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A Necessary and Sufficient Condition for Pareto-Optimal Security Strategies in Multicriteria Matrix Games[1,2]

D. GHOSE[3]

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Abstract. In this paper, a scalar game is derived from a zero-sum multicriteria matrix game, and it is proved that the solution of the new game with strictly positive scalarization is a necessary and sufficient condition for a strategy to be a Pareto-optimal security strategy (POSS) for one of the players in the original game. This is done by proving that a certain set, which is the extension of the set of security level vectors in the criterion function space, is convex and polyhedral. It is also established that only a finite number of scalarizations are necessary to obtain all the POSS for a player. An example is included to illustrate the main steps in the proof.

Key Words. Game theory, multicriteria games, games with vector payoffs, Pareto-optimal security strategies, multicriteria optimization, scalarization methods.

1. Introduction

A natural extension of the well-known classical zero-sum matrix game (with a scalar criteria), proposed and solved by von Neumann, is the zero-sum matrix game with a vector payoff, which is also known as the zero-sum multicriteria matrix game. In Blackwell's paper (Ref. 1), an asymptotic analog of the minimax theorem in scalar criterion games was established for repeated games with vector payoffs. The analysis was aimed

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[3] Lecturer, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India.

at answering the question as to whether a player will be able to force his average payoff to approach or exclude a given subset in the payoff space, if the game is repeated a large number of times. Shapley (Ref. 2) defined the concept of equilibrium points in these games and presented methods of obtaining them through the solution of scalarized bimatrix games. A similar approach was taken by Nieuwenhuis (Ref. 3) and Corley (Ref. 4) using the notion of Pareto optimality (efficiency). The equilibrium points, as obtained in Refs. 2–4, do not possess the important property of security in the individual criteria against opponent's deviations in strategy, unlike the equilibrium saddle points in zero-sum scalar criterion matrix games. In Ref. 5, a solution concept based on Pareto optimality and security was proposed which is independent of the notion of equilibrium. It was demonstrated that this concept is important in some areas of application. A similar concept has been used earlier by Schmitendorf and Moriarty (Ref. 6) for coalitive Pareto optimality and by Schmitendorf (Ref. 7) to analyze systems with disturbances. But these were in the context of differential games.

In Ref. 5, Pareto-optimal security strategies (POSS) were obtained for a player in a zero-sum multicriteria matrix game by scalarization of the original game. A necessary condition and a sufficient condition were separately obtained. In this paper, we prove that strictly positive scalarization is both a necessary and a sufficient condition for such games. In addition, we also prove that only a finite number of scalarizations are required to obtain all the Pareto-optimal security strategies of a player.

The paper is organized as follows. Section 2 formulates the multicriteria game and defines Pareto-optimal security strategies. Section 3 defines a new scalarized game and a set which is an extension of the set of security level vectors associated with the multicriteria game. It is established that, if the extended set is polyhedral, then the solution of the new game with strictly positive scalarization is a necessary and sufficient condition for a strategy to be a POSS. In Section 4, we prove that the extended set is always polyhedral for a multicriteria matrix game. Section 5 presents an example illustrating the major steps in the proof, and Section 6 concludes the paper.

2. Some Definitions and Remarks

Let $\mathscr{D} \subseteq R^n$ be a compact, convex subset of the n-dimensional real space. An element $v \in \mathscr{D}$ is called a vector and is an n-tuple $(v_1, \ldots, v_n)$ of real numbers. Let $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be two arbitrary vectors in $\mathscr{D}$. Then, using the definitions given in Lin (Ref. 8), but with a slight change in notation, we have:

(i) $y \geqq z$, iff $y_i \geq z_i$, for all i;

(ii) $y \geq z$, iff $y_i \geq z_i$, for all i, and $y_j > z_j$, for at least one j;
(iii) $y > z$, iff $y_i > z_i$, for all i.

A vector y is said to be positive if $y \geqq 0$, definitely positive if $y \geq 0$, and strictly positive if $y > 0$.

Consider a payoff matrix $A = \{a_{ij}\}$ with p number of rows and q number of columns. Each element a_{ij}, situated at the ith row and jth column, belongs to $\mathscr{D}$ and is an n-tuple represented by $(a_{ij}(1), \ldots, a_{ij}(n))$. We define individual matrices of dimension $p \times q$ as

$$A(k) = \{a_{ij}(k)\}, \qquad k = 1, \ldots, n.$$

There are two players: P1 (the minimizer, who chooses rows) and P2 (the maximizer, who chooses columns). The mixed strategy spaces of the players P1 and P2 are

$$\Gamma^1 = \left\{\gamma^1 \colon \sum_{i=1}^{p} \gamma_i^1 = 1, \gamma_i^1 \geq 0, i = 1, \ldots, p\right\}, \tag{1}$$

$$\Gamma^2 = \left\{\gamma^2 \colon \sum_{j=1}^{q} \gamma_j^2 = 1, \gamma_j^2 \geq 0, j = 1, \ldots, q\right\}, \tag{2}$$

where an element $\gamma^k \in \Gamma^k$, $k = 1, 2$, is of the form

$$\gamma^1 = (\gamma_1^1, \ldots, \gamma_p^1)' \in R^p, \tag{3}$$

$$\gamma^2 = (\gamma_1^2, \ldots, \gamma_q^2)' \in R^q. \tag{4}$$

The pure strategies of the players are the extreme points or the vertices of Γ^1 and Γ^2.

