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Iterative Computation of Noncooperative Equilibria in Nonzero-Sum Differential Games with Weakly Coupled Players[1,2]

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Abstract. We study the Nash equilibria of a class of two-person nonlinear, deterministic differential games where the players are weakly coupled through the state equation and their objective functionals. The weak coupling is characterized in terms of a small perturbation parameter ϵ. With $\epsilon = 0$, the problem decomposes into two independent standard optimal control problems, while for $\epsilon \neq 0$, even though it is possible to derive the necessary and sufficient conditions to be satisfied by a Nash equilibrium solution, it is not always possible to construct such a solution. In this paper, we develop an iterative scheme to obtain an approximate Nash solution when ϵ lies in a small interval around zero. Further, after requiring strong time consistency and/or robustness of the Nash equilibrium solution when at least one of the players uses dynamic information, we address the issues of existence and uniqueness of these solutions for the cases when both players use the same information, either closed loop or open loop, and when one player uses open-loop information and the other player uses closed-loop information. We also show that, even though the original problem is nonlinear, the higher (than zero) order terms in the Nash equilibria can be obtained as solutions to LQ optimal control problems or static quadratic optimization problems.

Key Words. Differential games, Nash equilibria, regular perturbation, policy iteration, information structures.

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1. Introduction

A challenging task in nonzero-sum differential games with nonlinear dynamics and nonquadratic cost functions is to establish the existence and uniqueness of a noncooperative equilibrium and to develop constructive procedures to obtain the corresponding strategies, especially when the information structure is dynamic. In this paper, we address these issues, under the Nash equilibrium solution concept, for a class of nonzero-sum differential games where the players are weakly coupled through the presence of a small parameter ϵ in the state equation. When $\epsilon = 0$, the problem decomposes into two independent single decision maker problems whose solutions can be obtained using known results in optimal control theory. The approach developed in this paper iterates on the zeroth order solution thus obtained, to arrive at better approximations to the true Nash equilibrium solution, if one exists. The procedure is developed for the cases where both players use the same information, either closed loop or open loop, and when one player uses open-loop information and the other player uses closed-loop information (Ref. 1). Further, we address the issues of convergence, existence, and uniqueness of the solutions obtained using this procedure.

The class of weakly coupled systems is a subclass of regularly perturbed systems, which have been studied extensively for the single decision maker case (see Refs. 2 and 3). In Ref. 2, the solution to regularly perturbed optimal control problems has been obtained using both the dynamic programming approach and Pontryagin's minimum principle. Here, we extend the results of Ref. 2 to differential games with weakly coupled decision makers. One of the sources of difficulty in attempting to extend the results from the single decision maker case to a game situation is the fact that the state trajectory under dynamic information patterns (e.g., feedback state information) is not the same as the state trajectory under the open-loop information pattern (Ref. 1). It is this equivalence of the optimal trajectories in the single decision maker problems under different information patterns that is exploited in Ref. 2 to prove the existence and uniqueness of the solution in the closed-loop information case.

Differential games with weakly coupled agents have been studied before in Refs. 8 and 10 for the special class of linear-quadratic (LQ) problems, but the approximate solutions have been obtained using (as the starting point) the solution of the perturbed problem, i.e., the one with $\epsilon \neq 0$. This implies, however, that the perturbed problem is solvable, which is an assumption that is difficult to justify *a priori*, especially in nonlinear differential games. In this paper, we will develop a method that will circumvent this difficulty, by making direct use of the weakness in the spatial coupling between the two independent subsystems. The idea of exploiting the presence of weak coupling among decision makers to solve otherwise unsolvable problems has been studied earlier in the context of stochastic multiple decision maker problems in Refs. 4, 5, and 6.

The rest of the present paper is organized as follows. Section 2 deals with the problem formulation, where we formally introduce the three types of information structures that we will be considering in this paper. In Section 3, we first assume that the Nash equilibrium solution is expandable as a function of ϵ, and obtain the various terms in this expansion as the solution of simpler optimization problems. Further, we establish the uniqueness of such an asymptotic expansion in the sense that, irrespective of the starting policy choices, a Cournot or Gauss–Seidel policy iteration yields uniquely the successive terms in the expansion at successive steps of the iteration. Section 4 justifies the assumption of the asymptotic expansion of the Nash equilibrium solution, by showing that, if we use the first $(k+1)$ terms of the above expansion, we have an $O(\epsilon^{2k+2})$ Nash equilibrium. Further, we establish the uniqueness of the various terms in the expansion. Section 5 provides the concluding remarks.

2. Problem Statement

In this section, we provide a precise mathematical formulation for the class of two-person, deterministic differential game problems which will be studied in this paper. We consider only the two-player case without much loss of conceptual generality, and simply note that the results derived here are readily extendible to the multiple player case in a rather straightforward manner, as further explained in Section 5.

Consider the game dynamics described by the following state equations:

$$\dot{x}_1(t) = f_1(x_1(t), u_1(t)) + \epsilon f_{12}(x_2(t)), \qquad x_1(t_0) = x_{10}, \tag{1a}$$

$$\dot{x}_2(t) = f_2(x_2(t), u_2(t)) + \epsilon f_{21}(x_1(t)), \qquad x_2(t_0) = x_{20}, \tag{1b}$$

where $x(t) = [x_1'(t), x_2'(t)]'$ is the state vector of dimension n, and $x_i(t)$ is the ith subsystem state of dimension n_i, $i = 1, 2$. The functions $f_1(.,.), f_{12}(.,.)$, $f_{21}(.,.), f_2(.)$ are infinitely many times differentiable in their arguments. The control of Player i, denoted by $u_i(t)$, belongs to $\mathscr{R}^{m_i}$, $i = 1, 2$. The scalar ϵ is a small parameter which (weakly) couples the two players.

The objective functional for Player i is given by

$$J_i(u_i, u_j) = g_{if}(x_i(t_f)) + \epsilon g_{ijf}(x_j(t_f)) + \int_{t_0}^{t_f} (g_i(x_i, u_i) + \epsilon g_{ij}(x_j, u_j))\, dt, \qquad i, j = 1, 2, j \neq i, \tag{2}$$

which he strives to minimize.

We will deal with three types of information structures for the differential game: (i) both players have open-loop (OL) information; (ii) both players have closed-loop perfect state (CLPS) information; and (iii) Player 1 uses open-loop information, and Player 2 uses closed-loop information (OL-CLPS). Under any one of these patterns, let the information available to Player i at time t be denoted by $I_i(t)$. For the open-loop information pattern,

$$I_i(t)=\{x(t_0)\};$$

and for closed-loop perfect-state information,

$$I_i(t)=\{x(s), 0\leq s\leq t\}.$$

In the open-loop case, for each fixed $x(t_0)$, a permissible strategy is a measurable mapping γ_i: $[t_0, t_f]\rightarrow\mathscr{R}^{m_i}$, $i=1, 2$. In the closed-loop case, by imposing strong time consistency or asymptotic robustness (Ref. 1), the information to Player i can be assumed, without loss of generality, to be

$$I_i(t)=\{x(t_0), x(t)\}.$$

Hence, in this case, for each fixed $x(t_0)$, a permissible strategy is a measurable mapping γ_i: $[t_0, t_f]X\mathscr{R}^n\rightarrow\mathscr{R}^{m_i}$, $i=1, 2$. We let Γ_i, $i=1, 2$, be the appropriate strategy space in each case. Then, the problem is to find a pair of policies $\{\gamma_1^*\in\Gamma_1, \gamma_2^*\in\Gamma_2\}$ that constitutes a Nash equilibrium solution, i.e., a pair $\{\gamma_1^*\in\Gamma_1, \gamma_2^*\in\Gamma_2\}$ such that the following inequalities are satisfied for all $\{\gamma_i\in\Gamma_i, i=1, 2\}$:

$$J_1^*:=J_1(\gamma_1^*, \gamma_2^*)\leq J_1(\gamma_1, \gamma_2^*), \tag{3a}$$

$$J_2^*:=J_2(\gamma_1^*, \gamma_2^*)\leq J_1(\gamma_1^*, \gamma_2). \tag{3b}$$

In the open-loop case, the Nash equilibrium is known to be weakly (but not strongly) time consistent. In the closed-loop or the mixed case, the Nash equilibrium is known to exhibit informational nonuniqueness (Refs. 1 and 11). In order to avoid this plethora of equilibria, we bring in the further refinement of strong time consistency or asymptotic robustness (to infinitesimal noise), as discussed in Ref. 9. The significance of these additional impositions on the Nash equilibrium concept should become clear in Sections 3.2 and 3.3 below, as we discuss equilibria under dynamic information patterns.

