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Differential Games of Fixed Duration with State Constraints[1]

K. H. GHASSEMI[2]

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Abstract. We consider differential games of fixed duration with phase coordinate restrictions on the players. Results of Ref. 1 on games with phase restrictions on only one of the players are extended. Using Berkovitz's definition of a game (Ref. 2), we prove the existence and continuity (or Lipschitz continuity) of the value under appropriate assumptions. We also note that the value can be characterized as the viscosity solution of the associated Hamilton-Jacobi-Isaacs equation.

Key Words. Differential games, phase restrictions, value function, viscosity solutions.

1. Introduction

Let E_1 and E_2 be two given closed sets in R^n. Consider a differential game of fixed duration governed by an ordinary differential equation and with terminal payoff in which the ith player, $i=1,2$, must choose his strategies in such a way that all resulting trajectories lie in E_i. Such restrictions can be formalized by using an extended real-valued payoff which penalizes each player for violating his constraint.

In introducing a new notion of a game in Ref. 2 (see also Ref. 3), Berkovitz considered a game with no state constraints, i.e., the case where $E_1=E_2=R^n$. In Ref. 1, we studied the case where only one of the E_i's equals R^n. Here, using Berkovitz's definitions, we examine the questions

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[2] Staff member, Rockwell International Space Systems Division, Downey, California.

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of the existence of value and its regularity properties in games where both E_1 and E_2 are allowed to be proper subsets of R^n. We show that, under appropriate assumptions, such games can be viewed as a finite sequence of games in each of which only one of the players faces the possibility of violating his constraint. We then use an inductive argument to extend the results of Ref. 1 to conclusions about the general case.

Games with phase coordinate restrictions as above have been considered previously by Friedman (Refs. 4 and 5), Scalzo (Ref. 6), Subbotin (Ref. 7), and Zaremba (Refs. 8 and 9). Each author, using a particular definition of a game, establishes the existence (Refs. 4–9) and continuity or Lipschitz continuity (Refs. 4–6) of the value. In Refs. 4, 5, 7, games with phase restrictions on essentially only the players are treated. In Ref. 6, Scalzo extends the results of Friedman to games with phase restrictions on both players. Our technique for extending the results of Ref. 1 is similar in spirit to that of Ref. 5, but we make fewer assumptions on the data.

The paper is organized as follows. In Section 2, we state the basic assumptions on the data of the problem and review the definitions and notations used in the paper. Section 3 recalls, for the convenience of the reader, the results of Refs. 1 and 2, which we will need in the later sections. In Section 4, we state and prove the main theorem (Theorem 4.1). In Section 5, we discuss properties which characterize the value of such games, as the unique constrained viscosity solution (see Refs. 10 and 11) of the associated Hamilton-Jacobi-Isaacs equation.

2. Definitions, Basic Assumptions, and Notation

The following basic notation is used in the paper. The letter B stands for the closed unit ball in R^n. If $E \subset R^n$, then

$$d(x, E) = \inf\{|x - e|: e \in E\}.$$

If E is a domain with C^2-boundary ∂E and $x \in \partial E$, then $n(x)$ denotes the unit inward normal to ∂E at x. $C([a, b])$ denotes the space of all continuous functions $x: [a, b] \to R^n$ with the uniform topology. If F and G are subsets of R^n, then $\mathrm{int}(F)$ denotes the interior of F, and $F \setminus G$, the set $\{x \in F: x \notin G\}$. Finally, if F is a subset of the domain of a given function f, then we will use $f \lceil F$ to denote the restriction of f to F, and $\mathrm{supp}(f)$ to denote the closure of $\{x: f(x) \neq 0\}$.

We now briefly review the definition of a differential game according to Berkovitz. For a more complete discussion and the relationship between this and other definitions, the reader should consult Refs. 2 and 3.

Let Y and Z be compact subsets of Euclidean spaces R^p and R^q, respectively, and $f:[0, T]\times R^n \times Y\times Z \to R^n$. If J is an interval in R, let $\mathcal{Y}(J)$ and $\mathcal{Z}(J)$ denote the set of all measurable functions from J into Y and Z, respectively. Let $T>0$ be given. A zero-sum two-person differential game of fixed duration ($\leq T$) is completely defined once we specify (a) what is meant by a strategy for each of the players and (b) what payoff is assigned to a pair of strategies. In Berkovitz's definition, these are done as follows:

(a) Strategies. A strategy Γ of player I in a game with initial point (t_0, x_0), $t_0 \in [0, T)$, is a sequence $\{\Gamma_n, \Pi_n\}$ where Π_n is a (not necessarily uniform) partition $\{t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{p_n} = T\}$ of the interval $[t_0, T]$ and, for each n, Γ_n is a collection of mappings $\{\Gamma_{n_0}, \Gamma_{n_1}, \ldots, \Gamma_{n_{p_n}}\}$ with

$$\Gamma_{n_0} \in \mathcal{Y}([\tau_0, \tau_1)),$$

$$\Gamma_{n_i} : \mathcal{Y}([\tau_0, \tau_i)) \times \mathcal{Z}([\tau_0, \tau_i)) \to \mathcal{Y}([\tau_0, \tau_{i+1})).$$

A strategy Δ for player II is defined in a parallel manner. It is not required that the players use the same partitions. The letters Γ and Δ will, throughout the paper, denote respectively strategies of player I and player II.

(b) Payoff. To each pair of strategies (Γ, Δ), one associates a set of absolutely continuous functions, called motions, as follows. First, note that, to a pair of strategies, we can associate a sequence of pairs of control functions $\{(u_n, v_n)\}$ where, letting $\{t_0 = s_0 < s_1 < \cdots < s_{q_n} = T\}$ be the nth partition for Δ, we have

$$u_n \in \mathcal{Y}([\tau_0, T)), \qquad v_n \in \mathcal{Z}([\tau_0, T)),$$

$$u_n|[\tau_i, \tau_{i+1}) = \Gamma_{n_i}(u_n|[\tau_0, \tau_i), v_n|[\tau_0, \tau_i)),$$

$$v_n|[s_i, s_{i+1}) = \Delta_{n_i}(v_n|[\tau_0, s_i), u_n|[\tau_0, s_i)).$$

Now, let x_{0n} be a sequence converging to x_0. For each n, consider the initial-value problem

$$\dot{x}(t) = f(t, x(t), u_n(t), v_n(t)), \qquad \forall t \in [t_0, T], \tag{1a}$$

$$x(t_0) = x_{0n}. \tag{1b}$$

Conditions are imposed (see below) on the dynamics to ensure that (1) has a unique solution φ_n defined on $[t_0, T]$ and that the sequence $\{\varphi_n\}$ is uniformly bounded and equicontinuous on $[t_0, T]$. Such a φ_n is called an nth stage trajectory, and the set of all nth stage trajectories is denoted by $\Phi(\cdot, t_0, x_{0n}, \Gamma, \Delta)$. It follows from Ascoli's theorem that the sequence $\{\varphi_n\}$ has accumulation points φ in $C([t_0, T])$. Such limit points are called motions. The set of all motions corresponding to a pair (Γ, Δ) is denoted by $\Phi[\cdot, t_0, x_0, \Gamma, \Delta]$. Note that $\Phi[\cdot, t_0, x_0, \Gamma, \Delta]$ is independent of the choice of the sequence x_{0n}. For notational brevity, we will write $\Phi[\cdot, t_0, x_0, \Delta]$ for $\bigcup_\Gamma \Phi[\cdot, t_0, x_0, \Gamma, \Delta]$ and $\Phi[\cdot, t_0, x_0, \Gamma]$ for $\bigcup_\Delta \Phi[\cdot, t_0, x_0, \Gamma, \Delta]$. We are now ready to define the payoff $P[t_0, x_0, \Gamma, \Delta]$ associated to a pair (Γ, Δ) for a game with initial point (t_0, x_0). Let $g: R^n \to R$ be a given function. Define

$$P[t_0, x_0, \Gamma, \Delta] = \{g(\varphi[T]): \varphi \in \Phi[t_0, x_0, \Gamma, \Delta]\}.$$

We assume the following concerning the data of the problem throughout the paper.

Assumption A1. (i) The function f in (1a) is continuous on $[0, T] \times R^n \times Y \times Z$.

(ii) The function $(\tau, \xi) \to f(\tau, \xi, y, z)$ is locally Lipschitz, uniformly in y and z, on $[0, T] \times R^n$.

(iii) The function g, used to define the payoff, is continuous on R^n.

(iv) Isaacs' Condition. For any $(t, x) \in [0, T] \times R^n$, and $s \in R^n$,

$$\max_y \min_z \langle s, f(t, x, y, z)\rangle = \min_z \max_y \langle s, f(t, x, y, z)\rangle.$$

Remark 2.1. For a discussion of the Isaacs' condition, see Section 9 of Ref. 2. The Isaacs condition may be dropped by allowing relaxed controls as admissible choices of control function for each player (see Ref. 12).