When P1 chooses a strategy $\gamma^1 \in \Gamma^1$ and P2 chooses $\gamma^2 \in \Gamma^2$, the expected payoff of the game is denoted by

$$J(\gamma^1, \gamma^2) = [J_1(\gamma^1, \gamma^2), \ldots, J_n(\gamma^1, \gamma^2)], \tag{5}$$

where

$$J(\gamma^1, \gamma^2) = \gamma^{1'} A \gamma^2, \tag{6}$$

$$J_k(\gamma^1, \gamma^2) = \gamma^{1'} A(k) \gamma^2, \qquad k = 1, \ldots, n. \tag{7}$$

Here, $J_k(\gamma^1, \gamma^2)$ is called the jth criterion. Thus, we have a two-person zero-sum multicriteria matrix game. From here onward, the transpose sign t will be omitted in the expression.

Associated with every strategy $\gamma^i \in \Gamma^i$ for player Pi, one can define security levels in terms of individual criteria as

$$\bar{J}_k(\gamma^1) = \max_{\gamma^2 \in \Gamma^2} J_k(\gamma^1, \gamma^2), \qquad k = 1, \ldots, n, \tag{8a}$$

$$\underline{J}_k(\gamma^2) = \min_{\gamma^1 \in \Gamma^1} J_k(\gamma^1, \gamma^2), \qquad k = 1, \ldots, n. \tag{8b}$$

Then, the security level vectors are denoted by

$$\bar{J}(\gamma^1)=[\bar{J}_1(\gamma^1),\ldots,\bar{J}_n(\gamma^1)], \tag{9a}$$

$$\underline{J}(\gamma^2)=[\underline{J}_1(\gamma^2),\ldots,\underline{J}_n(\gamma^2)], \tag{9b}$$

which represent the guaranteed payoffs in each of the criteria to the players P1 and P2, respectively.

Definition 2.1. A strategy $\gamma^{1*}\in\Gamma^1$ is a Pareto-optimal security strategy (POSS) for P1 iff, for all $\gamma^1\in\Gamma^1$, $\bar{J}(\gamma^{1*})\geqq\bar{J}(\gamma^1)$ implies $\bar{J}(\gamma^{1*})=\bar{J}(\gamma^1)$. Similarly, a strategy $\gamma^{2*}\in\Gamma^2$ is a POSS for P2 iff, for all $\gamma^2\in\Gamma^2$, $\underline{J}(\gamma^2)\geqq \underline{J}(\gamma^{2*})$ implies $\underline{J}(\gamma^2)=\underline{J}(\gamma^{2*})$.

The POSS are analogous to the security strategies in scalar criterion games (Ref. 9).

3. Solution through Scalarization Methods

The easiest way to determine the POSS of a player (say, P1) is through scalarization. A scalarized game is obtained from the original game as follows.

Game P1(α). In this game, P1 has a strategy $\gamma^1\in\Gamma^1$, but P2 has n number of strategies $\gamma^{21},\ldots,\gamma^{2n}\in\Gamma^2$. Here, α is a vector consisting of an n-tuple $(\alpha_1,\ldots,\alpha_n)$ of real numbers. The payoff function of the game is defined as

$$\hat{J}^1(\gamma^1,\underline{\gamma}^2)=\alpha_1 J_1(\gamma^1,\gamma^{21})+\cdots+\alpha_n J_n(\gamma^1,\gamma^{2n}), \tag{10}$$

where

$$\underline{\gamma}^2=(\gamma^{21},\ldots,\gamma^{2n})\in\underline{\Gamma}^2=\prod_{k=1}^{n}\Gamma^2, \tag{11}$$

$$J_k(\gamma^1,\gamma^{2k})=\gamma^1 A(k)\gamma^{2k},\qquad k=1,\ldots,n. \tag{12}$$

In this game, P1 tries to minimize and P2 tries to maximize the payoff.

In Ref. 5, it was proved that the game P1(α) can be transformed to a zero-sum scalar criterion matrix game; therefore, it is possible to obtain its saddle-point solution in mixed strategies. The following theorems were also proved there using standard results in vector optimization (Ref. 8 and Refs. 10–12).

Theorem 3.1. A strategy $\gamma^{1*}\in\Gamma^1$ is a POSS for P1 in the original multicriteria game if γ^{1*} is a saddle-point strategy for P1 in the game P1(α) with $\alpha>0$, $\alpha_1+\cdots+\alpha_n=1$.

Theorem 3.1 states a sufficient condition for a strategy to be a POSS for P1.

Theorem 3.2. If $\gamma^{1*} \in \Gamma^1$ is a POSS for P1 in the original multicriteria game, then there exists a vector $\alpha \geq 0$, $\alpha_1 + \cdots + \alpha_n = 1$, such that γ^{1*} is a saddle-point strategy for P1 in the game P1(α).

Theorem 3.2 states a necessary condition for a strategy to be a POSS for P1.

It is to be noted that, in Theorems 3.1 and 3.2, the condition $\alpha_1 + \cdots + \alpha_n = 1$ is not really required, in the sense that the same theorems can be stated and proved even without it. However, the condition is included here to ensure that the scalarization coefficients lie in a bounded set. This assists in the development of computational methods (Ref. 11).

The main objective of this paper is to prove that the saddle point solution of the game P1(α) with strictly positive scalarization (i.e., $\alpha > 0$) is both a necessary and sufficient condition for a strategy to be a POSS for P1. Thus, we wish to prove the following theorem.

Theorem 3.3. A strategy $\gamma^{1*} \in \Gamma^1$ is a POSS for P1 in the original multicriteria game if and only if γ^{1*} is a saddle point strategy for P1 in the game P1(α) with $\alpha > 0$.