3. Asymptotic Expansion of the Nash Equilibrium Solution

3.1. Open-Loop Information Structure. Toward studying Nash equilibria of the problem formulated in Section 2 under the OL information pattern, let us first write down the necessary conditions associated with such a solution. Suppose that $u_i^*(t) := \gamma_i^*(t, x_0)$, $i=1, 2$, provides an OL Nash equilibrium solution. Then, there exist costate vectors $p_i(t) := [p_{i1}'(t), p_{i2}'(t)]'$, $i=1, 2$, which satisfy the following equations (Ref. 1):

$$\dot{x}_1^* = f_1(x_1^*, u_1^*) + \epsilon f_{12}(x_2^*), \qquad x_1^*(t_0) = x_{10}, \tag{4a}$$

$$\dot{x}_2^* = f_2(x_2^*, u_2^*) + \epsilon f_{21}(x_1^*), \qquad x_2^*(t_0) = x_{20}, \tag{4b}$$

$$u_1^*(t; \epsilon) = \arg\min_{u_1 \in \mathscr{R}^{m_1}} H_1(t, p_1(t), x^*(t), u_1(t), u_2^*(t)), \tag{5a}$$

$$u_2^*(t; \epsilon) = \arg\min_{u_2 \in \mathscr{R}^{m_2}} H_2(t, p_2(t), x^*(t), u_1^*(t), u_2(t)), \tag{5b}$$

$$\dot{p}_i(t; \epsilon) = -(\partial/\partial x) H_i(t, p_i(t), x^*(t), u_1^*(t), u_2^*(t)), \tag{6a}$$

$$\begin{aligned} p_i(t_f; \epsilon) &= (d/dx) g_{if}(x_i^*(t_f)) \\ &\quad + \epsilon (d/dx) g_{ijf}(x_j(t_f)), \qquad i, j = 1, 2, j \neq i, \end{aligned} \tag{6b}$$

where

$$\begin{aligned} H_i(t, p_i, x, u_1, u_2) &= g_i(x_i, u_i) + \epsilon g_{ij}(x_j, u_j) \\ &\quad + \sum_{i=1}^{2} p_{ij}'(f_i(x_i, u_i) + \epsilon f_{ij}(x_j)), \qquad i, j = 1, 2, j \neq i. \end{aligned} \tag{7}$$

In the above expressions, we have used $\partial/\partial x$ to mean the first partial derivative with respect to the vector x, and it is expressed as a column vector. Since $f_i(.,.)$, $f_{ij}(.)$, $i, j = 1, 2$, $i \neq j$, are taken to be differentiable in their arguments, we can write down the first-order necessary condition for (5) as

$$(\partial/\partial u_i) g_i(x_i^*, u_i^*) + (\partial/\partial u_i) f_i(x_i^*, u_i^*) p_{ii} = 0, \qquad i = 1, 2. \tag{8}$$

Now, suppose that there exists an expansion for $x^*(t; \epsilon)$, $p_1^*(t; \epsilon)$, $p_2^*(t; \epsilon)$, $u^*(t; \epsilon)$ in terms of ϵ as

$$\begin{aligned} x^*(t) &= \sum_{k=0}^{\infty} x^{(k)}(t) \epsilon^k, \qquad p^*(t) = \sum_{k=0}^{\infty} p^{(k)}(t) \epsilon^k, \\ u^*(t) &= \sum_{k=0}^{\infty} u^{(k)}(t) \epsilon^k. \end{aligned} \tag{9}$$

Using (9), we can obtain a power series expansion of (4a), (4b), (6a), (6b), (8) in terms of ϵ, with the zeroth order terms satisfying:

$$\dot{x}_i^{(0)} = f_i(x_i^{(0)}, u_i^{(0)}), \qquad x_i^{(0)}(t_0) = x_{i0}, \tag{10a}$$

$$\begin{aligned} \dot{p}_{ii}^{(0)} &= -(\partial/\partial x_i) g_i(x_i^{(0)}, u_i^{(0)}) - (\partial/\partial x_i) f_i(x_i^{(0)}, u_i^{(0)}) p_{ii}^{(0)}, \\ p_{ii}^{(0)}(t_f) &= g_{if}(x_i^{(0)}(t_f)), \end{aligned} \tag{10b}$$

$$\dot{p}_{ij}^{(0)} = -(d/dx_j) f_j(x_j^{(0)}) p_{ij}^{(0)}, \qquad p_{ij}^{(0)}(t_f) = 0, \ i, j = 1, 2, j \neq i, \tag{10c}$$

$$(\partial/\partial u_i) g_i(x_i^{(0)}, u_i^{(0)}) + (\partial/\partial u_i) f_i(x_i^{(0)}, u_i^{(0)}) p_{ii}^{(0)} = 0. \tag{10d}$$

It should be noted that the unique solution to (10c) is $p_{ij}^{(0)}(t) \equiv 0, j \neq i$. Now, the ϵ terms in the expansion (that is, the first-order terms) yield

$$\dot{x}_i^{(1)} = (\partial/\partial x_i) f_i(x_i^{(0)}, u_i^{(0)}) x_i^{(1)} + (\partial/\partial u_i) f_i(x_i^{(0)}, u_i^{(0)}) u_i^{(1)} + f_{ij}(x_j^{(0)}),$$

$$x_i^{(1)}(t_0) = 0, \tag{11a}$$

$$\begin{aligned}\dot{p}_{ii}^{(1)} = &-(\partial^2/\partial x_i^2) g_i(x_i^{(0)}, u_i^{(0)}) x_i^{(1)} \\ &- [(\partial^2/\partial x_i^2) f_i(x_i^{(0)}, u_i^{(0)}) x_i^{(1)} + (\partial^2/\partial u_i \partial x_i) f_i(x_i^{(0)}, u_i^{(0)}) u_i^{(1)}] p_{ii}^{(0)} \\ &- (\partial/\partial x_i) f_i(x_i^{(0)}, u_i^{(0)}) p_{ii}^{(1)} - (\partial^2/\partial u_i \partial x_i) g_i(x_i^{(0)}, u_i^{(0)}) u_i^{(1)},\end{aligned}$$

$$p_{ii}^{(1)}(t_f) = (d^2/dx_{if}^2) g_{if}(x_{if}^{(0)}) x_{if}^{(1)}, \tag{11b}$$

$$\dot{p}_{ij}^{(1)} = -(d/dx_j) g_{ij}(x_j^{(0)}) - (d/dx_j) f_{ij}(x_j^{(0)}) p_{ii}^{(0)} - (d/dx_j) f_j(x_j^{(0)}) p_{ij}^{(1)},$$

$$p_{ij}^{(1)}(t_f) = (d/dx_{jf}) g_{ijf}(x_{jf}^{(0)}), \tag{11c}$$

$$\begin{aligned}&(\partial^2/\partial u_i^2) g_i(x_i^{(0)}, u_i^{(0)}) u_i^{(1)} \\ &+ [(\partial^2/\partial u_i^2) f_i(x_i^{(0)}, u_i^{(0)}) u_i^{(1)} + (\partial^2/\partial u_i\, \partial x_i) f_i(x_i^{(0)}, u_i^{(0)}) x_i^{(1)}] p_{ii}^{(0)} \\ &+ (\partial/\partial u_i) f_i(x_i^{(0)}, u_i^{(0)}) p_{ii}^{(1)} + (\partial^2/\partial x_i\, \partial u_i) g_i(x_i^{(0)}, u_i^{(0)}) x_i^{(1)} = 0,\end{aligned} \tag{11d}$$

where we have made use of the fact that $p_{ij}^{(0)}(t) \equiv 0$, $i \neq j$. It should be noted that the functions $x_i^{(1)}(t)$, $p_i^{(1)}(t)$, $u_i^{(1)}(t)$ can be solved independently of $x_j^{(1)}(t)$, $p_j^{(1)}(t)$, $u_j^{(1)}(t)$, $i \neq j$. Furthermore, the dependence on $x_j(t)$, $p_j(t)$, $u_j(t)$, $i \neq j$, is only through the zeroth order terms, which have already been determined in the previous step of the iteration.