3. Review of Known Results

Let E_1 and E_2 be two given closed subsets of R^n. Consider the game defined in the sense described in Section 2 with dynamics

$$\dot{x} = f(t, x, y, z), \tag{2}$$

where $y \in Y$ and $z \in Z$ denote the control parameters of palyers I and II, respectively, and the penalized payoff is defined by

$$P[t_0, x_0, \Gamma, \Delta] = \{V(\varphi): \varphi \in \Phi[t_0, x_0, \Gamma, \Delta]\},$$

where

$$V(\varphi) = \begin{cases} g(\varphi[T]), & \text{if } \varphi[t] \in E_1 \cap E_2, \quad \forall t \in [t_0, T], \\ -\infty & \text{if } \varphi \text{ leaves } E_1 \text{ first}, \\ +\infty, & \text{if } \varphi \text{ leaves } E_2 \text{ first}. \end{cases} \tag{3}$$

Here by "φ leaves E_1 first" we mean that there exists $\bar{t} \in [t_0, T]$ such that:

(i) for all $t \in [t_0, \bar{t}]$, $\varphi[t] \in E_1 \cap E_2$;
(ii) there exists $\delta > 0$ such that, for all $t \in (\bar{t}, \bar{t} + \delta)$, $\varphi[t] \in E_2 \backslash E_1$.

Similarly for "φ leaves E_2 first." Clearly, we need to impose conditions to ensure that $V(\cdot)$ is well-defined; see Remark 4.2 below.

For $(t, x) \in [0, T] \times R^n$, let us define the upper and lower values of the game with initial point (t, x), $t \in [0, T)$, by

$$w^+(t, x) = \inf_{\Delta} \sup_{\Gamma} P[t, x, \Gamma, \Delta],$$

$$w^-(t, x) = \sup_{\Gamma} \inf_{\Delta} P[t, x, \Gamma, \Delta],$$

respectively, and

$$w^+(T, x) = w^-(T, x) = g(x).$$

The game is said to have value if $w^+ = w^-$, and the common value is denoted by $w(t, x)$. For a real number α, let

$$C^+(\alpha) = \{(t, x): x \in E_1 \cap E_2, w^+(t, x) \geq \alpha\},$$

$$C^-(\alpha) = \{(t, x): x \in E_1 \cap E_2, w^-(t, x) \leq \alpha\}.$$

Suppose first that $E_1 = E_2 = R^n$. Then one can show (see Theorem 7.1 of Ref. 2) that Assumption A1 implies that w^+ and w^- are continuous on $[0, T] \times R^n$ (locally Lipschitz if g is locally Lipschitz). It follows that the sets $C^+(\alpha)$ and $C^-(\alpha)$ are closed. This allows the introduction of extremal pointing strategies Γ_e and Δ_e. We will briefly describe Γ_e for a game with initial point (t_0, x_0). For more detail, see Section 10 of Ref. 2.

Let $\Gamma_e = \{\Gamma_n, \Pi_n\}$, where Π_n is a uniform partition, $\{t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{p_n} = T\}$, and $\Gamma_n = \{\Gamma_{n_i}\}_{i=0}^{n}$. To define Γ_{n_i}, $i \geq 0$, let (u, v) be a pair of admissible control functions on $[t_0, \tau_i]$ and $\varphi(\cdot)$ the solution of (2) on $[t_0, \tau_i]$ with controls u and v and $\varphi(t_0) = x_0$. Let w_i satisfy

$$|w_i - \varphi(\tau_i)| = \min\{|w - \varphi(\tau_i)|: (\tau_i, w) \in C^+(w^+(t_0, x_0))\},$$

and set

$$s_i = w_i - \varphi(\tau_i).$$

Define $\Gamma_{n_i}(u, v) = y^*$ with y^* determined as follows. If $s_i = 0$, let y^* denote an arbitrarily picked element of Y. If $s_i \neq 0$, let $y^* \in Y$ be such that, for some $z^* \in Z$, (y^*, z^*) forms a saddle point on $Y \times Z$ for the function $(y, z) \mapsto \langle s_i, f(\tau_i, \varphi(\tau_i), y, z)\rangle$. The existence of (y^*, z^*) follows from A1(iv). The strategy Δ_e is defined similarly using $C^-(w^-((t_0, x_0)))$. The importance of these extremal strategies lies in that (cf. Lemma 10.1 of Ref. 2), if

$\varphi \in \Phi[t_0, x_0, \Gamma_e]$ (respectively, $\varphi \in \Phi[t_0, x_0, \Delta_e]$), then $(t, \varphi[t]) \in C^+(w^+(t_0, x)))(C^-(w^-(t_0, x_0)))$ for all $t \in [t_0, T]$. It follows from this property of the extremal strategies that (Γ_e, Δ_e) provides a saddle point for the game at (t_0, x_0). In particular, the game has value.

Theorem 3.1. (Theorem 10.1 of Ref. 2). If $E_1 = E_2 = R^n$, then the game above with any initial point $(t, x) \in [0, T) \times R^n$ has value $w(t, x)$ and $w(\cdot, \cdot)$ is continuous on $[0, T] \times R^n$ (locally Lipschitz if g is).

In the case when one of the E_i's is a proper closed subset of R^n, the upper and lower values are no longer continuous. However, one can show (see Lemma 1.2 of Ref. 1) that, if $E_2 = R^n$, then w^+ is upper semicontinuous. Thus, in that case, the sets $C^+(\alpha)$ are closed. Therefore, one can define Γ_e as before and obtain the following lemma.

Lemma 3.1. (Lemma 1.5 of Ref. 1). Suppose that $E_2 = R^n$. If $\varphi \in \Phi[t_0, x_0, \Gamma_e]$, then $(t, \varphi[t]) \in C^+(w^+(t_0, x_0))$ for all $t \in [t_0, T]$.

The existence of value follows (see Theorem 1.1 of Ref. 1). Similar results are obtained if only $E_1 = R^n$. We will also need the following result, obtained in Ref. 1 (parallel statements hold for the case $E_1 = R^n$).

Theorem 3.2. (Theorem 3.1 of Ref. 1). Let $E_2 = R^n$. Suppose that E_1 is the closure of a domain with C^2 boundary. If

$$\max_y \min_z \langle n(x), f(t, x, y, z)\rangle > 0, \qquad \text{for all } (t, x) \in [0, T] \times \partial E_1,$$

then w is continuous on $[0, T] \times E_1$ (locally Lipschitz if g is).

4. Phase Restrictions on Both Players

In this section, we allow for both of the E_i's to be proper subsets of R^n. We make the following assumption.

Assumption A2. (i) The phase sets E_1 and E_2 are closures of C^2 domains in R^n.

(ii) Assumption A1 holds for the data.

(iii) If $n_i(x)$ denotes the inward unit normal to ∂E_i at $x \in \partial E_i$, then

(a) $\max_y \min_z \langle n_1(x), f(t, x, y, z)\rangle > 0$, for all $(t, x) \in [0, T] \times \partial E_1$,

(b) $\max_z \min_y \langle n_2(x), f(t, x, y, z)\rangle > 0$, for all $(t, x) \in [0, T] \times \partial E_2$.

(iv) Either $E_1 \cap E_2 = \emptyset$ or, if $x \in E_1 \cap E_2$, then for all $t \in [0, T]$, $y \in Y$, and $z \in Z$,

$$\langle n_i(x), f(t, x, y, z)\rangle > 0, \qquad i = 1, 2.$$

Remark 4.1. Assumptions A2(i), (ii), and (iii)(a) imply by Theorem 3.2 that, if we take $E_2 = R^n$, then the corresponding games will have continuous value on $[0, T] \times E_1$. Similarly, by virtue of A2(iii)(b), the games with $E_1 = R^n$ have continuous value on $[0, T] \times E_2$.

Remark 4.2. Suppose that $E_1 \cap E_2 \neq \emptyset$. For $\lambda > 0$, define

$$\partial_\lambda E = \{x \in E_1 \cap \partial E_2 : d(x, \partial E_1) < \lambda\} \cup \{x \in E_2 \cap \partial E_1 : d(x, \partial E_2) < \lambda\}. \tag{4}$$

Then, it follows from our continuity assumptions on the dynamics f and A2(iv) that, for every $R > 0$, there exists $\lambda = \lambda(R) > 0$ such that, for every $x \in \partial_\lambda E$ with $|x| \leq R$,

$$\min\{\langle n_i(x), f(t, x, y, z)\rangle : i = 1, 2\} \geq 0, \tag{5}$$

for all $y \in Y$, $z \in Z$, and $t \in [0, T]$. Hence no nth stage trajectory (and therefore no motion), contained in $\{x: |x| \leq R\}$, can leave $E_1 \cap E_2$ through $\partial_\lambda E$. Hence, if a motion $\varphi \in \Phi[\cdot, t_0, x_0, \Gamma, \Delta]$ should leave $E_1 \cap E_2$, then either "φ leaves E_1 first" or "φ leaves E_2 first." Therefore, $V(\varphi)$ is defined unambiguously by (3). If $E_1 \cap E_2 = \emptyset$, then the same is true trivially.

