequalities, we can state the necessary conditions on the state function F using only the partitions f_i of F. In particular, we assume that each of the partitions of F satisfy the local strong monotonicity condition (32) and the local Lipschitz condition (33). We also assume that each partition f_i is continuously differentiable in both its argument x_i and the parameter λ. Theorem 3.2 along with (28) make it clear that these conditions on the partitions imply the differentiability, local strong monotonicity, and the local Lipschitz continuity required by Theorem 5.7.

The final simplification occurs in the expression for $\nabla_\lambda F$. It is clear from (28) that, in the case of partitionable functions,

$$\nabla_\lambda F = \begin{bmatrix} \nabla_\lambda F_1(\tilde{x}; \tilde{\lambda}) \\ \vdots \\ \nabla_\lambda F_m(\tilde{x}; \tilde{\lambda}) \end{bmatrix} = \begin{bmatrix} \nabla_\lambda f_1(\tilde{x}; \tilde{\lambda}) \\ \vdots \\ \nabla_\lambda f_m(\tilde{x}; \tilde{\lambda}) \end{bmatrix}.$$

Under these conditions, Theorem 5.7 can be restated for partitionable variational inequalities.

Theorem 5.8. Consider the l subspaces of dimension $l-1$, denoted by $H_1, \ldots, H_l$, where H_k is spanned by

$$\{\nabla_x^T g_1(\tilde{x}), \ldots, \nabla_x^T g_{k-1}(\tilde{x}), \nabla_x^T g_{k+1}(\tilde{x}), \ldots, \nabla_x^T g_l(\tilde{x})\}.$$

If

$$F(\tilde{x}: \tilde{\lambda}) \notin H_{s+1} \cup \cdots H_l,$$

then the parametric variational inequality (29) admits a locally unique solution $x(\lambda)$ which is continuously differentiable on some neighborhood Λ of $\tilde{\lambda}$ and $\nabla_\lambda x(\tilde{\lambda})$ is given by

$$\nabla_\lambda x(\tilde{\lambda}) = -\rho D_\rho^{-1}(I-Q) \begin{bmatrix} \nabla_\lambda f_1(\tilde{x}; \tilde{\lambda}) \\ \vdots \\ \nabla_\lambda f_m(\tilde{x}; \tilde{\lambda}) \end{bmatrix}, \tag{37}$$

where ρ is a sufficiently small positive number and D_ρ is the nonsingular matrix

$$D_\rho = Q - \rho(I-Q)(A - \nabla_x F).$$

One advantage of this formulation is that the effects of a perturbation in any of the partitions can be easily isolated in the case where $\nabla_x F$ is independent of λ.

6. Examples of Partitionable Variational Inequalities in the Current Literature

While partitionable variational inequalities have only been identified recently [see McKelvey (Ref. 12)], examples have appeared in the literature as far back as 1984. In this section, we present several of these formulations to demonstrate the broad applicability of partitionable variational inequalities. As we shall see, in applications the partition functions turn out to be the most interesting and important functions in the problem at hand. Thus, being able to state existence, uniqueness, and sensitivity analysis results in terms of these functions is very natural.

The first example of partitionable variational inequalities comes from the literature of general network equilibria. Dafermos and Nagurney (Ref. 8) consider a congested multimodal transportation network with a set of origin/destination pairs w, and a set P of paths broken down into subsets P_w containing all the paths connecting origin/destination pair w. The flow on path p using mode i is given by F_p^i. The flows on all paths for all modes are gathered together in the vector F. The cost of travel on path p using mode i is given by the function $C_p^i(F)$. The equilibrium cost of traveling between origin/destination pair w using mode i is given by v_w^i. As with the flow vector, the equilibrium costs for mode i are gathered together into a vector which is given by v^i. These, in turn, are gathered together to form the vector v of all equilibrium costs. The number of users traveling between origin/destination pair w using mode i is denoted $d_w^i = d_w^i(v)$. These demands are gathered together, by mode, to form the vectors d^i.

In this context, a flow is said to be in equilibrium if, for each mode i and origin/destination pair w, there is a number v_w^i so that all paths with positive flow between w have cost v_w^i, while unused paths have costs exceeding v_w^i. It is also required, of course, that the flows on paths serving an origin/destination pair meet the travel demand for that origin/destination pair.

Dafermos and Nagurney (Ref. 8) formulate this equilibrium problem as a variational inequality which turns out to be partitionable. Their variational inequality is

$$\sum_{i,w}\sum_{p\in P_w}[C_p^i(\tilde{F})-\tilde{v}_w^i][F_p^i-\tilde{F}_p^i]$$

$$+\sum_{i,w}\left[\sum_{p\in P_w}\tilde{F}_p^i-d_w^i(\tilde{v})\right][v_w^i-\tilde{v}_w^i]\geq 0, \qquad \forall (F,v)\in K, \tag{38a}$$

$$K=\{(F,v)\,|\,F\geq 0,\ v\geq 0\}. \tag{38b}$$

It is easily verified that this variational inequality is partitionable. The partitions turn out to be the two most important functions in the problem, namely $C(F)$, the vector of path costs, and $-d(v)$, the negative of the vector of origin/destination pair travel demands.