Before obtaining the general expression for the kth order terms, let us introduce some notation. Let $W(a(\epsilon), b(\epsilon))$ be a function which is infinitely many times differentiable in its arguments, where

$$a(\epsilon) := \sum_{i=0}^{\infty} a^{(i)} \epsilon^i, \qquad b(\epsilon) := \sum_{i=0}^{\infty} b^{(i)} \epsilon^i.$$

Then, the kth total derivative of $W(a(\epsilon), b(\epsilon))$ with respect to ϵ at $\epsilon = 0$ is given by

$$\begin{aligned}(1/k!)(d^k/d\epsilon^k) W = &(\partial/\partial a) W(a^{(0)}, b^{(0)}) a^{(k)} \\ &+ (\partial/\partial b) W(a^{(0)}, b^{(0)}) b^{(k)} + R_1(W, k),\end{aligned} \tag{12}$$

$$\begin{aligned}R_1(W, k) := &\sum_{j=2}^{k} \sum_{\substack{l_1 + \cdots + l_j = k \\ l_1, \ldots, l_j \geq 1}} (1/j!)(\partial^j/\partial a^j) W(a^{(0)}, b^{(0)}) a^{(l_1)} \cdots a^{(l_j)} \\ &+ \sum_{j=2}^{k} \sum_{\substack{l_1 + \cdots + l_j = k \\ l_1, \ldots, l_j \geq 1}} (1/j!)(\partial^j/\partial b^j) W(a^{(0)}, b^{(0)}) b^{(l_1)} \cdots b^{(l_j)},\end{aligned} \tag{13}$$

where $(\partial^j/\partial a^j) W(a^{(0)}, b^{(0)})$ and $(\partial^j/\partial b^j) W(a^{(0)}, b^{(0)})$ denote the jth partial derivative operators with respect to a and b, respectively, at the point $(a^{(0)}, b^{(0)})$. If W is a function of one variable [i.e., $W \equiv W(a(\epsilon))$], then the above expression simplifies to

$$(1/k!)(d^k/d\epsilon^k)W = (d/da)W(a^{(0)})a^{(k)} + R_2(W, k), \tag{14}$$

$$R_2(W, k) := \sum_{j=2}^{k} \sum_{\substack{l_1+\cdots+l_j=k \\ l_1,\ldots,l_j \geq 1}} (1/j!)(d^j/da^j)W(a^{(0)})a^{(l_1)} \cdots a^{(l_j)}, \tag{15}$$

where $(d^j/da^j)W(a^{(0)})$ denotes the jth derivative operator at the point $a^{(0)}$.

Now, the expressions for the kth order terms (the coefficients of ϵ^k) in (9) are given by

$$\dot{x}_i^{(k)} = (\partial/\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})x_i^{(k)} + (\partial/\partial u_i)f_i(x_i^{(0)}, u_i^{(0)})u_i^{(k)} + M_{xk}(t),$$
$$x_i^{(k)}(t_0) = 0, \tag{16}$$

$$\begin{aligned}\dot{p}_{ii}^{(k)} = &-(\partial^2/\partial x_i^2)g_i(x_i^{(0)}, u_i^{(0)})x_i^{(k)} - (\partial^2/\partial u_i\,\partial x_i)g_i(x_i^{(0)}, u_i^{(0)})u_i^{(k)} \\ &- [(\partial^2/\partial x_i^2)f_i(x_i^{(0)}, u_i^{(0)})x_i^{(k)}]p_{ii}^{(0)} - [(\partial^2/\partial u_i\,\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})u_i^{(k)}]p_{ii}^{(0)} \\ &- [(\partial/\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})]p_{ii}^{(k)} - M_{pk}(t),\end{aligned}$$
$$p_{ii}^{(k)}(t_f) = (\partial^2/\partial x_{if}^2)g_i(x_i^{(0)}(t_f))x_i^{(k)}(t_f) + M_{pkf}, \tag{17}$$

$$\begin{aligned}\dot{p}_{ij}^{(k)} = &-[1/(k-1)!](d^{k-1}/d\epsilon^{k-1})(d/dx_j)g_{ij}(x_j^{(0)}) \\ &- \sum_{l=0}^{k-1} (1/j!)(d^j/d\epsilon^j)[(d/dx_j)f_{ij}]p_{ii}^{(k-1-l)} \\ &- \sum_{l=0}^{k-1} (1/l!)(d^l/d\epsilon^l)[(\partial/\partial x_j)f_j]p_{ij}^{(k-l)},\end{aligned}$$
$$p_{ij}^{(k)}(t_f) = [1/(n-1)!](d^{(n-1)}/d\epsilon^{(n-1)})(d/dx_{jf})g_{ijf}, \tag{18}$$

$$\begin{aligned}&(\partial^2/\partial x_i\,\partial u_i)g_i(x_i^{(0)}, u_i^{(0)})x_i^{(k)} + (\partial^2/\partial u_i^2)g_i(x_i^{(0)}, u_i^{(0)})u_i^{(k)} \\ &+ (\partial/\partial u_i)f_i(x_i^{(0)}, u_i^{(0)})p_{ii}^{(k)} \\ &+ (\partial^2/\partial u_i^2)f_i(x_i^{(0)}, u_i^{(0)})u_i^{(k)}p_{ii}^{(0)} + [(\partial^2/\partial x_i^2)f_i(x_i^{(0)}, u_i^{(0)})x_i^{(k)}]p_{ii}^{(0)} \\ &+ M_{uk}(t) = 0,\end{aligned} \tag{19}$$

where

$$M_{xk}(t) = R_1(f_i, k) + [1/(k-1)!](d^{k-1}/d\epsilon^{k-1})f_{ij}, \tag{20}$$

$$\begin{aligned}M_{pk}(t) = &R_1((\partial/\partial x_i)f_i, k)p_{ii}^{(0)} \\ &+ \sum_{l=1}^{k-1} (1/l!)(d^l/d\epsilon^l)[(\partial/\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})]p_{ii}^{(k-1)} \\ &+ R_1((\partial/\partial x_i)g_i, k) + \sum_{l=0}^{k-1} (1/j!)(d^j/d\epsilon^j)f_{ji}p_{ij}^{(k-1-l)},\end{aligned} \tag{21}$$

$$M_{uk}(t)=R_1((\partial/\partial u_i)g_i, k) + \sum_{l=1}^{k-1}(1/l!)(d^l/d\epsilon^l)(\partial/\partial u_i)[f_i(x_i^{(0)}, u_i^{(0)})]p_{ii}^{(k-l)} + [R_1((\partial/\partial u_i)f_i, k)]p_{ii}^{(0)}, \tag{22}$$

$$M_{pkf}=R_1((\partial/\partial x_{if})g_i(x_i^{(0)}(t_f)), k). \tag{23}$$

Note that the term $p_{ij}^{(k)}(t)$ can be computed from the $(k-1)$th order terms, independently of $u_i^{(k)}(t)$; and the rest of the ϵ^k terms, for every $k\geq 0$, correspond to solutions of optimal control problems as follows: For $k=0$, (10a)–(10d) corresponds to the solution of two independent control problems, one for each $i\in\{1, 2\}$, defined by

$$\dot{x}_i=f(x_i, u_i), \qquad x_i(t_0)=x_{i0}, \tag{24a}$$

$$J_i^*=\min_{u_i\in\mathscr{R}^{m_i}}\left\{g_{if}(x_i(t_f))+\int_{t_0}^{t_f} g_i(x_i(t), u_i(t))\, dt\right\}. \tag{24b}$$

The above optimal control problems are those obtained by setting $\epsilon=0$ in the original differential game.