To prove this theorem we have to establish some intermediate steps first. Consider the payoff function in the game P1(α). The saddle-point solution of this game is also its minimax solution. So,

$$\min_{\gamma^1 \in \Gamma^1} \max_{\gamma^2 \in \Gamma^2} \hat{J}^1(\gamma^1, \gamma^2) \tag{13}$$

can be written as

$$\min_{\gamma^1 \in \Gamma^1} \left[\alpha_1 \max_{\gamma^2 \in \Gamma^2} J_1(\gamma^1, \gamma^2) + \cdots + \alpha_n \max_{\gamma^2 \in \Gamma^2} J_n(\gamma^1, \gamma^2) \right], \tag{14}$$

which further reduces to

$$\min_{\gamma^1 \in \Gamma^1} [\alpha_1 \bar{J}_1(\gamma^1) + \cdots + \alpha_n \bar{J}_n(\gamma^1)]. \tag{15}$$

Thus, the saddle-point strategy for P1 in the game P1(α) is a strategy which minimizes the expression within the brackets in (15).

In Section 2, the security level vector $\bar{J}(\gamma^1)$ was defined for a strategy $\gamma^1 \in \Gamma^1$. Let the set $S \subseteq R^n$ be the set of all security level vectors associated with all of P1's strategies,

$$S = \{\bar{J}(\gamma^1) \in R^n: \gamma^1 \in \Gamma^1\}. \tag{16}$$

We define an extension of the set S, denoted by S^E, as follows:

$$S^E = S + R^n_+, \tag{17}$$

where R^n_+ is the positive orthant cone in R^n, i.e.,

$$R^n_+ = \{x: x \in R^n, x \geqq 0\}.$$

In Ref. 5, it has been proved that S^E is a convex set. It is obvious that the noninferior (i.e., Pareto-minimum) points of S^E are the security level vectors corresponding to the POSS of P1. In order to prove Theorem 3.3, we have to prove that it is possible to obtain all the Pareto-minimum points of S^E, as defined in (17), by solving (15) with $\alpha > 0$.

Next, we state some definitions followed by a theorem (Theorem 3.4) which can be proved easily by invoking an extremely useful result on convex sets, obtained first by Arrow, Barankin, and Blackwell (Ref. 10) and later by others (e.g., Ref. 11).

Definition 3.1. A convex set $C \subseteq R^n$ is said to be polyhedral if it is an intersection of a finite number of half spaces in R^n.

Definition 3.2. A convex set $C \subseteq R^n$ is said to be finitely generated if there exists sets of vectors $X = \{x_1, \ldots, x_w \in R^n\}$ and $Y = \{y_1, \ldots, y_m \in R^n\}$ such that C can be expressed as

$$C = \{z: z = \lambda_1 x_1 + \cdots + \lambda_w x_w + \beta_1 y_1 + \cdots + \beta_m y_m, \lambda_1 + \cdots + \lambda_w = 1,$$
$$\lambda_i \geq 0, \beta_j \geq 0, i = 1, \ldots, w, j = 1, \ldots, m\}.$$

In such a case, C is said to be finitely generated by (X, Y) and is denoted by $G(X, Y)$. Further, X and Y are called the set of vertices (or extreme points) and the set of generators (or extreme directions), respectively.

It has been proved (Ref. 13, Theorem 19.1) that Definitions 3.1 and 3.2 are equivalent; thus, polyhedral convex sets are finitely generated convex sets, and vice versa.

Theorem 3.4. If the set S^E is closed, convex, and polyhedral, then all its noninferior (Pareto-minimum) points can be obtained by solving (15) with $\alpha > 0$.

In Ref. 5, it has been proved that S^E is a closed and convex set. The primary objective of this paper is to prove that S^E is also a polyhedral set.

Definition 3.3. The epigraph of a function $f: X \to R$, $X \subseteq R^p$, is a set of vectors denoted by Epi f and is defined as

$$\text{Epi} f = \{(x, \mu): x \in X, \mu \in R, \mu \geq f(x)\} \subseteq R^{p+1}.$$

Definition 3.4. A polyhedral convex function is a function whose epigraph is a polyhedral convex set.

Lastly, we state another theorem, the proof of which will be found in Rockafellar (Ref. 13, Theorem 19.3).

Theorem 3.5. If C is a polyhedral convex set in R^n and $L: R^n \rightarrow R^m$ is a linear transformation, then $L(C)$ is also a polyhedral convex set.

4. Polyhedrality of S^E

The above results are now used to prove that S^E is a polyhedral convex set. This is done by first defining a set E which, when a linear transformation is applied to it, gives the set S^E. Then, we proceed to prove that E is a polyhedral convex set, so that, by using Theorem 3.5, we can prove that S^E is also a polyhedral convex set.

Theorem 4.1. The set Epi $\bar{J}_i$ is convex and polyhedral, $i = 1, \ldots, n$, $\bar{J}_i : \Gamma^1 \rightarrow R$.

Proof. Observe that

$$\bar{J}_i(\gamma^1) = \max_{\gamma^2 \in \Gamma^2} \gamma^1 A(i) \gamma^2.$$

Let the column vectors of the Matrix $A(i)$ be denoted by $C_{i1}, \ldots, C_{iq}$. Then,

$$\gamma^1 A(i) = (\gamma^1 C_{i1}, \ldots, \gamma^1 C_{iq}),$$

where

$$\gamma^1 C_{ij} \in R, \qquad j = 1, \ldots, q.$$

So,

$$\bar{J}_i(\gamma^1) = \max_{\gamma^2 \in \Gamma^2} (\gamma^1 C_{i1}, \ldots, \gamma^1 C_{iq}) \gamma^2 = \max\{\gamma^1 C_{i1}, \ldots, \gamma^1 C_{iq}\}.$$

The function

$$f_j(\gamma^1) = \gamma^1 C_{ij}, \qquad j = 1, \ldots, q,$$

is linear on the convex set Γ^1. Thus, using a result in Rockafellar (Ref. 13, p. 172), we arrive at the conclusion that $\bar{J}_i$ is a polyhedral convex function, and thus Epi $\bar{J}_i$ is a polyhedral convex set. □