Remark 4.3. It will be seen below that A2(i), (iii), (iv) may be replaced by any other assumptions which guarantee: (a) E_1 and E_2 are closed; (b) if $E_1 = R^n$, then the corresponding game has continuous value (and similarly if $E_2 = R^n$); and (c) for every $R > 0$, there exists a $\lambda = \lambda(R) > 0$ such that, if φ is an nth stage trajectory with $|\varphi(t)| \leq R$ for all t, then φ does not leave $E_1 \cap E_2$ through a λ-neighborhood of $\partial E_1 \cap \partial E_2$. Thus, for example, Theorems 3.2 and 3.3 of Ref. 1 provide alternative hypotheses to A2(iii) if E_1 and E_2 are arbitrary closed subsets of R^n.

Let $R > 0$. Denote by $S(R)$ the set of all initial points $(t_0, x_0) \in [0, T] \times E_1 \cap E_2$ such that any solution $\varphi(\cdot)$ of (2) with $\varphi(t_0) = x_0$ satisfies $|\varphi(t)| \leq R$ for all $t \in [t_0, T]$. It follows from assumption A1 that, given any $(t_0, x_0) \in [0, T] \times E_1 \cap E_2$, there exists $R_0 > 0$ such that $(t_0, x_0) \in S(R_0)$. Hence,

$$[0, T] \times E_1 \cap E_2 = \bigcup \{S(R): R > 0\}.$$

Furthermore, we have the following lemma.

Lemma 4.1. $S(R)$ is compact. If $(t_0, x_0) \in S(R)$, then for every $t \in [t_0, T]$ and for every motion $\varphi \in \Phi[\cdot, t_0, x_0, \Gamma, \Delta]$, we have $(t, \varphi[t]) \in S(R)$.

Proof. The proof is straightforward; see "note iv" in Section 3 of Ref. 1. □

For notational convenience, we will write S_t for $S(R) \cap \{(t, x): x \in E_1 \cap E_2\}$ and $S_{[\sigma,\tau]}$ for $S(R) \cap \{(t, x): t \in [\sigma, \tau], x \in E_1 \cap E_2\}$.

Set

$$M = M(R) = \max\{|f(t, x, y, z)|: t \in [0, T], |x| \leq R, y \in Y, z \in Z\}.$$

Let $\lambda = \lambda(R)$ be as in Remark 4.2 above. Define

$$h = h(R) = \lambda/4M. \tag{6}$$

It is clear that, if $(t_0, x_0) \in S(R)$ and $\varphi[\cdot]$ is a motion in a game starting at (t_0, x_0), then

$$|\varphi[t] - x_0| \leq \lambda/4, \qquad \text{for all } t \in [t_0, t_0 + h]. \tag{7}$$

Lemma 4.2. Let $R > 0$ and $S = S(R)$. Then, there exists $h = h(R) > 0$ such that a game with initial point $(t, x) \in S_{[T-h,T]}$ has value $w(t, x)$ and w is continuous on $S_{[T-h,T]}$.

Proof. Let R be given. Take λ and h to be as defined above with respect to R. Let $(t_0, x_0) \in S_{[T-h,T]}$. Define

$$\Omega_1 = \{x \in E_1 \cap E_2: d(x, \partial E_2) \geq 3\lambda/4, d(x, \partial E_1) \leq \lambda/4\},$$

$$\Omega_2 = \{x \in E_1 \cap E_2: d(x, \partial E_2) \leq \lambda/4, d(x, \partial E_1) \geq 3\lambda/4\},$$

$$\Omega_3 = \{x \in E_1 \cap E_2: d(x, \partial E_2) \leq 3\lambda/4, d(x, \partial E_1) \leq 3\lambda/4\},$$

$$\Omega_4 = \{x \in E_1 \cap E_2: x \notin \Omega_1 \cup \Omega_2 \cup \Omega_3\}.$$

If $x_0 \in \Omega_3 \cup \Omega_4$, then by (7) and Remark 4.2, all motions starting at (t_0, x_0) remain in $E_1 \cap E_2$ for the duration of the game. Therefore, the game starting at such a point is a game with no phase restrictions since $V(\varphi) = g(\varphi[T])$ for all motions φ. By Theorem 3.1, $w(t_0, x_0)$ exists. Moreover, w is continuous on $(\Omega_3 \cup \Omega_4) \cap S_{[T-h,T]}$.

If $x_0 \in \Omega_1$, then by (7), all motions stay in E_2 for the duration of the game. Hence, the game is one with phase restrictions on only the maximizer, player I, since $V(\varphi) \neq \infty$ for all motions φ. By Theorem 1.1 of Ref. 1, such a game has value $w(t_0, x_0)$. Moreover, in view of assumption A2(iii)(a) and Theorem 3.3, w is continuous on $\Omega_1 \cap S_{[T-h,T]}$. Observe that, if $x_0 \in \Omega_1 \cap (\Omega_3 \cup \Omega_4)$, then $V(\varphi) = g(\varphi[T])$ for all motions. Therefore, $w(t_0, x_0)$ is the same whether the game is regarded as one with no phase restrictions or as one with phase restrictions on the maximizer. It follows that w is continuous on $S_{[T-h,T]} \cap (\Omega_1 \cup \Omega_3 \cup \Omega_4)$.

Similarly, if $x_0 \in \Omega_2$, the game is one with phase restrictions on the minimizer only. Hence, $w(t, x)$ exists and, by assumption A2(iii)(b), is continuous on $\Omega_2 \cap S_{[T-h,T]}$. Since $\Omega_1 \cap \Omega_2 = \varnothing$, we get, as in the case of Ω_1, that w is defined and continuous on $S_{[T-h,T]} \cap (\Omega_1 \cup \Omega_3 \cup \Omega_4)$.

Since $E_1 \cap E_2 = \bigcup_{i=1}^{4} \Omega_i$, w is continuous on all of $S_{[T-h,T]}$. □

Remark 4.4. If in Assumption A1 we require that $g(\cdot)$ be locally Lipschitz, then the same proof shows that w will also be locally Lipschitz on $S_{[T-h,T]}$.

Remark 4.5. Examining the proof of Theorem 3.2, one finds that it is sufficient, for continuity or local Lipschitz continuity of w on $S_{[T-h,T]}$, to assume $g(\cdot)$ to be continuous or locallly Lipschitz relative to the closed set S_T, rather than on all of $E_1 \cap E_2$.

Remark 4.6. The choice of h above depends only on the dynamics, the compact control sets Y and Z, and the constant R. Hence, Lemma 4.2 can be applied to games with initial points in $S_{[T-kh,T-(k-1)h]}$ and with terminal time $T-(k-1)h$, provided the payoff is defined through a function which is continuous relative to $S_{T-(k-1)h}$.

We will need the following lemma in the proof of Lemma 4.4 below. This is parallel to Lemma 1.3 of Ref. 1, to which we refer the reader for proof.

Lemma 4.3. Consider a game with phase restrictions on only the minimizer, i.e., $E_1 = R^n$. Let $t_1 \in [0, T]$, and let $X \subset E_1 \cap E_2$ be a compact set such that, for some real number α,

$$w^{-}(t_1, x) > \alpha, \qquad \text{for every } x \in X.$$

Then, there exists a strategy Γ^*, defined on $[t_1, T]$, such that

$$V(\varphi) > \alpha, \qquad \text{for all } \varphi \in \{\Phi[\cdot, t_1, x, \Gamma^*]: x \in X\}.$$

Lemma 4.4. Let X be a compact subset of $E_1 \cap E_2$ with $(T-h, x) \in S_{T-h}$ for all $x \in X$. Suppose that, for some $\alpha \in R$,

$$w(T-h, x) > \alpha, \qquad \text{for all } x \in X.$$

Then, there exists a strategy Γ on $[T-h, T]$, independent of x, such that

$$V(\varphi) > \alpha, \qquad \text{for all } \varphi \in \bigcup \{\Phi[\cdot, T-h, x, \Gamma]: x \in X\}.$$

Similarly, if $w(T-h, x) < \alpha$ for all $x \in X$, then there exists Δ such that

$$V(\varphi) < \alpha, \qquad \text{for all } \varphi \in \bigcup \{\Phi[\cdot, T-h, x, \Delta]: x \in X\}.$$

Proof. We will prove the first statement. The second statement can be treated in a similar manner. By Lemma 4.2, w is continuous on X. Since X is compact, there exists an α' such that

$$w(T-h, x) \geq \alpha', \qquad \text{for all } x \in X. \tag{8}$$

Let Ω_i, $i=1, \ldots, 4$, be as in the proof of Lemma 4.2. Define $X_1 = X \cap \Omega_1$ and $X_2 = X \backslash X_1$. Clearly, X_i's are compact and $X = X_1 \cup X_2$. Furthermore, if $x \in X_i$, then the game with initial point $(T-h, x)$ is a game with phase restrictions, if any, on only the ith player (see the proof of Lemma 4.2). Consider the case when $x \in X_1$. Then, the game with initial point $(T-h, x)$ is a game with phase restrictions on the maximizer, as considered in Section 1. Let Γ_1 be the extremal strategy defined with respect to $C^+(\alpha')$. Note that, by (8),

$$\{(T-h, x): x \in X_1\} \subset C^+(\alpha').$$

Since X_1 is compact, we may apply Lemma 3.1 to obtain

$$V(\varphi) \geq \alpha' > \alpha, \qquad \text{for all } \varphi \in \bigcup \{\Phi[\cdot, T-h, x, \Gamma_1]: x \in X_1\}. \tag{9}$$