For $k\geq 1$, (16), (17), and (19) correspond to the solution of the following LQ problem:

$$\dot{x}_i=(\partial/\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})x_i+(\partial/\partial u_i)f_i(x_i^{(0)}, u_i^{(0)})u_i+M_{xk}(t),$$
$$x_i(t_0)=0, \tag{25a}$$

$$\begin{aligned} J_i^*=\min_{u_i\in\mathscr{R}^{m_i}}\Big\{&(1/2)x_i'(t_f)(\partial^2/\partial x_{if}^2)g_{if}(x_i^{(0)}(t_f))x_i(t_f)\\ &+M_{pkf}x_i(t_f)+(1/2)\int_{t_0}^{t_f}(u_i'[(\partial^2/\partial u_i^2)g_i(x_i^{(0)}, u_i^{(0)})\\ &\qquad+p_{ii}^{(0)'}(\partial^2/\partial u_i^2)f_i(x_i^{(0)}, u_i^{(0)})]u_i\\ &+x_i'[(\partial^2/\partial x_i^2)g_i(x_i^{(0)}, u_i^{(0)})+p_{ii}^{(0)'}(\partial^2/\partial x_i^2)f_i(x_i^{(0)}, u_i^{(0)})]x_i\\ &+2x_i'[(\partial^2/\partial x_i\partial u_i)g_i(x_i^{(0)}, u_i^{(0)})+p_{ii}^{(0)'}(\partial^2/\partial u_i\,\partial x_i)f_i(x_i^{(0)}, u_i^{(0)})]u_i\\ &+2x'M_{pk}(t)+2u_i'M_{uk}(t))dt\Big\}. \end{aligned} \tag{25b}$$

Therefore, the original differential game has been decomposed into two nonlinear optimal control problems (the zeroth order problems) and a sequence of LQ control problems. The issues of existence and uniqueness of the original problem can now be studied by analyzing these simpler problems. This situation is analogous to the case of nonlinear, regularly perturbed optimal control problems studied in Ref. 2. What we have shown here though, is that, in spite of the fact that there is an additional co-state vector, the game problem also admits a decomposition that is similar to the case of the usual optimal control problems.

The above decomposition procedure for constructing a Nash equilibrium solution can be summarized as follows:

Procedure 3.1.

Step 1. Use (10a)–(10d) to calculate $u_1^{(0)}(t)$ and $u_2^{(0)}(t)$. Set $k=1$.

Step 2. Calculate $u_i^{(k)}(t)$, $i=1,2$, using (16)–(19). Set $k=k+1$.

Step 3. If $O(\epsilon^{l+1})$ accuracy is required, stop at $k=l+1$. Otherwise, go back to Step 2 and iterate.

Notice that the computation of $u_i^{(k)}(t)$, $i=1,2$, $k\geq 0$, using (16)–(19) is equivalent to the solution of a Riccati equation and two linear equations, because (16)–(19) correspond to the solution of the LQ control problem (25). The solution to (25) is explicitly given by

$$u_i^{(k)}(t) = -R_{ik}^{-1}(S'_{ik}x^*_{ik} + B'_{ik}P_{ik}x^*_{ik} + B'_{ik}p_{ik} + l_{ik}), \tag{26a}$$

$$\dot{P}_{ik} + Q_{ik} + P_{ik}B_{ik}R_{ik}^{-1}B'_{ik}P_{ik} - S_{ik}R_{ik}S'_{ik} + P_{ik}\tilde{A}_{ik} + \tilde{A}'_{ik}P_{ik} = 0,$$

$$P_{ik}(t_f) = Q_{fik}, \tag{26b}$$

$$\dot{p}_{ik} + S_{ik}R_{ik}^{-1}r_{ik} + P_{ik}B_{ik}R_{ik}^{-1}r_{ik} - P_{ik}c_{ik} - q_{ik} = 0,$$

$$p_{ik}(t_f) = q_{fik}, \tag{26c}$$

$$\dot{l}_{ik} + (1/2)(r'_{ik} + P_{ik}B_{ik})R_{ik}^{-1}(B'_{ik}P_{ik} + r_{ik})$$
$$+ P_{ik}(-B_{ik}R_{ik}^{-1}B'_{ik}P_{ik} - B_{ik}R_{ik}^{-1}r_{ik} + c_{ik}) = 0, \qquad l_{ik}(t_f) = 0, \tag{26d}$$

$$\dot{x}^*_{ik} = \tilde{A}_{ik}x^*_{ik} - B_{ik}(R_{ik}^{-1}B'_{ik}p_{ik} + R_{ik}^{-1}r_{ik}) + c_{ik}, \qquad x_{ik}(t_0) = 0, \tag{26e}$$

where

$$A_{ik} := (\partial/\partial x_i)f_i(x_i^{(0)}, u_i^{(0)}), \qquad B_{ik} := (\partial/\partial u_i)f_i(x_i^{(0)}, u_i^{(0)}),$$

$$C_{ik} := M_{xk}(t), \qquad Q_{fik} := (\partial^2/\partial x_{if}^2)g_{if}(x_i^{(0)}(t_f)), \qquad q_{fik} := M_{pkf},$$

$$Q_{ik} := [(\partial^2/\partial x_i^2)g_i(x_i^{(0)}, u_i^{(0)}) + p_{ii}^{(0)'}(\partial^2/\partial x_i^2)f_i(x_i^{(0)}, u_i^{(0)})],$$

$$R_{ik} := [(\partial^2/\partial u_i^2)g_i(x_i^{(0)}, u_i^{(0)}) + p_{ii}^{(0)'}(\partial^2/\partial u_i^2)f_i(x_i^{(0)}, u_i^{(0)})],$$

$$S_{ik} := [(\partial^2/\partial x_i\,\partial u_i)g_i(x_i^{(0)}, u_i^{(0)}) + p_{ii}^{(0)'}(\partial^2/\partial x_i\,\partial u_i)f_i(x_i^{(0)}, u_i^{(0)})],$$

$$\tilde{A}_{ik} := A_{ik} - B_{ik}R_{ik}^{-1}S'_{ik} - B_{ik}R_{ik}^{-1}B'_{ik}P_{ik}, \qquad q_{ik} := M_{pk}, \qquad r_{ik} := M_{uk}.$$

3.2. Closed-Loop Perfect State Information Structure. When the information structure is CLPS for both the players, there exist a plethora of (informationally nonunique) Nash equilibrium solutions as mentioned before. However, by restricting the class of admissible Nash equilibrium solutions to those of the feedback type (Ref. 1) or by requiring strong time consistency (Ref. 9), we can remove this informational nonuniqueness. A pair of strategies $(\gamma_1^* \in \Gamma_1, \gamma_2^* \in \Gamma_2)$ is in feedback Nash equilibrium, under the CLPS information structure, if there exist functions V_i: $[t_0, t_f] X \mathscr{R}^n \to \mathscr{R}$, $i=1,2$, satisfying the following coupled partial differential equations (Ref. 1):

$$\begin{aligned}
&-(\partial/\partial t) V_i(t, x_1, x_2) \\
&= \min_{u_i \in \mathscr{R}^{m_i}} [((\partial/\partial x_i) V_i)'(f_i(x_i, u_i) + \epsilon f_{ij}(x_j)) \\
&\qquad + ((\partial/\partial x_j) V_j)'(f_j(x_j, \gamma_j^*) + \epsilon f_{ji}(x_i)) \\
&\qquad + g_i(x_i, u_i) + \epsilon g_{ij}(x_i, \gamma_j^*(t, x))], \\
&V_i(t_f, x_1, x_2) = g_{if}(x_i) + \epsilon g_{ij}(x_j), \qquad i=1,2.
\end{aligned} \tag{27}$$

Since $f_i(.,.)$ and $g_i(.,.)$ were taken to be differentiable, we can write (27) in terms of the first-order necessary condition as

$$\begin{aligned}
-(\partial/\partial t) V_i(t, x_1, x_2) &= ((\partial/\partial x_i) V_i)'(f_i(x_i, \gamma_i^*) + \epsilon f_{ij}(x_j)) \\
&\quad + ((\partial/\partial x_j) V_j)'(f_j(x_j, \gamma_j^*) + \epsilon f_{ji}(x_i)) \\
&\quad + g_i(x_i, \gamma_i^*) + \epsilon g_{ij}(x_i, \gamma_j^*(t, x)), \\
V_i(t_f, x_1, x_2) &= g_{if}(x_1) + \epsilon g_{ij}(x_2), \qquad i=1,2,
\end{aligned} \tag{28a}$$

where $u_i^*(t) = \gamma_i^*(t, x)$ solves

$$\begin{aligned}
&(\partial/\partial u_i) f_i(x_i, u_i^*)(\partial/\partial x_i) V_i(t, x_1, x_2) \\
&+ (\partial/\partial u_i) g_i(x_i, u_i^*) = 0, \qquad i=1,2.
\end{aligned} \tag{28b}$$