Now, Epi $\bar{J}_i$, denoted here by E_i, is defined according to Definition 3.3 as follows:

$$\text{Epi } \bar{J}_i = E_i = \{(\gamma^1, \mu_i) : \gamma^1 \in \Gamma^1, \mu_i \in R, \mu_i \geq \bar{J}_i(\gamma^1)\}. \tag{18}$$

Define a set $E \subseteq R^{n+p}$ as follows:

$$E = \{(\gamma^1, \mu_1, \ldots, \mu_n): \gamma^1 \in \Gamma^1, \mu_i \in R, \mu_i \geq \bar{J}_i(\gamma^1), i = 1, \ldots, n\}. \tag{19}$$

Let E^p be a projection of E onto the criterion function space,

$$E^p = \{(\mu_1, \ldots, \mu_n): (\gamma^1, \mu_1, \ldots, \mu_n) \in E, \gamma^1 \in \Gamma^1\}. \tag{20}$$

It is easily seen that

$$E^p = S^E. \tag{21}$$

Theorem 4.2. The set E is convex.

Proof. Let $a, b \in E$. Then, there exists $\gamma_a^1 \in \Gamma^1$ and $\gamma_b^1 \in \Gamma^1$ such that

$$a = (\gamma_a^1, \mu_{1a}, \ldots, \mu_{na}) \in E,$$

$$b = (\gamma_b^1, \mu_{1b}, \ldots, \mu_{nb}) \in E,$$

where

$$\bar{J}_i(\gamma_a^1) \leq \mu_{ia}, \quad \bar{J}_i(\gamma_b^1) \leq \mu_{ib}, \qquad i = 1, \ldots, n. \tag{22}$$

For $\lambda \in [0, 1]$, let

$$c = \lambda a + (1-\lambda) b$$

$$= (\lambda \gamma_a^1 + (1-\lambda)\gamma_b^1, \lambda\mu_{1a} + (1-\lambda)\mu_{1b}, \ldots, \lambda\mu_{na} + (1-\lambda)\mu_{nb}).$$

Since Γ^1 is convex,

$$\gamma_c^1 = \lambda \gamma_a^1 + (1-\lambda)\gamma_b^1 \in \Gamma^1.$$

Let

$$\mu_{ic} = \lambda\mu_{ia} + (1-\lambda)\mu_{ib}, \qquad i = 1, \ldots, n.$$

From (22), we have

$$\lambda\mu_{ia} + (1-\lambda)\mu_{ib} \geq \lambda\bar{J}_i(\gamma_a^1) + (1-\lambda)\bar{J}_i(\gamma_b^1). \tag{23}$$

But since $\bar{J}_i$ is a convex function,

$$\lambda\bar{J}_i(\gamma_a^1) + (1-\lambda)\bar{J}_i(\gamma_b^1) \geq \bar{J}_i(\lambda\gamma_a^1 + (1-\lambda)\gamma_b^1). \tag{24}$$

Putting (23) and (24) together, we have

$$\mu_{ic} \geq \bar{J}_i(\gamma_c^1), \qquad i = 1, \ldots, n.$$

So,

$$c = (\gamma_c^1, \mu_{1c}, \ldots, \mu_{nc}) \in E.$$

Hence, E is a convex set. □

The sets E_i, as defined in (18), were proved to be convex and polyhedral in Theorem 4.1. Thus, there exists a pair of vector sets (X_i, Y_i) for each E_i which finitely generates it. These are of the form

$$X_i = \{x_{i,1}, \ldots, x_{i,w(i)}\}, \tag{25}$$

$$Y_i = \{y_{i,1}, \ldots, y_{i,m(i)}\}, \tag{26}$$

with $i = 1, \ldots, n$ and $w(i)$, $m(i)$ integers ≥ 1. Thus,

$$E_i = G(X_i, Y_i). \tag{27}$$

Theorem 4.3. (i) Each vector in Y_i is of the form $(0, \ldots, 0, \eta)$, $\eta \geq 0$, $\eta \in R$.

(ii) There exists at least one vector in Y_i of the form $(0, \ldots, 0, \eta)$, $\eta > 0$, $\eta \in R$.

Proof. (i) Let $y \in Y_i$ be of the form

$$y = (\alpha_1, \ldots, \alpha_p, \alpha_{p+1}).$$

Now, consider a vector

$$z = \lambda_1 x_{i,1} + \cdots + \lambda_{w(i)} x_{i,w(i)} + \beta y,$$

with

$$\lambda_1 + \cdots + \lambda_{w(i)} = 1, \qquad \lambda_j \geq 0, \quad j = 1, \ldots, w(i), \quad \beta \geq 0.$$

Obviously, by definition, $z \in E_i$ for all $\beta \geq 0$. Define

$$z_1 = \lambda_1 x_{i,1} + \cdots + \lambda_{w(i)} x_{i,w(i)}.$$

Obviously, $z_1 \in E_i$ is true; therefore, there exists $\gamma^1 \in \Gamma^1$ and $\hat{\mu} \geq \bar{J}_i(\gamma^1)$, such that

$$z_1 = (\gamma^1, \hat{\mu}).$$

Then,

$$z = (\gamma^1 + \beta(\alpha_1, \ldots, \alpha_p), \hat{\mu} + \beta\alpha_{p+1}).$$

Let there be at least one $\alpha_i \neq 0$, $1 \leq i \leq 0$. Then, by taking a large enough β, we can have

$$\gamma^1 + \beta(\alpha_1, \ldots, \alpha_p) \notin \Gamma^1.$$

But this would mean that $z \notin E_i$. Thus,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0.$$

Suppose that $\alpha_{p+1} < 0$. Then again, by choosing a large enough β, we can have

$$\hat{\mu} + \beta\alpha_{p+1} < \bar{J}_i(\gamma^1),$$

which would again imply that $z \notin E_i$. Thus, each vector in Y_i is of the form $(0, \ldots, 0, \eta)$, $\eta \geq 0$, $\eta \in R$.