In the case when $x \in X_2$, the game with initial point $(T-h, x)$ is a game with phase restrictions on the minimizer. Note that, by (8), $w(T-h, x) > \alpha$, for all $x \in X_2$, and X_2 is compact. By Lemma 4.3, there exists a strategy Γ_2 such that

$$V(\varphi) > \alpha', \qquad \text{for all } \varphi \in \bigcup \{\Phi[\cdot, T-h, x, \Gamma_2]: x \in X_2\}. \tag{10}$$

Now, take Γ to be the strategy whose nth stage partition is the common refinement of those of Γ_1 and Γ_2. Let the nth stage of Γ play the nth stage of Γ_i if the initial point of the game, $(T-h, x)$, has $x \in X_i$. In the case $x \in X_1 \cap X_2$, the corresponding game is one with no phase restrictions and can play either Γ_1 or Γ_2 at the nth stage. For the sake of definiteness, let us say Γ plays Γ_1 in such a case. Note that any motion φ in $\Phi[\cdot, T-h, x, \Gamma]$ is either in $\Phi[\cdot, T-h, x, \Gamma_1]$ or in $\Phi[\cdot, T-h, x, \Gamma_2]$. Therefore, it follows from (9) and (10) that Γ has the desired property. □

We now state and prove the main theorem of this section.

Theorem 4.1. Assume A2. Then, the game starting at a point $(t_0, x_0) \in [0, T] \times E_1 \cap E_2$ has value $w(t_0, x_0)$. Moreover, w is continuous on $[0, T] \times E_1 \cap E_2$ [locally Lipschitz if $g(\cdot)$ is locally Lipschitz].

Proof. Note that

$$[0, T] \times E_1 \cap E_2 = \bigcup \{S(R): R > 0\}.$$

Hence, it suffices to show that, for an arbitrarily chosen R, the statement of the theorem holds with $[0, T]\times E_1\cap E_2$ replaced by $S(R)$. Let $R>0$ and $S=S(R)$. Let $h=h(R)$ be as in (6). Then, for some positive integer σ, we have

$$S\subset \bigcup_{k=1}^{\sigma} S_{[T-kh,T]}.$$

We will prove, by induction on k, that:

(i) For $(t, x)\in S_{[T-kh,T]}$, $w(t, x)$ is defined [i.e., the game starting at (t, x) has value] and w is continuous on $S_{[T-kh,T]}$.

(ii) If X is a compact subset of E such that, for some $\alpha\in R$ and all $(T-kh, x)\in([0, T]\times X)\cap S_{[T-kh,T]}$, $w(T-kh, x)>\alpha$, then there exists a strategy $\Gamma(\alpha, k)$ such that

$$V(\varphi)>\alpha, \qquad \text{for all } \varphi\in\bigcup\{\Phi[\cdot, T-kh, x, \Gamma(\alpha, k)]: x\in X\}.$$

Similarly, if $w(T-kh, x)<\alpha$, then there exists $\Delta(\alpha, k)$ such that

$$V(\varphi)<\alpha, \qquad \text{for all } \varphi\in\bigcup\{\Phi[\cdot, T-kh, x, \Delta(\alpha, k)]: x\in X\}.$$

Note that statement (i) is the primary goal. Statement (ii) will be needed in the inductive step of the proof of statement (i). Let $k=1$. Then, statements (i) and (ii) are exactly the assertions of Lemma 4.2 and Lemma 4.4, respectively.

Suppose, as our inductive hypothesis, that (i) and (ii) hold for k. Consider a game starting at $(t_0, x_0)\in S_{[T-(k+1)h,T-kh]}$ with final time $T-kh$, dynamics (2), and payoff (3) defined using, in place of $V(\cdot)$,

$$V_1(\varphi)=\begin{cases} w(T-hk, \varphi[T-kh]), & \text{if } \varphi[t]\in E_1\cap E_2, \quad \forall t\in[t_0, T-kh],\\ -\infty, & \text{if } \varphi \text{ leaves } E_1 \text{ first},\\ +\infty & \text{if } \varphi \text{ leaves } E_2 \text{ first}.\end{cases}$$

Note that by Lemma 4.1, $(T-kh, \varphi[T-kh])\in S_{T-kh}$. Hence, by our induction hypothesis, $w(T-kh, \varphi[T-kh])$ is defined for any motion φ with initial point (t_0, x_0). Let us call such games G_1 and write Γ_1, Δ_1 for strategies and w_1^-, w_1^+ for lower and upper values in G_1. Since, by the induction hypothesis, $w|S_{T-kh}$ is continuous and the duration of a G_1-game is less than or equal to h, we have by Lemma 4.2 (see Remark 4.6) that $w_1(t_0, x_0)$ exists and w_1 is continuous on $S_{[T-(k+1)h,T-kh]}$. Also, note that, by the definition of V_1,

$$w_1|S_{T-kh}=w|S_{T-kh}. \tag{11}$$

Now, we need only show that

$$w^-(t_0, x_0)\geq w_1(t_0, x_0)\geq w^+(t_0, x_0).$$

Once this is established, we will have that $w(t_0, x_0)$ exists and equals $w_1(t_0, x_0)$. Moreover, the continuity of w on $S_{[T-(k+1)h, T-kh]}$ follows from that of w_1.

Claim C1. $w^-(t_0, x_0) \geq w_1(t_0, x_0)$.

Proof. Suppose that the claim is false: $w_1(t_0, x_0) > w^-(t_0, x_0)$. Let $v_0 = w^-(t_0, x_0)$. Then, by definition, there exists a strategy Γ_1, defined on $[t_0, T-kh]$, and $\delta > 0$ such that

$$V_1(\varphi_1) > v_0 + \delta, \qquad \text{for all } \varphi_1 \in \Phi[\cdot, t_0, x_0, \Gamma_1]. \tag{12}$$

In particular, for all $\varphi_1 \in \Phi[\cdot, t_0, x_0, \Gamma_1]$,

$$\varphi_1[t] \in E_1, \qquad \text{for all } t \in [t_0, T-kh]. \tag{13}$$

Let

$$X_1 = \{\varphi_1[T-kh]: \varphi \in \Phi[\cdot, t_0, x_0, \Gamma_1], V_1(\varphi_1) \neq \infty\}.$$

By (12) and the definition of V_1,

$$w(T-kh, x) > v_0 + \delta, \qquad \text{for all } x \in X_1. \tag{14}$$

Note that the statement "$V_1(\varphi_1) \neq \infty$" is equivalent to the statement "$\varphi_1[t] \in E_2, \forall t \in [t_0, T-kh]$." Since $\Phi[\cdot, t_0, x_0, \Gamma_1]$ is compact as a subset of $C[t_0, T]$, and since $E_2 \subset R^n$ is closed, the set

$$\{\varphi \in \Phi[\cdot, t_0, x_0, \Gamma_1]: \varphi[t] \in E_2, \forall t \in [t_0, T-kh]\},$$

is compact in $C[t_0, T]$. It follows that X_1 is compact. Therefore, by our induction hypothesis, there exists $\Gamma(v_0, k)$, defined on $[T-kh, T]$, such that

$$V(\varphi) > v_0 + \delta, \qquad \text{for all } \varphi \in \Phi[\cdot, T-kh, x, \Gamma(v_0, k)], \quad x \in X_1.$$

Let Γ^* be the concatenation of Γ_1 defined on $[t_0, T-kh]$ and $\Gamma(v_0, k)$ defined on $[T-kh, T]$. Let $\varphi^* \in \Phi[\cdot, t_0, x_0, \Gamma^*]$. By the definition of Γ^*, there exist motions $\varphi_1 \in \Phi[\cdot, t_0, x_0, \Gamma_1]$ and $\varphi \in \Phi[\cdot, T-kh, \varphi^*[T-kh], \Gamma(v_0, k)]$ such that

$$\varphi^*[t] = \begin{cases} \varphi_1[t], & \text{for } t \in [t_0, T-kh], \\ \varphi[t], & \text{for } t \in [T-kh, T]. \end{cases} \tag{15}$$

By (13), $\varphi^*[t] \in E_1$ for all $t \in [t_0, T-kh]$. Now, if $\varphi^*[\bar{t}] \notin E_2$ for some $\bar{t} \in [t_0, T-kh]$, then

$$V(\varphi^*) = \infty. \tag{16}$$

If $\varphi^*[t] \in E_2$ for all $t \in [t_0, T-kh]$, then by (15) and the definition of V,

$$\varphi_1[t] \in E_2, \qquad \text{for all } t \in [t_0, T-kh], \tag{17a}$$

$$V(\varphi^*) = V(\varphi). \tag{17b}$$

Because of (17a), $V_1(\varphi_1) \neq \infty$. Therefore,

$$\varphi^*[T-kh] = \varphi_1[T-kh] \in X_1.$$

By the defining property of $\Gamma(v_0, k)$ and (17b), we get

$$V(\varphi^*) = V(\varphi) > v_0 + \delta. \tag{18}$$

Since $\varphi^* \in \Phi[\cdot, t_0, x_0, \Gamma^*]$ was chosen arbitrarily, we conclude from (16) and (18) that

$$\inf_{\Delta} P[t_0, x_0, \Gamma^*, \Delta] > v_0.$$

Therefore, $w^-(t_0, x_0) > v_0$. This contradiction proves Claim C1. □

Claim C2. $w_1(t_0, x_0) \geq w^+(t_0, x_0)$.