The study of equilibria in large-scale market activities has generated several formulations involving partitionable variational inequalities. The earliest is found in Freisz *et al.* (Ref. 4) in a study of the spatial price equilibrium problem with one commodity. The partitionable variational inequality is derived while transformating a nonlinear complementarity problem into a variational inequality for the limited purpose of developing numerical methods for the problem's solution. In a follow-up paper by Tobin (Ref. 5), the variational inequality formulation plays a key role in the analysis of the spatial price equilibrium model.

Freisz and Tobin consider the case of several markets l, each of which has a supply function $S_l(\pi)$ and a demand function $D_l(\pi)$, where π is a vector consisting of the prices of the commodity at every market. The price at a particular market, say market l, is denoted π_l.

In addition to the markets, there is an underlying transportation network with arc-node incidence matrix A. For each arc in this network, there is an arc flow f_a and an arc cost function $c_a(f)$ which depends on the flows along all the arcs in the network.

An equilibrium, in this context, consists of nonnegative arc flows and nonnegative prices such that the markets are exactly cleared and there is no incentive for a unilateral shift in the trading pattern. Freisz and Tobin show that, under certain reasonable technical conditions, this equilibrium problem can be cast as the variational inequality

$$[c(\tilde{f})+A^T\tilde{\pi}]^T(f-\tilde{f})+[S(\tilde{\pi})-D(\tilde{\pi})-A\tilde{f}]^T(\pi-\tilde{\pi})\geq 0,$$

for all $f\geq 0$ and $\pi\geq 0$.

This variational inequality is partitionable with partitions $c(f)$ and $S(\pi)-D(\pi)$. As was the case in the traffic equilibrium problem, the partitions are the central functions in the study of the problem. In this case, $c(f)$ represents the transportation cost, and $S(\pi)-D(\pi)$ is the net supply function for each market.

In their remarks on this variational inequality, Freisz and Tobin noted that the Jacobian of the state function is not symmetric, regardless of the form of $c(f)$, $S(\pi)$, and $D(\pi)$. This fact holds for almost all partitionable variational inequalities for reasons developed earlier in this paper. The only exceptions are those cases where all the matrices M_{ij} of Theorem 2.1 are zero matrices and the state function F is separable.

While the formulations suggested by Dafermos and Nagurney as well as the work by Freisz and Tobin result in partitionable variational inequalities, the authors of these papers did not identify the general structure common to the entire class of partitionable variational inequalities. The first work to use the more general characteristics of partitionable variational inequalities was that of Dafermos and McKelvey (Ref. 9).

Dafermos and McKelvey consider a general multicommodity market equilibrium problem. Each market is designated as a supply market, denoted α, or a demand market, denoted β. For commodity i, each supply market has a corresponding supply function $S^i_\alpha(\pi)$, where π is the vector of prices for all commodities at all supply markets. Each demand market has an associated demand function for commodity i, denoted $D^i_\beta(\rho)$, where ρ is the vector of prices for all commodities at all demand markets.

In addition to supplies and demands, the market allows transactions between supply and demand markets. If we let $Q^i_{\alpha\beta}$ denote the quantity of commodity i purchased from supply market α by demand market β, then we can define the unit transaction cost function $t^i_{\alpha\beta}(Q)$, which we note depends on the entire transaction pattern Q.

In this context, an equilibrium exists if no supplier or consumer has an incentive to unilaterally change their trading partners or the level of transaction with those partners. In addition, we require in the face of strictly positive prices that all markets clear exactly.

Dafermos and McKelvey do not require market clearing in the case of zero prices. If a given commodity at a supply market has zero price, we allow the possibility of excess production over exports. If a given commodity has zero price at a demand market, we allow the possibility of excess imports over consumption.

With this notion of equilibrium, Dafermos and McKelvey show that a combination of transaction pattern $\tilde{Q}$ and prices $\tilde{\pi}$ and $\tilde{\rho}$ form an equilibrium if and only if they satisfy the following variational inequality:

$$\begin{aligned}&\sum_{i,\alpha,\beta} [\tilde{\pi}^i_\alpha + t^i_{\alpha\beta}(\tilde{Q}) - \tilde{\rho}^i_\beta](Q^i_{\alpha\beta} - \tilde{Q}^i_{\alpha\beta})\\ &-\sum_{i,\alpha}\left[\sum_\beta \tilde{Q}^i_{\alpha\beta} - S^i_\alpha(\tilde{\pi})\right](\pi^i_\alpha - \tilde{\pi}^i_\alpha)\\ &+\sum_{i,\beta}\left[\sum_\alpha \tilde{Q}^i_{\alpha\beta} - D^i_\beta(\tilde{\rho})\right](\rho^i_\beta - \tilde{\rho}^i_\beta) \geq 0, \quad \text{for all} \begin{bmatrix} Q \\ \pi \\ \rho \end{bmatrix} \geq 0.\end{aligned}$$

This variational inequality is partitionable of order three. The partitions are the three key functions in the problem, namely, the vector of transaction costs $t(Q)$, the vector of supplies $S(\pi)$, and the negative of the vector of demands, $-D(\rho)$. Thus, statements concerning existence, uniqueness, and sensitivity analysis can be made in terms of these three key quantities.

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