(ii) From (i), we know that each vector in Y_i is of the form $(0, \ldots, 0, \eta)$, $\eta \geq 0$, $\eta \in R$. Let $\eta = 0$ for all the vectors in Y_i. This would mean that (X_i, Y_i) will generate only a bounded convex set, since all vectors in E_i would be convex combinations of the vectors in X_i. But, by definition, E_i is an unbounded set. Thus, $\eta > 0$ must be true for some vector in Y_i. □

The above Theorem 4.3 suggests that it is enough for Y_i to be a singleton with only a single vector $(0, 0, \ldots, 0, 1) \in R^{p+1}$, in order that $E_i = G(X_i, Y_i)$.

Theorem 4.4. In order that $E_i = G(X_i, Y_i)$, the vectors in X_i may be chosen in such a way that

$$x_{i,j} = (\gamma^1_{i,j}, \bar{J}_i(\gamma^1_{i,j})), \tag{28}$$

with $\gamma^1_{i,j} \in \Gamma^1$, $j = 1, \ldots, w(i)$.

Proof. Let $x \in X_i$. Then, by definition, $x \in E_i$; and so, there exists $\gamma^1 \in \Gamma^1$ and $\mu \in R$, such that

$$x = (\gamma^1, \mu), \qquad \gamma^1 \in \Gamma^1, \qquad \mu \geq \bar{J}_i(\gamma^1).$$

This can be rewritten as

$$x = (\gamma^1, \bar{J}_i(\gamma^1)) + (\mu - \bar{J}_i(\gamma^1))(0, \ldots, 0, 1).$$

Thus, for any finite generation process, in which the coefficient of x is nonzero, x can be replaced by the above expression. The first part will now contribute to the convex combination, while the second part can be added to the positive coefficient of the vector in Y_i. Thus, it is the first part which can now become an element in X_i; i.e., the original x can be replaced by a vector of the form (28). □

Remark 4.1. Theorems 4.3 and 4.4 prove that there exist strategies $\gamma^1_{i,1}, \ldots, \gamma^1_{i,w(i)} \in \Gamma^1$ such that the sets of vectors

$$X_i = \{(\gamma^1_{i,1}, \bar{J}_i(\gamma^1_{i,1})), \ldots, (\gamma^1_{i,w(i)}, \bar{J}_i(\gamma^1_{i,w(i)}))\}, \tag{29}$$

$$Y_i = \{(0, \ldots, 0, 1)\} \tag{30}$$

finitely generate the set E_i, $i=1,\ldots,n$. It is to be noted that X_i is not necessarily the minimal set of vectors necessary to finitely generate E_i. In order to obtain the minimal set, we have to eliminate those vectors in X_i which can be expressed as a convex combination of the other vectors in X_i. But, for our purposes, it is not necessary to obtain the minimal set. It is also obvious that, if we include additional vectors of the form $(\gamma^1, \bar{J}_i(\gamma^1))$ in X_i, the new (X_i, Y_i) still finitely generates E_i.

Each vector in X_i and Y_i has dimension $p+1$. We now derive sets of vectors P_i and Q_i from X_i and Y_i, respectively, but having vectors of dimension $p+n$. Let the jth vector in X_i be

$$x_{i,j} = (\gamma^1_{i,j}, \bar{J}_i(\gamma^1_{i,j})).$$

Then, the corresponding vector $p_{i,j} \in P_i \subseteq R^{p+n}$ will be defined as

$$p_{i,j} = (\gamma^1_{i,j}, \bar{J}_1(\gamma^1_{i,j}), \ldots, \bar{J}_n(\gamma^1_{i,j})). \tag{31}$$

The vector $(0,\ldots,0,1) \in Y_i$ will have the corresponding vector $q_i \in Q_i \subseteq R^{p+n}$ as

$$q_i = (0,\ldots,0,1,0,\ldots,0), \tag{32}$$

with the 1 in the $(p+i)$th position.

By Theorem 4.1, the set $E_i = \text{Epi}\, \bar{J}_i$ is convex and polyhedral in R^{p+1} space. Thus, all its faces are also polyhedral convex sets (see Ref. 13). Suppose that all these faces are projected onto the subset formed by Γ^1 in R^{p+1}. Then, according to Theorem 3.5, these projections are also polyhedral convex sets. Actually, they are bounded (and so polytopes), because Γ^1 is bounded. Obviously, the union of these polytopes constitutes Γ^1. If all such polytopes are considered by taking projections of all E_i, $i=1,\ldots,n$, onto Γ^1, then the whole of Γ^1 will be just a collection of a finite number of polytopes. Let us denote these polytopes as $T_1,\ldots,T_r$. They have the property that

$$\bigcup_{k=1}^{r} T_k = \Gamma^1, \qquad \text{int}\, T_i \cap \text{int}\, T_j = \varnothing, \qquad i \neq j. \tag{33}$$

Since each T_k is a polytope, it can be generated by the convex combinations of a finite number of vectors in Γ^1. Let this set of vectors be defined as

$$\Gamma^1(T_k) = \{\gamma^{1k}_1, \ldots, \gamma^{1k}_{t(k)}\}, \tag{34}$$

where $k=1,\ldots,r$ and $t(k)$ is a positive integer.