Proof. The proof of this claim is parallel to that of Claim C1. Here, one uses $\Delta(v_0, k)$ with $v_0 = w^+(t_0, x_0)$. We omit the details. □

To complete the proof of Theorem 4.1, it remains to prove statement (ii) in this case. Let X be a compact subset of E such that, for all $x \in X$,

$$(T-(k+1)h, x) \in S_{T-(k+1)h},$$

and, for some $\alpha \in R$,

$$w(T-(k+1)h, x) > \alpha.$$

We need to show that a strategy $\Gamma(\alpha, k+1)$, as described in statement (ii), exists. Since $w \neq w_1$ on $S_{T-(k+1)h}$, we have

$$w_1(T-(k+1)h, x) > \alpha, \qquad \text{for all } x \in X.$$

Arguing as in the proof of Lemma 4.4 [using the fact that G_1-games with initial points $(T-(k+1)h, x)$, $x \in X$, are of duration $\leq h$], we obtain that there exists a strategy Γ_1 on $[T-(k+1)h, T-kh]$ such that

$$V_1(\varphi) > \alpha, \qquad \text{for all } \varphi \in \Phi[\cdot, T-(k+1)h, x, \Gamma_1], \quad x \in X.$$

Let

$$X_1 = \{\varphi_1[T-kh]: \varphi_1 \in \Phi[\cdot, T-(k+1)h, x, \Gamma_1], V_1(\varphi_1) \neq \infty, x \in X\}.$$

Using the compactness of X, we get, as before, that X_1 is compact. Also, by the definition of V_1, $w(T-kh, x) > \alpha$, for all $x \in X_1$. Let $\Gamma(\alpha, k)$ be as guaranteed by the induction hypothesis. Take $\Gamma(\alpha, k+1)$ to be the concatenation of Γ_1 and $\Gamma(\alpha, k)$. Then, $\Gamma(\alpha, k+1)$ has the desired property; the proof of this is similar to that concerning Γ^*, above.

In a similar manner, one can obtain the strategy $\Delta(\alpha, k+1)$. This concludes our induction and proves the theorem. □

Note that, as a corollary to the proof of the theorem, we have the following statement.

Corollary 4.1. For every compact subset X of $E_1 \cap E_2$, there exists $\bar{\lambda} > 0$ such that, for any $x_0 \in X$, $w(t, x) = w_1(t, x)$ for all $(t, x) \in [0, T] \times (x_0 + \bar{\lambda} B)$, where w_1 is the value in a G_1-game, as defined above, where only one of the players is restricted to a phase set.

Proof. Let $R > 0$ be such that $[0, T] \times X \subset S(R)$. Corresponding to this R, let $\lambda > 0$, $h > 0$, and Ω_i, $i = 1, \ldots, 4$, be as in Lemma 1.3. Note that any $x_0 \in X$ belongs to some Ω_i, $i = 1, \ldots, 4$. Take $\bar{\lambda} = \lambda$. The conclusion follows as in the proof of Theorem 4.1 using the definitions of λ and h; see Remark 4.2 and (6). □

5. Characterization of the Value

Set

$$H(t, x, p) = \max_y \min_z \langle p, f(t, x, y, z)\rangle.$$

Let $C^1(F)$, $F \subset R^n$, denote the set of all functions which are continuously differentiable on a neighborhood of F. Recall (Ref. 13 and 14) that a function $w(t, x)$, continuous on $[0, T] \times F$, is called a viscosity supersolution (subsolution) of

$$u_t + H(t, x, D_x u) = 0, \tag{19a}$$

$$w(T, x) = g(x), \qquad \text{for all } x \in F, \tag{19b}$$

if for every $\varphi \in C^1([0, T] \times F)$ such that $w - \varphi$ has a local maximum (minimum) at a point $(t_0, x_0) \in [0, T) \times F$, then

$$\varphi_t(t_0, x_0) + H(t_0, x_0, D_x \varphi(t_0, x_0)) \geq 0 \ (\leq 0)$$

holds. Note that the inequalities are the reverse of those in Ref. 13 and Ref. 14, since the values of w are prescribed at the terminal, instead of the initial, time. In the case when $E_1 = E_2 = R^n$, Berkovitz showed (Ref. 15) that w is a viscosity solution of (19) with $F = R^n$. Under Assumption A1, one may apply standard uniqueness results (Ref. 13, 14, 16) to characterize w as the only such function. When $E_2 = R^n$ (that is, state constraints are imposed on at most the maximizing player) and the corresponding value function

w is continuous, one can verify by essentially the same arguments as in Ref. 15 that w is a viscosity supersolution on $[0, T) \times E_1$ and a viscosity subsolution on $[0, T) \times \text{int}(E_1)$; see Ref. 1 or Ref. 17. It has also been shown that such conditions, together with prescribed terminal values, determine w uniquely on $[0, T] \times E_1$; see Ref. 18, Ref. 11. When $E_1 = R^n$, similar statements hold with "super" and "sub" switched. Such uniqueness results were first obtained by Soner (Ref. 10) in the context of an infinite-horizon optimal control problem with discounted cost and the associated Hamilton–Jacobi–Bellman equation. Further investigations of such problems, under various and more general assumptions on the data, have been carried out by Capuzzo-Dolcetta and Lions in Ref. 18 and Ref. 11. In particular, uniqueness results for the Cauchy problem are proved in Ref. 11 (cf. Theorems iii.2 and iii.4).

Here, let us note that it follows from Corollary 4.1 that the local properties of the value function w are the same as those of the value w_1 of a game with state constraints on at most one of the players. Therefore, combined with the results of Ref. 1, we may conclude the following proposition.

Proposition 5.1. For every compact subset X of $E_1 \cap E_2$, there exists $\bar{\lambda} > 0$ such that, for every $x \in X$, letting $\mathcal{U} = X \cap (x + \bar{\lambda} B)$, then:

(A) If $d(x, \partial E_1) < \bar{\lambda}$), then w is a viscosity supersolution of (19a) on $[0, T) \times \mathcal{U}$ and a viscosity subsolution on $[0, T) \times (\mathcal{U} \cap \text{int}(E_1))$.

(B) If $d(x, \partial E_2) < \bar{\lambda}$, then w is a viscosity subsolution of (19a) on $[0, T) \times \mathcal{U}$ and a viscosity supersolution on $[0, T) \times (\mathcal{U} \cap \text{int}(E_2))$.

(C) If $d(x, (\partial E_1 \cup \partial E_2)) \geq \bar{\lambda}$, then w is a viscosity solution of (19a) on $[0, T) \times \mathcal{U}$.

Remark 5.1. It follows from this proposition that w is a viscosity solution on $[0, T) \times \text{int}(E_1 \cap E_2)$, supersolution on $[0, T) \times \partial E_1$, and subsolution on $[0, T) \times E_2$.

Thus, in view of the uniqueness results mentioned above, one expects the following theorem to hold.

Theorem 5.1. Let $E = E_1 \cap E_2$. Properties (A), (B), (C) above and the terminal condition $w(T, x) = g(x)$, for all $x \in E$, determine w uniquely if E satisfies assumption A3 below.

Assumption A3. For every compact subset X of E, there exists $\lambda^* > 0$ and a function $\nu : X \to B$ such that, for any $\bar{x}, x \in X$ with $|x - \bar{x}| < \lambda^*$, we have:

(i) there exists $\alpha_0 = \alpha_0(\bar{x}) > 0$ such that

$$\bar{x} + \alpha \nu(x) \in \text{int}(E), \ \forall \alpha \in (0, \alpha_0);$$

(ii) if $\bar{x} \in \partial E$, then there exists $\bar{c} = \bar{c}(\bar{x})$ such that

$$d(\bar{x} + \alpha \nu(x), \partial E) \geq \bar{c}\alpha, \quad \alpha \in (0, \alpha_0).$$

Indeed, this can be proved by modifying the arguments of uniqueness proofs in problems without state constraints (e.g., Theorem 2.5 of Ref. 16 or Theorem V.4 of Ref. 13) along the lines suggested by Soner in Ref. 10. Since no single reference contains all the details for our setting, we give a sketch of the proof and leave some of the details for the appendix.

We will need the following lemma which follows easily from Assumption A1. The proof is omitted.

Lemma 5.1. Assume A1. Then:

(i) $H(t, x, p)$ is continuous on $[0, T] \times E \times R^n$.