As the counterpart of our assumption in the OL case, we will assume here that $\gamma_i^*(t, x; \epsilon)$ and $V_i(t, x_1, x_2; \epsilon)$, $i=1,2$, are expandable in ϵ as

$$\gamma_i^*(t, x; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \gamma_i^{(k)}(t, x), \quad V_i(t, x; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k V_i^{(k)}(t, x). \tag{29}$$

Using this, we can expand (28) in terms of ϵ, and retaining only the ϵ^0 terms yields

$$\begin{aligned}
-(\partial/\partial t) V_i^{(0)}(t, x_1, x_2) &= ((\partial/\partial x_i) V_i^{(0)})' f_i(x_i, \gamma_i^{(0)}(t, x)) \\
&\quad + ((\partial/\partial x_j) V_i^{(0)})' f_j(x_j, \gamma_j^{(0)}(t, x)) \\
&\quad + g_i(x_i, \gamma_i^{(0)}(t, x)), \\
V_i^{(0)}(t_f, x_1, x_2) &= g_{if}(x_i),
\end{aligned} \tag{30a}$$

where $u_i^{(0)}=\gamma_i^{(0)}(t,x)$ solves

$$(\partial/\partial u_i)f_i(x_i,u_i^{(0)})(\partial/\partial x_i)V_i^{(0)}$$
$$+(\partial/\partial u_i)g_i(x_i,u_i^{(0)})=0, \qquad i=1,2. \tag{30b}$$

We first have the following useful observation.

Lemma 3.1. $(\partial/\partial x_j)V_i^{(0)}=0,\ i\neq j.$

Proof. Equation (30a)–(30b) provides a set of necessary and sufficient conditions for solution of the optimal control problem defined by (24), which is the zeroth order equivalent problem for the OL case; in addition, the Hamilton–Jacobi–Bellman equation in (30) is driven by the following state equation:

$$\dot{x}_j=f_j(x_j,\gamma^{(0)}(t,x)). \tag{31}$$

Since x_j does not appear in the cost function (25b), and due to the assumption of strong-time consistency of the Nash equilibrium solution, the cost is independent of x_j. But $V_i^{(0)}(t,x)$ is nothing but the cost-to-go at time t; hence, it should be independent of x_2. In other words, $(\partial/\partial x)V_i^{(0)}=0,\ i\neq j.$ □

Making use of the above, we can rewrite (30) as

$$-(\partial/\partial t)V_i^{(0)}(t,x_1,x_2)=((\partial/\partial x_i)V_i^{(0)})'f_i(x_i,\gamma_i^{(0)}(t,x))$$
$$+g_i(x_i,\gamma_i^{(0)}(t,x)),$$
$$V_i^{(0)}(t_f,x_1,x_2)=g_{if}(x_i), \tag{32a}$$

where $u_i^{(0)}=\gamma_i^{(0)}(t,x)$ solves

$$(\partial/\partial u_i)f_i(x_i,u_i^{(0)})(\partial/\partial x_i)V_i^{(0)}+(\partial/\partial u_i)g_i(x_i,u_i^{(0)})=0. \tag{32b}$$

The coefficients of ϵ in (29) are given by

$$(\partial/\partial t)V^{(1)}(t,x_1,x_2)$$
$$=((\partial/\partial x_i)V_i^{(0)})'[(\partial/\partial u_i)f_i(x_i,\gamma_i^{(0)}(t,x))\gamma_i^{(1)}(t,x)]$$
$$+((\partial/\partial x_i)V_i^{(1)})'f_i(x_i,\gamma_i^{(0)}(t,x))+((\partial/\partial x_i)V_i^{(0)})'f_{ij}(x_j)$$
$$+((\partial/\partial x_j)V_i^{(1)})'f_j(x_j,\gamma_j^{(0)}(t,x))+(\partial/\partial u_i)g_i(x_i,\gamma_i^{(0)}(t,x))\gamma_i^{(1)}(t,x)$$
$$+g_{ij}(x_j,\gamma_j^{(0)}(t,x)),$$
$$V_i^{(1)}(t_f,x_1,x_2)=g_{ijf}(x_j), \tag{33}$$

and $u_i^{(1)} = \gamma_i^{(1)}(t, x)$ solves

$$(\partial/\partial u_i) f_i(x_i, \gamma_i^{(0)}(t, x))(\partial/\partial x_i) V_i^{(1)}$$
$$+ ((\partial/\partial x_i) V_i^{(0)})'(\partial^2/\partial u_i^2) f_i(x_i, \gamma_i^{(0)}(t, x)) u_i^{(1)}$$
$$+ (\partial^2/\partial u_i^2) g_i(x_i, \gamma_i^{(0)}(t, x)) u_i^{(1)} = 0, \qquad i = 1, 2. \tag{34}$$

Using (32b) in (33), we note that the terms containing $\gamma_i^{(1)}(t, x)$ drop out. Hence, (33) can be rewritten as

$$(\partial/\partial t) V_i^{(1)} = ((\partial/\partial x_i) V_i^{(1)})' f_i(x_i, \gamma_i^{(0)}(t, x)) + ((\partial/\partial x_i) V_i^{(0)})' f_{ij}(x_j)$$
$$+ ((\partial/\partial x_j) V_i^{(1)})' f_j(x_j, \gamma_j^{(0)}(t, x)) + g_{ij}(x_j, \gamma_j^{(0)}(t, x)),$$
$$V_i(t_f, x_1, x_2) = g_{ijf}(x_j). \tag{35}$$

Before we provide the general expression for the kth order terms, $k \geq 2$, let us introduce some notation parallel to the one in the previous subsection. Let $W(a, b(\epsilon))$ be a function which is infinitely many times differentiable in b, with a being independent of ϵ, and $b(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k b^{(k)}$. Then, the kth total derivative of $W(a, b(\epsilon))$, with respect to ϵ, at $\epsilon = 0$ is given by

$$(1/k!)(d^k/d\epsilon^k) W = (\partial/\partial b) W(a, b^{(0)}) b^{(k)} + R_3(W, k), \tag{36}$$

$$R_3(W, k) = \sum_{j=2}^{k} \sum_{\substack{l_1 + \cdots + l_j = k \\ l_1, \ldots, l_j \geq 1}} (1/j!)(d^j/db^j) W(a, b^{(0)}) b^{(l_1)} \cdots b^{(l_j)}, \tag{37}$$

where $(\partial^j/\partial b^j) W(a, b^{(0)})$ denotes the jth partial derivative operator, with respect to b, at the point $(a, b^{(0)})$.

For $k \geq 2$, the ϵ^k terms are given by

$$-(\partial/\partial t) V_i^{(k)}(t, x_1, x_2)$$
$$= ((\partial/\partial x_i) V_i^{(0)})'(\partial/\partial u_i) f_i(x_i, \gamma_i^{(0)}(t, x)) \gamma_i^{(k)}(t, x)$$
$$+ ((\partial/\partial x_i) V_i^{(k)})' f_i(x_i, \gamma_i^{(0)}(t, x)) + ((\partial/\partial x_j) V_i^{(k)})' f_j(x_j, \gamma_j^{(0)}(t, x))$$
$$+ (\partial/\partial u_i) g_i(x_i, \gamma_i^{(0)}(t, x)) \gamma_i^{(k)}(t, x) + M_{vk}(t, x),$$
$$V_i^{(k)}(t_f, x_1, x_2) = 0, \tag{38}$$

and $u_i^{(k)} = \gamma_i^{(k)}(t, x)$ solves

$$[((\partial^2/\partial u_i^2) f_i(x_i, \gamma_i^{(0)}(t, x)) u_i^{(k)})](\partial/\partial x_i) V_i^{0)}$$
$$+ (\partial^2/\partial u_i^2) g(x_i, \gamma_i^{(0)}(t, x)) u_i^{(k)} + M_{uk}^{CL}(t, x) = 0, \tag{39}$$

where

$$
\begin{aligned}
M_{vk}(t, x) = {} & R_3(f_i, k) \\
& + \sum_{l=1}^{k-1} ((\partial/\partial x_i) V_i^{(l)})'[1/(k-l)!](d^{k-l}/d\epsilon^{k-l}) f_i(x_i, \gamma_i^{(0)}(t, x)) \\
& + ((\partial/\partial x_i) V_i^{(k-1)})' f_{ij}(x_j) + \sum_{l=1}^{k-1} ((\partial/\partial x_j) V_i^{(l)})' \\
& \times [1/(k-l)!](d^{k-l}/d\epsilon^{k-l}) f_j(x_j, \gamma_j^{(0)}(t, x)) \\
& + R_3(g_i, k) + ((\partial/\partial x_j) V_i^{(k-1)})' f_{ji}(x_j) \\
& + [1/(k-1)!](d^{k-1}/d\epsilon^{k-1}) g_{ij}(x_j, \gamma_j^{(0)}(t, x)),
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
M_{uk}^{CL}(t, x) = {} & R_3((\partial/\partial u_i) f_i, k)(\partial/\partial x_i) V_i^{(0)} \\
& + \sum_{l=1}^{k} [1/(k-l)!](d^{k-l}/d\epsilon^{k-l})(\partial/\partial u_i) f_i(x_i^{(0)}, \gamma_i^{(0)}(t, x)) \\
& \times (\partial/\partial x_i) V_i^{(l)} + R_3((\partial/\partial u_i) g_i, k).
\end{aligned} \tag{41}
$$