Now, for each $\gamma^{1k}_j \in \Gamma^1(T_k)$, define a $(p+n)$-dimensional vector

$$v_{k,j} = [\gamma^{1k}_j, \bar{J}_1(\gamma^{1k}_j), \ldots, \bar{J}_n(\gamma^{1k}_j)], \qquad j=1,\ldots,t(k). \tag{35}$$

Thus, from each $\Gamma^1(T_k)$, we obtain a finite set of vectors,

$$V_k = \bigcup_{j=1}^{t(k)} \{v_{k,j}\}, \qquad k=1,\ldots,r. \tag{36}$$

Now, define

$$P = \left[\bigcup_{i=1}^{n} P_i\right] \cup \left[\bigcup_{k=1}^{r} V_k\right], \tag{37}$$

$$Q = \bigcup_{i=1}^{n} Q_i. \tag{38}$$

The polyhedral convex set generated by these two sets is denoted by $G(P, Q)$. Our objective is to prove that $E = G(P, Q)$, which would imply that E is a finitely generated set and thus convex and polyhedral.

Let the total number of vectors in P be denoted by a; then,

$$a \leq \sum_{i=1}^{n} w(i) + \sum_{k=1}^{r} t(k). \tag{39}$$

Let the total number of vectors in Q be denoted by b; then,

$$b = n. \tag{40}$$

Let a vector in P be of the form

$$\tilde{p}_j = (\gamma_j^1, \bar{J}_1(\gamma_j^1), \ldots, \bar{J}_n(\gamma_j^1)), \qquad j=1,\ldots,a; \tag{41}$$

and let a vector in Q be of the form

$$\tilde{q}_j = (0,\ldots,0,1,0,\ldots,0), \qquad j=1,\ldots,n, \tag{42}$$

with the 1 in the $(p+j)$th position.

Theorem 4.5. $G(P, Q) \subseteq E$.

Proof. Let $z \in G(P, Q)$. Then, there exist weights $\lambda_j \geq 0$, $j=1,\ldots,a$, $\lambda_1 + \cdots + \lambda_a = 1$, and $\beta_j \geq 0$, $j=1,\ldots,b$, such that

$$z = \lambda_1 \tilde{p}_1 + \cdots + \lambda_a \tilde{p}_a + \beta_1 \tilde{q}_1 + \cdots + \beta_b \tilde{q}_b,$$

where each $\tilde{p}_j$ is of the form given in (41). Let

$$\gamma_0^1 = \lambda_1 \gamma_1^1 + \cdots + \lambda_a \gamma_a^1.$$

Obviously,

$$\gamma_0^1 \in \Gamma^1.$$

Let

$$\hat{\mu}_k = \lambda_1 \bar{J}_k(\gamma_1^1) + \cdots + \lambda_a \bar{J}_k(\gamma_a^1), \qquad k = 1, \ldots, n.$$

By Theorem 3.2 and Definition 3.4, $\bar{J}_k$ is a convex function. Then, we have

$$\hat{\mu}_k \geq \bar{J}_k(\gamma_0^1), \qquad k = 1, \ldots, n.$$

So, z can be written as follows:

$$\begin{aligned} z &= (\gamma_0^1, \hat{\mu}_1, \ldots, \hat{\mu}_n) + \beta_1 \tilde{q}_1 + \cdots + \beta_n \tilde{q}_n \\ &= (\gamma_0^1, \hat{\mu}_1 + \beta_1, \ldots, \hat{\mu}_n + \beta_n). \end{aligned}$$

Letting

$$\mu_k = \hat{\mu}_k + \beta_k, \qquad k = 1, \ldots, n,$$

we have

$$z = (\gamma_0^1, \mu_1, \ldots, \mu_n).$$

Since $\hat{\mu}_k \geq \bar{J}_k(\gamma_0^1)$, $\beta_k \geq 0$, and $\gamma_0^1 \in \Gamma^1$, we have

$$\mu_k \geq \bar{J}_k(\gamma_0^1), \qquad k = 1, \ldots, n.$$

This implies that

$$z \in E.$$

Thus,

$$G(P, Q) \subseteq E. \qquad \square$$

Theorem 4.6. $E \subseteq G(P, Q)$.

Proof. Let $z \in E$. Then, there exists $\gamma^1 \in \Gamma^1$ and $\mu_j \in R$, with $\mu_j \geq \bar{J}_j(\gamma^1)$, $j = 1, \ldots, n$, such that

$$z = (\gamma^1, \mu_1, \ldots, \mu_n),$$

which can be rewritten as

$$\begin{aligned} z &= (\gamma^1, \bar{J}_1(\gamma^1), \ldots, \bar{J}_n(\gamma^1)) \\ &\quad + (\mu_1 - \bar{J}_1(\gamma))\tilde{q}_1 + \cdots + (\mu_n - \bar{J}_n(\gamma^1))\tilde{q}_n. \end{aligned}$$

Since $\gamma^1 \in \Gamma^1$, there exists at least one $T_k \subseteq \Gamma^1$ (the polytopes defined earlier) such that $\gamma^1 \in T_k$. Then, there exists weights $\lambda_1, \ldots, \lambda_{t(k)}$, such that

$$\gamma^1 = \lambda_1 \gamma_1^{1k} + \cdots + \lambda_{t(k)} \gamma_{t(k)}^{1k}.$$

Also,

$$\begin{aligned} &\bar{J}_j(\lambda_1 \gamma_1^{1k} + \cdots + \lambda_{t(k)} \gamma_{t(k)}^{1k}) \\ &= \max_{\gamma^2 \in \Gamma^2} \{\lambda_1 \gamma_1^{1k} A(j) \gamma^2 + \cdots + \lambda_{t(k)} \gamma_{t(k)}^{1k} A(j) \gamma^2\}. \end{aligned}$$