(ii) For every compact subset X of E, there exist constants $K > 0$ and $C > 0$ such that

$$\begin{aligned} &|H(t, x, p) - H(t, x, q)| \\ &\leq (K|x| + C)|p - q|, \qquad \forall t \in [0, T], \quad x \in X, \quad p, q \in R^n, \end{aligned}$$

$$\begin{aligned} &|H(t, x, p) - H(t, \bar{x}, p)| \\ &\leq K|x - \bar{x}||p|, \qquad \forall t \in [0, T], \quad x, \bar{x} \in X, \quad p \in R^n. \end{aligned}$$

Proof of Theorem 5.1. Let w_1 and w_2 be two functions continuous on $[0, T] \times E$, satisfying (A), (B), (C), and $w_i(T, x) = g(x)$, for all $x \in E$. It suffices to show that, for every compact subset X of E, $w_1 = w_2$ on $[0, T] \times X$. Hence, without loss of generality, we may assume that E itself is compact. Let C and K be as in the above lemma. Let

$$T_0 = T - 1/(2K).$$

It suffices to show that $w_1 = w_2$ on $[T_0, T] \times E$, since then by similar arguments one obtains inductively that

$$w_1 = w_2, \qquad \text{on } [T - i/(2K), T] \times E, \quad i = 2, 3, \ldots.$$

We proceed to show that $w_1 = w_2$ on $[T_0, T] \times E$. Let L be any positive number satisfying

$$L \geq C/(1 - K(T - T_0)).$$

Consider the following compact set:

$$\Omega = \{(t, x) \colon t \in [T_0, T], |x| \leq L(T_0 - T), x \in E\}.$$

It follows from statement (iia) of Lemma 5.1 that

$$|H(t, x, p) - Ht, x, q)| \leq |p - q|, \qquad \text{for all } (t, x) \in \Omega, \text{ and } p, q \in R^n. \tag{20}$$

Suppose that we prove that $w_1 = w_2$ on Ω. Then, since L was chosen arbitrarily, it follows that this equality holds on $(T_0, T] \times E$ and the continuity of w_i's extends the equality to $[T_0, T] \times E$. Hence, we will only show $w_1 = w_2$ on Ω.

Now assume, for contradiction, that

$$\max_{(t,x)\in\Omega} \{w_1(t, x) - w_2(t, x)\} \geq \sigma_0,$$

for some $\sigma_0 > 0$. Let

$$M > \max\{\|w_1\|_\infty, \|w_2\|_\infty, \sigma_0\},$$

where

$$\|w_i\|_\infty = \max_{(t,x)\in\Omega} |w_i(t, x)|.$$

For $\epsilon > 0$, define $\Omega_\epsilon \subset \Omega$ by

$$\Omega_\epsilon = \{(t, x) \in \Omega: t \in [T_0 + \epsilon/L, T], |x| \leq L(t - T_0) - \epsilon\}. \tag{21}$$

By the continuity of w_i's there exists $\epsilon_0 < 1$ such that, for all $\epsilon \in (0, \epsilon_0)$,

$$\max_{(t,x)\in\Omega_\epsilon} \{w_1(t, x) - w_2(t, x)\} \geq \sigma_0/2. \tag{22}$$

Fix $\epsilon \in (0, \epsilon_0)$ and let $\eta \in C^\infty(R)$, depending on ϵ, be such that

$$\eta(r) = \begin{cases} 0, & \text{if } r \leq -\epsilon^2, \\ -4M, & \text{if } r \geq 0, \end{cases} \tag{23}$$

with $\eta'(r) \leq 0$ for all r. Let $\sigma \in (0, \sigma_0/4(T - T_0))$. Let

$$(t^*, x^*) \in \arg\max_{(t,x)\in\Omega_\epsilon} \{w_1(t, x) - w_2(t, x) + \sigma(t - T_0)\}.$$

Let

$$\bar{\eta}(t, x) = \eta(|x|^2 - L^2(t - T_0)^2).$$

Note that, since $|x| \leq L(t - T_0) - \epsilon$ and $t \geq T_0 + \epsilon/L$ imply that

$$|x|^2 - L^2(t - T_0)^2 \leq -\epsilon^2,$$

we have

$$\text{supp}(\eta) \subset \Omega \backslash \Omega_\epsilon. \tag{24}$$

Let $\bar{\lambda}$ and λ^* be respectively as in Proposition 5.1 and Assumption A3 with $X = E$. Set $\lambda = \min(\bar{\lambda}, \lambda^*)$. Fix $\gamma \in (0, \sigma_0/4)$. For $\alpha, \beta > 0$, define $\phi_{\alpha,\beta}: \Omega \times \Omega \to R$ by

$$\begin{aligned}\phi_{\alpha,\beta}(t, x, y, z) = w_1(t, x) - w_2(s, y) - (1/\beta)|t - s|^2 - \gamma\Psi_\alpha(x, y)\\ - (3M/\lambda^2)[|x - x^*|^2 + |y - y^*|^2]\\ + \sigma(t - T_0) + \bar{\eta}(t, x) + \bar{\eta}(s, y),\end{aligned} \tag{25}$$

where $\Psi_\alpha(x, y)$ is defined as follows [$\nu = \nu(x^*)$ is as in Assumption A3]:

if $d(x^*, \partial E_1) < \lambda$ and $d(x^*, \partial E_2) \geq \lambda$,

$$\text{then } \Psi_\alpha(x, y) = |(y - x)/\alpha - \nu|^2; \tag{26a}$$

if $d(x^*, \partial E_2) < \lambda$ and $d(x^*, \partial E_1) \geq \lambda$,

$$\text{then } \Psi_\alpha(x, y) = |(x - y)/\alpha - \nu|^2; \tag{26b}$$

$$\text{otherwise, } \Psi_\alpha(x, y) = (1/\alpha)|x - y|^2. \tag{26c}$$

Note the dependence of $\phi_{\alpha,\beta}$ on x^* and therefore on ϵ. By its continuity, $\phi_{\alpha,\beta}$ achieves its maximum on the compact set $\Omega \times \Omega$ at some point (t_1, x_1, s_1, y_1). We will need the following statements about (t_1, x_1, s_1, y_1), see the appendix for proofs.

Claim C1. For every fixed $\epsilon \in (0, \epsilon_0)$, we have:

(a) $\exists \delta \in (0, \epsilon)$ such that, for all α, β,

$|x_1| \leq L(t_1 - T_0) - \delta$ and $|y_1| \leq L(s_1 - T_0) - \delta$.

(b) For all α and β, $|x_1 - x^*| < \lambda$, $|y - x^*| < \lambda$.

(c) $|t_1 - s_1| \to 0$ as $\beta \to 0$ and $|x_1 - y_1| \to 0$ as $\alpha \to 0$.

(d) $(1/\beta)|t_1 - s_1|^2 + \gamma\Psi_\alpha(x, y) \to 0$ as $\alpha, \beta \to 0$.

(e) There exists $\tau_0 > 0$ such that, for all α, β sufficiently small, we have $t_1, s_1 \leq T - \tau_0$.

(f) if $d(x^*, \partial E - 1) < \lambda$, then either $d(y_1, \partial E_2) < \lambda$ or, for all α, β sufficiently small, $y_1 \in \text{int}(E_1)$. Similarly, if $d(x^*, \partial E_2) < \lambda$, then either $d(x_1, \partial E_1) < \lambda$ or, for all α, β sufficiently small, $x_1 \in \text{int}(E_2)$.

Claim C2. $|x - x^*|^2 + |y - x^*|^2 \to 0$ as $\alpha \to 0$ and $\beta \to 0$.

We now show that conditions (A), (B), (C) and the above claims lead to $\sigma = 0$, a contradiction which proves the theorem. Since the definition of $\phi_{\alpha,\beta}$ depends on x^*, for fixed ϵ, we distinguish three cases.