Using (32b) in (38), we note that the terms containing γ_i^k drop out. Hence, (38) can be rewritten as

$$
\begin{aligned}
& -(\partial/\partial t) V_i^{(k)}(t, x_1, x_2) \\
& = ((\partial/\partial x_i) V_i^{(k)})' f_i(x_i, \gamma_i^{(0)}(t, x)) \\
& + ((\partial/\partial x_j) V_i^{(k)})' f_j(x_j, \gamma_j^{(0)}(t, x)) + M_{vk}(t, x), \\
& V_i^{(k)}(t_f, x_1, x_2) = 0.
\end{aligned} \tag{42}
$$

Unlike the OL case, we do not have equivalent optimal control problems at each iteration. As we mentioned in the proof of Lemma 3.1, the zeroth order equivalent problems are identical to the zeroth order equivalent problems of the OL case. But, for $k \geq 1$, the evaluation of $V_i^{(k)}$, $i = 1, 2$, corresponds to the evaluation of a cost function subject to a state equation (this does not involve any optimization), and the evaluation of $u_i^{(k)}(t)$ corresponds to the necessary condition for a static quadratic optimization problem. Identifying these equivalent problems for the CLPS case is important in the game context as opposed to the single player optimal control problem. This is because, as mentioned earlier, in the one-player case, existence of the higher-order terms in the CLPS strategy can be shown using the fact that the optimal state trajectory is the same, irrespective of the information pattern (Ref. 2), whereas this property does not hold in nonzero-sum differential games.

For $k=1$, the equivalent problems are given by

$$V_i^{(1)}(s, x_1, x_2)$$
$$=g_{ij}(\xi_j(t_f))+\int_s^{t_f}[((\partial/\partial\xi_i)V_i^{(0)}(s, \xi_1, \xi_2)')f_{ij}(\xi_j)$$
$$+g_{ij}(\xi_j, \gamma_j^{(0)}(t, \xi))]\,dt, \tag{43a}$$
$$\dot{\xi}_i=f_i(\xi_i, \gamma_i^{(0)}(t, \xi)), \qquad \xi_i(s)=x_i, \tag{43b}$$
$$\dot{\xi}_j=f_j(\xi_j, \gamma_j^{(0)}(t, \xi)), \qquad \xi_j(s)=x_j, \tag{43c}$$

and

$$\min_{u_i\in\mathscr{R}^{m_i}}\{(1/2)u_i'[((\partial/\partial x_i)V_i^{(0)})'(\partial^2/\partial u_i^2)f_i(x_i, \gamma_i^{(0)}(t, x))$$
$$+(\partial^2/\partial u_i^2)g_i(x_i, \gamma_i^{(0)}(t, x))]u_i$$
$$+u_i'(\partial/\partial u_i)f_i(x_i, \gamma_i^{(0)}(t, x))(\partial/\partial x_i)V_i^{(1)}\}. \tag{44}$$

For $k\geq 2$, the equivalent problems are given by

$$V_i^{(k)}(s, x_1, x_2)=\int_s^{t_f}M_{vk}(t, \xi)\,dt, \tag{45a}$$
$$\dot{\xi}_i=f_i(\xi_i, \gamma_i^{(0)}(t, \xi)), \qquad \xi_i(t_0)=x_i, \tag{45b}$$
$$\dot{\xi}_j=f_j(\xi_j, \gamma_j^{(0)}(t, \xi)), \qquad \xi_j(t_0)=x_j, \tag{45c}$$

and

$$\min_{u_i\in\mathscr{R}^{m_i}}\{(1/2)u_i'[((\partial/\partial x_i)V_i^{(0)})'(\partial^2/\partial u_i^2)f_i(x_i, \gamma_i^{(0)}(t, x))$$
$$+(\partial^2/\partial u_i^2)g_i(x_i, \gamma_i^{(0)}(t, x))]u_i+u_i'M_{uk}^{CL}(t, x)\}, \tag{46}$$

where $M_{uk}^{CL}(t, x)$, given by (41), depends on $V_i^{(k)}$. Again, we have decomposed the original game problem into a sequence of simpler problems. We can now summarize the computation of the feedback Nash equilibrium using the above decomposition procedure as follows:

Procedure 3.2.

Step 1. Use (30) to calculate $\gamma_1^{(0)}(t, x)$ and $\gamma_2^{(0)}(t, x)$. Set $k=1$.

Step 2. Calculate $\gamma_i^{(k)}(t, x)$, $i=1, 2$, using (34) and (35) if $k=1$, or using (39)–(42) if $k\geq 2$. Set $k=k+1$.

Step 3. If $O(\epsilon^{l+1})$ accuracy is required, stop at $k=l+1$. Otherwise, go back to Step 2 and iterate.

3.3. Mixed Information Structure. In this section, we study the situation where Player 1 has open loop information, while Player 2 has closed-loop perfect state information (OL-CLPS case). As in the CLPS case, it is well known that this problem admits multiple Nash equilibria. To alleviate this problem and ensure informational uniqueness, we require that the Nash solution be robust to the presence of asymptotically diminishing noise in the system equation (Ref. 11). A pair of strategies $\{\gamma_1^*, \gamma_2^*\}$ constitute a robust OL-CLPS Nash equilibrium if there exist costate vectors $\lambda_1(t)$, $\lambda_2(t)$, and a function $V(t, x_1, x_2)$ such that the following equations are satisfied, with $u_1^*(t) = \gamma_1^*(t, x_0)$:

$$\dot{x}_1 = f_1(x_1, u_1^*) + \epsilon f_{12}(x_1), \qquad x_1(t_0) = x_{10}, \tag{47a}$$

$$\dot{x}_2 = f_2(x_2, \gamma_2^*(t, x)) + \epsilon f_{21}(x_1), \qquad x_2(t_0) = x_{20}, \tag{47b}$$

$$\begin{aligned} \dot{\lambda}_1 = &-(\partial/\partial x_1)g_1(x_1, u_1^*) - (\partial/\partial x_1)f_1(x_1, u_1^*)\lambda_1 \\ &-(\partial/\partial u_2)f_2(x_2, \gamma_2^*(t, x))(\partial/\partial x_1)\gamma_2^*(t, x)\lambda_2 - \epsilon(d/dx_1)f_{21}(x_1)\lambda_2, \end{aligned}$$

$$\lambda_1(t_f) = (d/dx_1)g_{1f}(x_1(t_f)), \tag{48a}$$

$$\begin{aligned} \dot{\lambda}_2 = &-\epsilon(\partial/\partial x_2)g_{12}(x_2, \gamma_2^*) - (\partial/\partial x_2)f_2(x_2, \gamma_2^*)\lambda_2 \\ &-(\partial/\partial u_2)f_2(x_2, \gamma_2^*)(\partial/\partial x_2)\gamma_2^*(t, x)\lambda_2 - \epsilon(d/dx_2)f_{12}(x_2)\lambda_1, \end{aligned}$$

$$\lambda_2(t_f) = \epsilon(d/dx_2)g_{12f}(x_2(t_f)), \tag{48b}$$

$$(\partial/\partial u_1)g_1(x_1, u_1^*) + (\partial/\partial u_1)f_1(x_1, u_1^*)\lambda_1 = 0, \tag{49}$$

$$\begin{aligned} &-(\partial/\partial t)V(t, x_1, x_2) \\ &= ((\partial/\partial x_1)V)'(f_1(x_1, u_1^*) + \epsilon f_{12}(x_2)) \\ &+ ((\partial/\partial x_2)V)'(f_2(x_2, \gamma_2^*(t, x)) + \epsilon f_{21}(x_1)) \\ &+ g_2(x_2, \gamma_2^*(t, x)) + \epsilon g_{21}(x_1, u_1^*), \end{aligned}$$

$$V(t_f, x_1, x_2) = g_{2f}(x_2) + \epsilon g_{21f}(x_1), \tag{50}$$

$$[(\partial/\partial u_2)f_2(x_2, u_2^*)](\partial/\partial x_2)V + (\partial/\partial u_2)g_2(x_2, u_2^*) = 0. \tag{51}$$

In the above expressions, unlike the case when both players have closed-loop information, $\gamma_2^*(t, x)$ explicitly depends on the initial state x_0, although we have suppressed this dependence in the notation.