Now, the linearity of $\bar{J}_j(\gamma^1)$ in T_k is obvious from the fact that each E_j is a polyhedral convex set (as proved in Theorem 4.1) and each of its faces is a segment of a hyperplane in the $(p+1)$-dimensional space. Thus, we can rewrite the above expression as

$$\lambda_1 \max_{\gamma^2\in\Gamma^2} \gamma_1^{1k} A(j)\gamma^2 + \cdots + \lambda_{t(k)} \max_{\gamma^2\in\Gamma^2} \gamma_{t(k)}^{1k} A(j)\gamma^2$$
$$= \lambda_1 \bar{J}_j(\gamma_1^{1k}) + \cdots + \lambda_{t(k)} \bar{J}_j(\gamma_{t(k)}^{1k}), \qquad j = 1, \ldots, n.$$

Using the above, we can write z as

$$z = \lambda_1[\gamma_1^{1k}, \bar{J}_1(\gamma_1^{1k}), \ldots, \bar{J}_n(\gamma_1^{1k})] + \cdots + \lambda_{t(k)}[\gamma_{t(k)}^{1k}, \bar{J}_1(\gamma_{t(k)}^{1k}), \ldots, \bar{J}_n(\gamma_{t(k)}^{1k})]$$
$$+ (\mu_1 - \bar{J}_1(\gamma^1))\tilde{q}_1 + \cdots + (\mu_n - \bar{J}_n(\gamma^1))\tilde{q}_n.$$

Since each of the vectors $\gamma_1^{1k}, \ldots, \gamma_{t(k)}^{1k}$ belongs to $\Gamma^1(T_k)$, we can define the vectors $v_{k,1}, \ldots, v_{k,t(k)}$, respectively, as given in (35). All these vectors belong to P. Thus, z can be expressed as the sum of the convex combination of these vectors (with the weights on the other vectors of P equal to zero) and the positive linear combination of the vectors $\tilde{q}_1, \ldots, \tilde{q}_n$, with weights $(\mu_1 - \bar{J}_1(\gamma^1)), \ldots, (\mu_n - \bar{J}_n(\gamma^1)) \geq 0$, respectively. Thus, z can be generated by the sets of vectors (P, Q). Hence,

$$z \in G(P, Q),$$

which implies that

$$E \subseteq G(P, Q). \qquad \square$$

Theorems 4.5 and 4.6 together imply that

$$E = G(P, Q). \tag{43}$$

Thus, E is finitely generated, and so is a polyhedral convex set. According to Theorem 3.5, as E^p is obtained by a linear transformation on E, E^p is also a polyhedral convex set. Further, since $E^p = S^E$, S^E is also a polyhedral convex set.

This completes the proof of polyhedrality of S^E. Now, using the result of Theorem 3.4, the proof of Theorem 3.3 is immediate.

This completes the proof for the solution of the game $\mathrm{P1}(\alpha)$ with strictly positive scalarization being the necessary and sufficient condition for a strategy to be a Pareto-optimal security strategy for player P1. Obviously, similar results can be obtained for player P2 too. $\square$

Since the set S^E is polyhedral, it has a finite number of faces in the form of segments of hyperplanes. Thus, only a finite number of strictly positive scalarizations are required to obtain all the POSS (Ref. 13). A major question is the identification of the scalarization vectors which would reduce the computational burden considerably. This question is not addressed in this paper. There could be many ways of obtaining these vectors, but they are mainly based on the identification of the extreme points of S^E. Some algorithms to identify the vertices of a polytope are discussed in Ref. 14.

5. Example

Consider the payoff matrix to be

$$A = \begin{bmatrix} (1,3) & (2,1) \\ (3,1) & (1,2) \\ (1,1) & (3,3) \end{bmatrix}.$$

The strategy set Γ^1 for P1 is shown in Fig. 1a. The projection of Epi $\bar{J}_1$ on Γ^1 produces the convex polytopes as shown in Fig. 1b and that of Epi $\bar{J}_2$ produces the convex polytopes as shown in Fig. 1c. When these two are

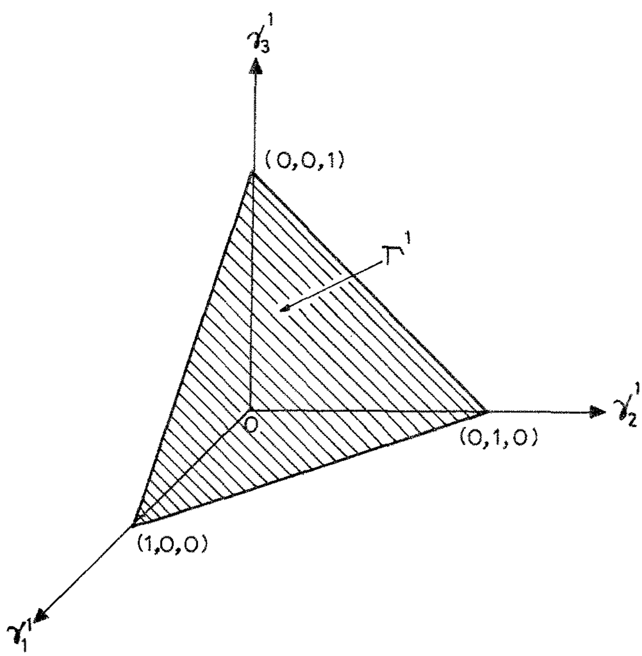

Fig. 1a. P1's strategy set Γ^1.