Case 1. $d(x^*, \partial E_1) < \lambda$. Then, by condition (A), w is a viscosity supersolution of (19a) on $[0, T) \times \mathcal{U}$; here, $\mathcal{U} = E \cap (x^* + \bar{\lambda} B)$. Note that, by Claim C1(b) and the choice of λ, $x_1 \in \mathcal{U}$. Now, the function $(t, x) \mapsto \phi_{\alpha,\beta}(t, x, s_1, y_1)$ has an absolute maximum relative to Ω at (t_1, x_1). By Claim C1(e), if α, β are sufficiently small, then $t_1 < T - \tau_0$. By Claim C1(a), $x_1 \notin \partial\Omega$. Therefore, this function has a local maximum on $[0, T) \times E$ at (t_1, x_1). We may therefore conclude from (A) that, for all α, β sufficiently small,

$$\begin{aligned} &-\sigma + (2/\beta)(t_1 - s_1) - \bar{\eta}_t(t_1, x_1) \\ &+ H(t_1, x_1, z_1 + (6M/\lambda^2)(x_1 - x^*)) \geq 0, \end{aligned} \tag{27}$$

where, setting $r(t, x) = |x|^2 - L^2(t - T_0)^2$, we have

$$\bar{\eta}_t(t_1, x_1) = -2\eta'(r(t_1, x_1))L^2(t_1 - T_0)$$

and

$$z_1 = \begin{cases} -(2\gamma/\alpha)[(y_1 - x_1)/\alpha - \nu] - 2\eta'(r(t_1, x_1))x_1, & \text{if } d(x^*, \partial E_2) \geq \lambda, \\ (2\gamma/\alpha)(x_1 - y_1) - 2\eta'(r(t_1, x_1))x_1, & \text{if } d(x^*, \partial E_2) < \lambda. \end{cases}$$

Furthermore, the function $(s, y) \mapsto \phi_{\alpha,\beta}(t_1, x_1, s, y)$ has an absolute minimum relative to Ω at (s_1, y_1). By Claims C1(a) and C1(e), $(s_1, y_1) \notin \partial\Omega$. Suppose that $d(y_1, \partial E_2) \geq \lambda$. Then by Claim C1(f), $y_1 \in \text{int}(E_1)$ for all α, β sufficiently small. Therefore, this function has a relative minimum on $[0, T) \times \mathcal{U} \cap \text{int}(E_1)$ at (s_1, y_1). Hence, again by (A), for sufficiently small α, β,

$$(2/\beta)(t_1 - s_1) + \bar{\eta}_t(s_1, y_1) + H(s_1, y_1, z_2 + (6M/\lambda^2)(y_1 - x^*)) \leq 0, \tag{28}$$

where

$$z_2 = \begin{cases} -(2\gamma/\alpha)[(y_1 - x_1)/\alpha - \nu] + 2\eta'(r(s_1, y_1))y_1, & \text{if } d(x^*, \partial E_2) \geq \lambda, \\ (2\gamma/\alpha)(x_1 - y_1) + 2\eta'(r(s_1, y_1))y_1, & \text{if } d(x^*, \partial E_2) < \lambda. \end{cases}$$

If $d(y_1, \partial E_2) < \lambda$, then $(s, y) \mapsto -\phi_{\alpha,\beta}(t_1, x_1, s, y)$ has a local minimum on $[0, T) \times \mathcal{U}$. Using (B), we again have (28).

Case 2. $d(x^*, \partial E_2) < \lambda$. This case is handled as in Case 1 using conditions (A) and (B) and the appropriate form of Ψ_α. This leads to (27) and (28) with

$$z_1 = \begin{cases} (2\gamma/\alpha)[(y_1 - x_1)/\alpha - \nu] - 2\eta'(r(t_1, x_1))x_1, & \text{if } d(x^*, \partial E_1) \geq \lambda, \\ (2\gamma/\alpha)(x_1 - y_1) - 2\eta'(r(t_1, x_1))x_1, & \text{if } d(x^*, \partial E_1) < \lambda, \end{cases}$$

and

$$z_2 = \begin{cases} (2\gamma/\alpha)[(y_1 - x_1)/\alpha - \nu] + 2\eta'(r(s_1, y_1))y_1, & \text{if } d(x^*, \partial E_1) \geq \lambda, \\ (2\gamma/\alpha)(x_1 - y_1) + 2\eta'(r(s_1, y_1))y_1, & \text{if } d(x^*, \partial E_1) < \lambda. \end{cases}$$

Case 3. $d(x^*, \partial E_1) \geq \lambda$, and $d(x^*, \partial E_2) \geq \lambda$. Then, by claim C1(b), x_1 and y_1 lie in $\text{int}(E_1 \cap E_2)$. Recalling Remark 5.1, we can obtain (27) and (28) with

$$z_1 = (2\gamma/\alpha)(x_1 - y_1) - 2\eta'(r(t_1, x_1))x_1,$$

$$z_2 = (2\gamma/\alpha)(x_1 - y_1) + 2\eta'(r(s_1, y_1))y_1.$$

Combining (27) and (28), we have that, for sufficiently small α, β,

$$\begin{aligned}\sigma \leq\, & 2L[\eta'(r(s_1, y_1)L(s_1 - T_0) + \eta'(r(t_1, x_1)L(t_1 - T_0)] \\ & + H(t_1, x_1, z_1 + (6M/\lambda^2)(x_1 - x^*)) \\ & - H(s_1, y_1, z_2 - (6M/\lambda^2)(y_1 - x^*)). \end{aligned} \tag{29}$$

We now estimate the right side of (29). Define

$$\Delta_1 = |H(t_1, x_1, z_1 + (6M/\lambda^2)(x_1 - x^*)) - H(t_1, x_1, z_1)|,$$

$$\Delta_2 = |H(t_1, x_1, z_1) - H(t_1, x_1, z_2)|,$$

$$\Delta_3 = |H(t_1, x_1, z_2) - H(t_1, y_1, z_2)|,$$

$$\Delta_4 = |H(t_1, y_1, z_2) - H(s_1, y_1, z_2)|,$$

$$\Delta_5 = |H(s_1, y_1, z_2 - (6M/\lambda^2)(y_1 - x^*)) - H(s_1, y_1, z_2)|.$$

Then, using (20), we have

$$\Delta_1 \leq 6LM|x_1 - x^*|/\lambda^2, \qquad \Delta_5 \leq 6LM|y_1 - x^*|/\lambda^2, \tag{30a}$$

$$\Delta_2 \leq L|z_1 - z_2| \leq 2L(|\eta'(r(s_1, y_1)||y_1| + |\eta'(r(t_1, x_1))||x_1|). \tag{30b}$$

By statement (ii)(b) of Lemma 5.1,

$$\begin{aligned}\Delta_3 &\leq K|x_1 - y_1||z_2| \\ &\leq 2K[\gamma(\Psi_\alpha(x_1, y_1) + \sqrt{\Psi_\alpha(x_1, y_1)}) + |y_1 - x_1||\eta'(r(s_1, y_1))||y_1|]. \end{aligned} \tag{31}$$

Using (30b), and noting that $\eta'(r) \leq 0$ for all r, by construction, we can write (29) as

$$\begin{aligned}\sigma \leq \Delta_1 + \Delta_3 + \Delta_4 + \Delta_5 + 2L[&|\eta'(r(s_1, y_1))|(|y_1| - L(s_1 - T_0)) \\ &+ |\eta'(r(t_1, x_1))|(|x_1| - L(t_1 - T_0))].\end{aligned}$$

Note that, by Claim C1(a), the quantity in the square bracket is negative. Therefore,

$$\sigma \leq \Delta_1 + \Delta_3 + \Delta_4 + \Delta_5. \tag{32}$$

Now, let $\vartheta > 0$. Using (30a) and Claim C2, we have that there exist α_1 and β_1 such that

$$\Delta_1 + \Delta_5 \leq \vartheta/4, \qquad \text{if } \alpha < \alpha_1 \text{ and } \beta < \beta_1.$$

Note that, by Claim C1(a), $|\eta'(r(s_1, y_1))|$ remains bounded as $\alpha, \beta \to 0$. Therefore, using (31) and Claim C1(c) and C1(d), we have that there exist $\alpha_2 < \alpha_1$, $\beta_2 < \beta_1$ such that

$$\Delta_3 \leq \vartheta/4, \qquad \text{if } \alpha < \alpha_2 \text{ and } \beta < \beta_2.$$

Finally, with $\epsilon < \epsilon_0$ and $\alpha < \alpha_2$ fixed, we may choose, by Claim C1(c), β sufficiently small to obtain

$$\Delta_4 \leq \vartheta/4.$$

Thus from (32), we obtain $\sigma \leq \vartheta$. Since $\vartheta > 0$ was chosen arbitrarily, we have $\sigma = 0$, contradicting the choice of σ. □

6. Appendix

In this section, we provide the proofs of Claims C1 and C2. These arguments are adaptations to our case of the arguments in Ishii (Ref. 16) and Soner (Ref. 10).