Now, as in the case of the other two information structures that we previously considered, we will assume that the optimal strategies are expandable in an infinite series in ϵ. Using this in (47)–(49), and comparing the ϵ^0

terms yields the zeroth order solution for Player 1, which is given by

$$\dot{x}_1^{(0)} = f_1(x_1^{(0)}, u_1^{(0)}), \qquad x_1^{(0)}(t_0) = x_{10}, \tag{52a}$$

$$\dot{x}_2^{(0)} = f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)})), \qquad x_2^{(0)}(t_0) = x_{20}, \tag{52b}$$

$$\begin{aligned} \dot{\lambda}_1^{(0)} &= -(\partial/\partial x_1)g_1(x_1^{(0)}, u_1^{(0)}) - (\partial/\partial x_1)f_1(x_1^{(0)}, u_1^{(0)})\lambda_1^{(0)} \\ &\quad - (\partial/\partial u_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))\lambda_2^{(0)}, \\ \lambda_1^{(0)}(t_f) &= (d/dx_1)g_{1f}(x_1^{(0)}(t_f)), \end{aligned} \tag{53}$$

$$\begin{aligned} \dot{\lambda}_2^{(0)} &= -(\partial/\partial x_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))\lambda_2^{(0)} \\ &\quad - (\partial/\partial u_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))(\partial/\partial x_2)\gamma_2^{(0)}(t, x^{(0)})\lambda_2^{(0)}, \\ \lambda_2^{(0)}(t_f) &= 0, \end{aligned} \tag{54}$$

$$(\partial/\partial u_1)g_1(x_1^{(0)}, u_1^{(0)}) + (\partial/\partial u_1)f_1(x_1^{(0)}, u_1^{(0)})\lambda_1^{(0)} = 0. \tag{55}$$

Notice that the unique solution to (54) is $\lambda_2^{(0)}(t) \equiv 0$. Using this in (53), it follows that the problem of computing $u_1^{(0)}(t)$ is equivalent to the following one-player optimal control problem:

$$u_1^{(0)}(t) = \arg\min_{u_1} \left\{ g_{1f}(x_1^{(0)}(t_f)) + \int_{t_0}^{t_f} g_1(x_1^{(0)}, u_1)\, dt \right\}, \tag{56a}$$

$$\dot{x}_1^{(0)} = f_1(x_1^{(0)}, u_1), \qquad x_1^{(0)}(t_0) = x_{10}. \tag{56b}$$

Comparing the coefficients of the ϵ^0 terms in (50) and (51) yields the following pair of equations to be satisfied by the zeroth order solution of Player 2:

$$\begin{aligned} -(\partial/\partial t)V^{(0)} &= ((\partial/\partial x_1)V^{(0)})' f_1(x_1, u_1^{(0)}) \\ &\quad + ((\partial/\partial x_2)V^{(0)})' f_2(x_2, \gamma_2^{(0)}(t, x)) + g_2(x_2, \gamma_2^{(0)}(t, x)), \\ V^{(0)}(t_f, x_1, x_2) &= g_{2f}(x_2), \end{aligned} \tag{57}$$

$$(\partial/\partial u_2)f_2(x_2, u_2^{(0)})(\partial/\partial x_2)V^{(0)} + (\partial/\partial u_2)g_2(x_2, u_2^{(0)}) = 0. \tag{58}$$

The equivalent problem for obtaining $\gamma_2^{(0)}(t, x)$ is given by

$$\gamma_2^{(0)}(t, x) = \arg\min_{\gamma_2 \in \Gamma_2} J(\gamma_2(t, x)), \tag{59a}$$

$$\dot{x}_2 = f_2(x_2, u_2), \qquad x_2(t_0) = x_{20}, \tag{59b}$$

$$J(u_2) = g_{2f}(x_2(t_f)) + \int_{t_0}^{t_f} g_2(x_2, u_2)\, dt. \tag{59c}$$

Comparing the ϵ terms in (47)–(49) yields the following equations for the computation of $u_1^{(1)}(t)$:

$$\dot{x}_1^{(1)} = (\partial/\partial x_1)f_1(x_1^{(0)}, u_1^{(0)})x_1^{(1)} + (\partial/\partial u_1)f_1(x_1^{(0)}, u_1^{(0)})u_1^{(1)} + f_{12}(x_2^{(0)}),$$

$$x_1^{(1)}(t_0) = 0, \tag{60}$$

$$\begin{aligned}\dot{\lambda}_1^{(1)} = &-(\partial^2/\partial x_1^2)g_1(x_1^{(0)}, u_1^{(0)})x_1^{(1)} - (\partial^2/\partial u_1\,\partial x_1)g_1(x_1^{(0)}, u_1^{(0)})u_1^{(1)}\\ &-[(\partial^2/\partial x_1^2)f_1(x_1^{(0)}, u_1^{(0)})x_1^{(1)} + (\partial^2/\partial u_1\,\partial x_1)f_1(x_1^{(0)}, u_1^{(0)})u_1^{(1)}]\lambda_1^{(0)}\\ &-(\partial/\partial x_1)f_1(x_1^{(0)}, u_1^{(0)})\lambda_1^{(1)} - (\partial/\partial u_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))\\ &\times(\partial/\partial x_1)\gamma_2^{(0)}(t, x^{(0)})\lambda_2^{(1)},\end{aligned}$$

$$\lambda_1^{(1)}(t_f) = (\partial^2/\partial x_1^2)g_{1f}(x_1^{(0)}(t_f))x_1^{(1)}(t_f), \tag{61}$$

$$\begin{aligned}\dot{\lambda}_2^{(1)} = &-(\partial/\partial x_2)g_{12}(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)})) - (d/dx_2)f_{12}(x_2^{(0)})\lambda_1^{(0)}\\ &-(\partial/\partial x_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))\lambda_2^{(1)}\\ &-(\partial/\partial u_2)f_2(x_2^{(0)}, \gamma_2^{(0)}(t, x^{(0)}))(\partial/\partial x_2)\gamma_2^{(0)}(t, x^{(0)})\lambda_2^{(1)},\end{aligned}$$

$$\lambda_2^{(1)}(t_f) = (d/dx_2)g_{1f}(x_2^{(0)}(t_f)). \tag{62}$$

If we make use of the fact that $(\partial/\partial x_1)\gamma_2^{(0)}(t, x) = 0$, then the $u_1^{(1)}(t)$ obtained above is identical to the first-order policy of Player 1 in the OL case. This is to be expected because of two reasons. One is that, in both cases, the first-order policy of Player 1 depends only on the zeroth order term of the open-loop representation of the zeroth order policy of Player 2. Secondly, the open-loop representations of Player 2's zeroth order policy are identical in both the OL case and the OL-CLPS case, since both are different representations of the solution of the same one-player optimal control problem.