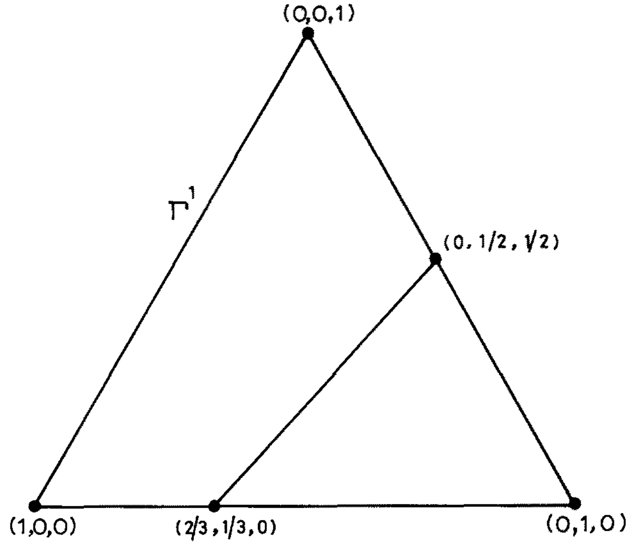

Fig. 1b. Projection of Epi $\bar{J}_1$ on Γ^1.

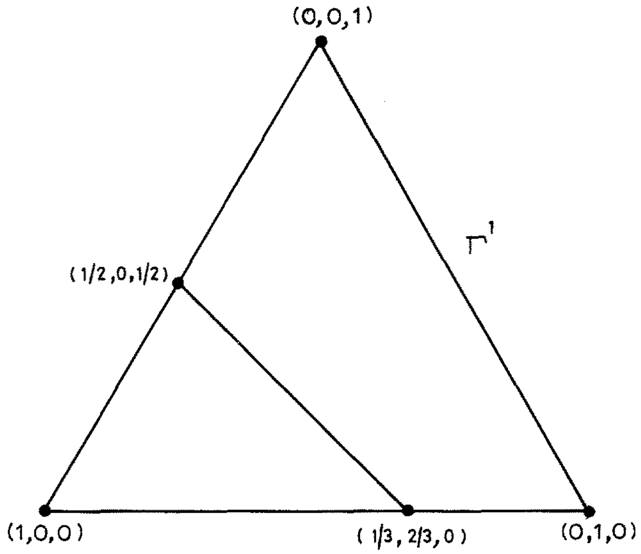

Fig. 1c. Projection of Epi $\bar{J}_2$ on Γ^1.

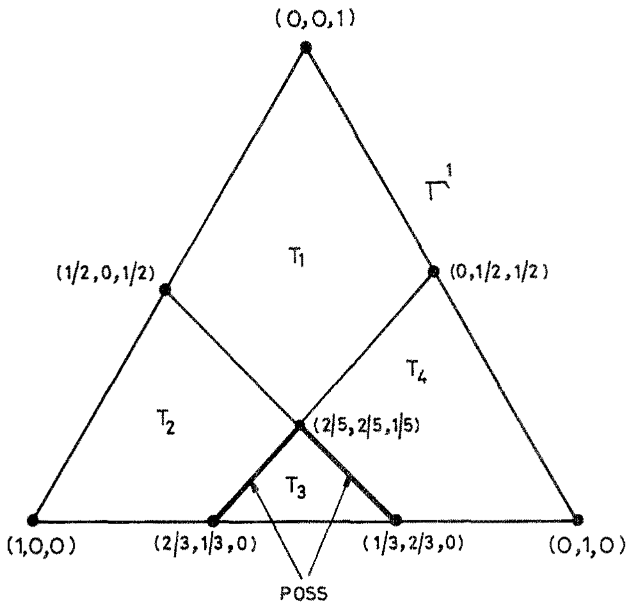

Fig. 1d. Polytopes obtained by projecting both Epi $\bar{J}_1$ and Epi $\bar{J}_2$ on Γ^1.

superimposed, we obtain the collection of polytopes as shown in Fig. 1d, from which we obtain

$$\Gamma^1(T_1) = \{(0, 0, 1), (1/2, 0, 1/2), (0, 1/2, 1/2), (2/5, 2/5, 1/5)\},$$
$$\Gamma^1(T_2) = \{(1, 0, 0), (1/2, 0, 1/2), (2/3, 1/3, 0), (2/5, 2/5, 1/5)\},$$
$$\Gamma^1(T_3) = \{(2/3, 1/3, 0), (1/3, 2/3, 0), (2/5, 2/5, 1/5)\},$$
$$\Gamma^1(T_4) = \{(0, 1/2, 1/2), (0, 1, 0), (1/3, 2/3, 0), (2/5, 2/5, 1/5)\}.$$

The sets P and Q are

$$P = \{(0, 0, 1, 3, 3), (1, 0, 0, 2, 3), (0, 1, 0, 3, 2), (1/2, 0, 1/2, 5/2, 2), (0, 1/2, 1/2, 2, 2), (2/3, 1/3, 0, 5/3, 7/3), (1/3, 2/3, 0, 7/3, 5/3), (2/5, 2/5, 1/5, 9/5, 9/5)\},$$
$$Q = \{(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}.$$

Further, the POSS for P1 are found to be the closed sets

$$[(2/3, 1/3, 0), (2/5, 2/5, 1/5)]$$

and

$$[(1/3, 2/3, 0), (2/5, 2/5, 1/5)],$$

which can be obtained by using the scalarization vectors

$$\alpha = (4/5, 1/5) \quad \text{and} \quad \alpha = (1/5, 4/5)$$

in the game $P1(\alpha)$, respectively. Thus, only two scalarizations are sufficient to obtain all the POSS for P1. These POSS are also shown in Fig. 1d.

6. Conclusions

In this paper, a necessary and sufficient condition was proved by which Pareto-optimal security strategies (POSS) of a player in a two-person, zero-sum multicriteria matrix game can be obtained as the saddle-point solution of a scalarized zero-sum matrix game parametrized by strictly positive coefficients. In the process, it was also shown that, in such a game, only a finite number of scalarizations are required to obtain all the POSS for a player. The specific method by which the scalarization coefficients could be obtained was not addressed here.

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