Proofs of Claims C1(a), C1(b), C1(c). Note that $(T, x^*) \in \Omega_\epsilon$. Therefore, by (24) and using $w_1(T, \cdot) = g(\cdot) = w_2(T, \cdot)$, we have that, for any α and β,

$$\phi_{\alpha,\beta}(T, x^*, T, x^*) = -\gamma\Psi_\alpha(x^*, x^*) + \sigma(T - T_0) \geq -\gamma + \sigma(T - T_0).$$

By the definition of (t_1, x_1, s_1, y_1),

$$\phi_{\alpha,\beta}(t_1, x_1, s_1, y_1) \geq \phi_{\alpha,\beta}(T, x^*, T, x^*).$$

This gives

$$\begin{aligned}
&(1/\beta)|t_1 - s_1|^2 + \gamma\Psi_\alpha(x_1, y_1) + (3M/\lambda^2)[|x_1 - x^*|^2 + |y_1 - x^*|^2] \\
&- \bar{\eta}(t_1, x_1) - \bar{\eta}(s_1, y_1) \\
&\leq w_1(t_1, x_1) - w_2(s_1, y_1) + \gamma - \sigma(T - t_1) \\
&\leq 2M + \sigma/4 < 3M. \qquad (33)
\end{aligned}$$

Now, Claim C1(a) follows from the fact that $-\bar{\eta}(t, x) \to 4M$ as $|x|^2 - L^2(t - T_0)^2 \to 0$; Claim C1(b) is clear from (33); and Claim C1(c) follows because the right side of (33) is independent of α, β and, if $|x - y| > 0$, then $\Psi_\alpha(x, y) \to \infty$ as $\alpha \to 0$. □

Proof of Claim C1(d). Let δ be as in Claim C1(a) and $\alpha_0(x_1)$ as in Assumption A3. Since δ is independent of α, β, we have that $(t_1, x_1, t_1, x_1 + \alpha\nu) \in \Omega$, if $\alpha \in (0, \bar{\epsilon})$, where $\bar{\epsilon} = \min(\delta, \alpha_0(x_1))$. Hence, if $\alpha \in (0, \bar{\epsilon})$, then by the definition of (t_1, x_1, s_1, y_1),

$$\phi_{\alpha,\beta}(t_1, x_1, s_1, y_1) \geq \phi_{\alpha,\beta}(t_1, x_1, t_1, x_1 + \alpha\nu).$$

This gives

$$(1/\beta)|t_1 - s_1|^2 + \gamma\Psi_\alpha(x_1, y_1)$$
$$\leq |w_2(s_1, y_1) - w_1(t_1, x_1 + \alpha\nu)| + |\bar{\eta}(s_1, y_1) - \bar{\eta}(t_1, x_1 + \alpha\nu)|$$
$$+ (3M/\lambda^2)[|y_1 - x^*|^2 - |x_1 - x^*|^2]. \quad (34)$$

Using the continuity of w_2 and $\bar{\eta}$ and Claim C1(c), we conclude that the right side of (34) tends to zero as $\alpha, \beta \to 0$. □

Proof of Claim C1(e). Suppose that the statement is false. Then, there exist $\alpha_n, \beta_n \to 0$ as $n \to \infty$ with corresponding t_{1n} and s_{1n} such that either t_{1n} or s_{1n} tends to T as $n \to \infty$. But, by Claim C1(c), both s_{1n} and $t_{1n} \to T$. For notational convenience, we will write ϕ_n and (t_n, x_n, s_n, y_n) instead of ϕ_{α_n, β_n} and $(t_{1n}, x_{1n}, s_{1n}, y_{1n})$.

By (22),

$$\max\{w_1(t, x) - w_2(t, x): (t, x) \in \Omega\} \geq \sigma_0/2,$$

and the maximum is achieved at (t^*, x^*). Also, since

$$w_1(t, x) - w_2(t, x) + \sigma(t - T_0) \geq w_1(t, x) - w_2(t, x),$$
$$\text{for all } (t, x) \text{ with } t \geq T_0,$$

we get

$$w_1(t^*, x^*) - w_2(t^*, x^*) + \sigma(t^* - T_0)$$
$$\geq \max\{w_1(t, x) - w_2(t, x): (t, x) \in \Omega_\epsilon\}.$$

Therefore,

$$\max\{\phi_n(t, x, t, x): (t, x) \in \Omega_\epsilon\} \geq \phi_n(t^*, x^*, t^*, x^*)$$
$$\geq \sigma_0/2 - \gamma, \quad (35)$$

where for the last inequality we have also used (24) and the fact that $(t^*, x^*) \in \Omega_\epsilon$. On the other hand,

$$\max \phi_n(t, x, t, x) \leq \phi_n(t_n, x_n, s_n, y_n)$$
$$\leq w_1(t_n, x_n) - w_2(s_n, y_n) + \sigma(t_n - T_0).$$

Since $w_1(T, \cdot) = w_2(T, \cdot)$, the right side of the above equals

$$[w_1(t_n, x_n) - w_1(T, x_n)] + [w_2(T, x_n) - w_2(s_n, y_n)] + \sigma(t_n - T_0). \quad (36)$$

Since the w_i's are uniformly continuous over the compact set Ω, since $s_n, t_n \to T$, by assumption, and $|x_n - y_n| \to 0$, by Claim C1(c), combining (35) and (36) and letting $n \to \infty$, we obtain

$$\sigma(T - T_0) \geq \sigma_0/2 - \gamma.$$

Since $\gamma \in (0, \sigma_0/4)$, we have $\sigma(T - T_0) > \beta_0/4$. But this is impossible, since

$$\sigma < \sigma_0/(4(T - T_0)). \qquad \square$$

Proof of Claim C1(f). Suppose that $d(x^*, \partial E_1) < \lambda$, but $d(y_1, E_2) \geq \lambda$. Then,

$$\Psi_\alpha(x, y) = |(y - x)/\alpha - \nu|^2.$$

Now, by Claim C1(d), since γ is fixed,

$$z_\alpha = (y - x)/\alpha - \nu \to 0, \qquad \text{as } \alpha, \beta \to 0. \tag{37}$$

By Claim C1(b), $x_1 \in x^* + \lambda B$. Since $\lambda \leq \lambda^*$, we have by Assumption A3 that:

$$x_1 + \alpha\nu \in \text{int}(E_1), \qquad \text{for } \alpha \in (0, \alpha_0(x_1)); \tag{38a}$$

there exists $\bar{c} > 0$, such that $d(x_1 + \alpha\nu, \partial E_1) \geq \bar{c}\alpha$, if $x_1 \in \partial E_1$. (38b)

Now,

$$y_1 = x_1 + \alpha(\nu + z_\alpha).$$

If $x_1 \in \text{int}(E_1)$, then $y_1 \in \text{int}(E_1)$ for sufficiently small α. Suppose that $x_1 \in \partial E_1$. By the triangle inequality,

$$d(x_1 + \alpha\nu, \partial E_1) \leq d(y_1, \partial E_1) + |y_1 - (x_1 + \alpha\nu)|.$$

Hence,

$$d(y_1, \partial E_1) \geq d(x_1 + \alpha\nu, \partial E_1) - \alpha|z_\alpha|.$$

By (37) and (38b), for all α, β sufficiently small, $d(y_1, \partial E_1) > 0$; therefore, $y_1 \in \text{int}(E_1)$. Similarly, if $d(x^*, \partial E_2) < \lambda$, but $d(x_1, E_1) \geq \lambda$, one obtains $x_1 \in \text{int}(E_2)$. $\square$

Proof of Claim C2. Recall that $(t^*, x^*) \in \Omega_\epsilon$. Let

$$0 < \alpha < \min(\epsilon, \alpha_0(x^*)).$$

Then, $(t^*, x^* + \alpha\nu) \in \Omega$. Therefore,

$$\phi(t_1, x_1, s_1, y_1) \geq \phi(t^*, x^*, t^*, x^* + \alpha\nu).$$

This gives

$$\begin{aligned}
&(1/\beta)|t_1 - s_1|^2 + \gamma\Psi_\alpha(x_1, y_1) + (3M/\lambda^2)[|x_1 - x^*|^2 + |y_1 - x^*|^2] \\
&\leq [w_1(t_1, x_1) - w_2(s_1, y_1) + \sigma(t_1 - T_0)] \\
&-[w_1(t^*, x^*) - w_2(t^*, x^* + \alpha\nu) + \sigma(t^* - T_0)] - \bar{\eta}(t^*, x^* + \alpha\nu).
\end{aligned}$$

Adding and subtracting $w_2(t_1, x_1)$ and $w_2(t^*, x^*)$ to the first and the second expressions in the square bracket, respectively, we obtain

$$\begin{aligned}&(3M/\lambda^2)[|x_1 - x^*|^2 + |y_1 - x^*|^2]\\&\leq \Delta_\epsilon + [w_2(s_1, y_1) - w_2(t_1, x_1)]\\&+ [w_2(t^*, x^*) - w_2(t^*, x^* + \alpha\nu)] - \bar{\eta}(t^*, x^* + \alpha\nu),\end{aligned} \tag{39}$$

where

$$\begin{aligned}\Delta_\epsilon &= [w_1(t_1, x_1) - w_2(t_1, x_1) + \sigma(t_1 - T_0)]\\&- [w_1(t^*, x^*) - w_2(t^*, x^*) + \sigma(t^* - T_0)].\end{aligned}$$

Note that, by the definition of (t^*, x^*), the last expression on the right is equal to

$$\max_{(t,x)\in\Omega_\epsilon} \{w_1(t, x) - w_2(t, x) + \sigma(t - T_0)\}.$$

Hence, Δ_ϵ is bounded above by

$$\max_{(t,x)\in\Omega_\epsilon} \{w_1(t, x) - w_2(t, x) + \sigma(t - T_0)\}$$

$$- \max_{(t,x)\in\Omega_\epsilon} \{w_1(t, x) - w_2(t, x) + \sigma(t - T_0))\}.$$

Thus, it follows that either $\Delta_\epsilon < 0$ or, if $\Delta_\epsilon \geq 0$, then $\Delta_\epsilon \to 0$ as $\epsilon \to 0$. Going back to (39), using the continuity of w_2, (24), and Claim C1(c), we observe that the remaining terms on the right of (39) tend to zero as $\alpha, \beta \to 0$. This is the desired conclusion. □

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