Now, let us study the first-order policy of Player 2. The associated equations are:

$$\begin{aligned}-(\partial/\partial t)V^{(1)} = &((\partial/\partial x_1)V^{(1)})'f_1(x_1, u_1^{(0)})\\ &+((\partial/\partial t)V^{(0)})'((\partial/\partial u_1)f_1(x_1, u_1^{(0)})u_1^{(1)} + f_{12}(x_2))\\ &+((\partial/\partial x_2)V^{(1)})'f_2(x_2, \gamma_2^{(0)}(t, x)) + ((\partial/\partial x_2)V^{(0)})'\\ &\times((\partial/\partial u_2)f_2(x_2, \gamma_2^{(0)}(t, x))\gamma_2^{(1)}(t, x) + f_{21}(x_1))\\ &+(\partial/\partial u_2)g_2(x_2, \gamma_2^{(0)}(t, x))\gamma_2^{(1)}(t, x) + g_{21}(x_1, u_1^{(0)}),\end{aligned}$$

$$V^{(1)}(t_f, x_1, x_2) = g_{21f}(x_1), \tag{63}$$

$$\begin{aligned}&(\partial^2/\partial u_2^2)f_2(x_2, \gamma_2^{(0)}(t, x))u_2^{(1)}(\partial/\partial x_2)V^{(0)}\\ &+(\partial/\partial u_2)f_2(x_2, \gamma_2^{(0)}(t, x))(\partial/\partial x_2)V^{(1)}\\ &+(\partial^2/\partial u_2^2)g_2(x_2, \gamma_2^{(0)}(t, x_2))u_2^{(1)} = 0.\end{aligned} \tag{64}$$

Making use of (58) and the fact that $(\partial/\partial x_1)V^{(0)}=0$, (63) can be rewritten as

$$-(\partial/\partial t)V^{(1)}=((\partial/\partial x_1)V^{(1)})'f_1(x_1, u_1^{(0)})$$
$$+((\partial/\partial x_2)V^{(1)})'f_2(x_2, \gamma_2^{(0)}(t, x))$$
$$+((\partial/\partial x_2)V^{(0)})'f_{21}(x_1)+g_{21}(x_1, u_1^{(0)}),$$

$$V^{(1)}(t_f, x_1, x_2)=g_{21f}(x_1). \tag{65}$$

The problem of computing $V^{(1)}(t, x_1, x_2)$, using (65), can be viewed as the computation of a cost functional subject to a state equation constraint, as follows:

$$\dot{x}_1=f_1(x_1, u_1^{(0)}), \qquad x_1(t_0)=x_{10}, \tag{66a}$$

$$\dot{x}_2=f_2(x_2, \gamma_2^{(0)}(t, x)), \qquad x_2(t_0)=x_{20}, \tag{66b}$$

$$J=g_{21f}(x_1)+\int_{t_0}^{t_f}[((\partial/\partial x_2)V^{(0)})'f_{21}(x_1)+g_{21}(x_1, u_1^{(0)})]\,dt. \tag{66c}$$

Notice that this is not the same as the first-order equivalent problem in the CLPS case, because the zeroth order strategy for Player 1 depends on OL information here, which makes the state equation for the equivalent problem different from the CLPS case. The computation of $\gamma_2^{(1)}(t, x)$, from (65), is equivalent to the following static problem:

$$\min_{u_2}\{u_2[(\partial^2/\partial u_2^2)g_2(x_2, \gamma_2^{(0)}(t, x))+((\partial/\partial x_2)V^{(0)})'(\partial^2/\partial u_2^2)f_2(x_2,\gamma_2^{(0)}(t, x))]u_2$$
$$+((\partial/\partial x_2)V^{(1)})'(\partial/\partial u_2)f_2(x_2, \gamma_2^{(0)}(t, x))u_2\}. \tag{67}$$

The higher-order terms in the expansion of the optimal strategies can be obtained in a similar fashion, but the expressions are very lengthy, and hence, will not be provided here. We note, however, that the computation of these higher-order terms for Player 1 can equivalently be viewed as the solution of an LQ optimization problem; for Player 2, it is equivalent to the evaluation of a cost functional subject to a state equation, together with a static optimization problem.

3.4. Policy Iteration. In this section, we show that the asymptotic expansion of the Nash equilibrium strategies can be interpreted as a policy iteration algorithm. Let us first consider a policy iteration of the Cournot type (Ref. 1):

$$\gamma_{1(k+1)} = \arg\min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_{2(k)}), \tag{68a}$$

$$\gamma_{2(k+1)} = \arg\min_{\gamma_2 \in \Gamma_2} J_2(\gamma_{1(k)}, \gamma_2), \tag{68b}$$

where $k=0, 1, \ldots$, $\gamma_{1(0)} \in \Gamma_1$, $\gamma_{2(0)} \in \Gamma_2$ are specified, and Γ_i is chosen to be compatible with the given information pattern. In what follows, we show that, after k-steps of the Cournot iteration, the strategies $\gamma_{1(k)}$ and $\gamma_{2(k)}$ are $O(\epsilon^k)$ close to the Nash equilibrium strategies under all three information patterns.

To show this, let us first consider the OL case. Clearly, the pair $(u_{1(0)}(t), u_{2(0)}(t))$ is $O(\epsilon^0)$ close to the Nash equilibrium solution pair $(u_1^*(t), u_2^*(t))$. Now, we shall proceed by induction. Assume that the pair $(u_{1(k)}(t), u_{2(k)}(t))$ is $O(\epsilon^k)$ close to the Nash equilibrium solution, i.e.,

$$u_{i(k)}(t) = \sum_{j=0}^{k-1} \epsilon^j u_i^{(j)}(t) + O(\epsilon^k), \qquad i=1, 2. \tag{69}$$

To obtain $u_{1(k+1)}(t)$, using the Cournot iteration, we fix $u_2(t) = u_{2(k)}(t)$, and minimize $J_1(u_1(t), u_2(t))$ with respect to $u_1(t)$. The necessary conditions for this minimization are given by (4), (6), (8), with $i=1$ and $u_2^*(t)$ replaced by $u_{2(k)}(t)$. The $(k+1)$th order term in the expansion of $u_{1(k+1)}(t)$ is given by (16)–(19), where k is replaced by $k+1$. Upon examination of (16)–(19), it is clear that the $(k+1)$th order term of $u_{1(k+1)}(t)$ depends on $u_2^{(j)}(t)$, $j \leq k$, and does not depend on $u_2^{(j)}(t)$, $j > k$. Therefore, from the induction hypothesis whereby

$$u_{2(k)}(t) = u_2^*(t) + O(\epsilon^k),$$

we have that

$$u_{1(k+1)}(t) = u_1^*(t) + O(\epsilon^{k+1}).$$

Similarly, one can show that

$$u_{2(k+1)}(t) = u_2^*(t) + O(\epsilon^{k+1}).$$

A similar argument shows that

$$\gamma_{i(k)} = \gamma_i^* + O(\epsilon^k),$$

even under the CLPS or OL-CLPS information patterns, where the DM's use dynamic information, because the asymptotic expansions in the previous two sections were conducted in the policy space (and not on the open-loop representations of the policies).

Now, we can state the following theorem.

Theorem 3.1. Suppose that the Nash equilibrium strategies, under OL, CLPS, or OL-CLPS information, are expandable as a power series in ϵ. Then, after k steps of the Cournot iteration,

$$\gamma_{i(k)} = \gamma_i^* + O(\epsilon^k),\ i = 1, 2.$$

Proof. See the discussion before the theorem. □

Suppose that, instead of the Cournot iteration, we use the Gauss–Seidel iteration (Ref. 1), given by

$$\gamma_{1(k+1)} = \arg\min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_{2(k)}), \tag{70a}$$

$$\gamma_{2(k+1)} = \arg\min_{\gamma_2 \in \Gamma_2} J_2(\gamma_{1(k+1)}, \gamma_2), \tag{70b}$$

where $k = 0, 1, \ldots, \gamma_{2(0)} \in \Gamma_2$ is specified. Then, we have the following theorem.

Theorem 3.2. Suppose that the Nash equilibrium strategies, under OL, CLPS, or OL-CLPS information, are expandable as a power series in ϵ. Denoting the strategies generically by (γ_1^*, γ_2^*), we have after k steps of the Gauss–Seidel iteration,

$$\gamma_{1(k)} = \gamma_1^* + O(\epsilon^{2k-1}), \qquad \gamma_{2(k)} = \gamma_2^* + O(\epsilon^{2k}).$$

Proof. The proof is similar to that of Theorem 3.1, with a minor modification. When $k = 1$, since $\gamma_{1(1)} = \gamma_1^* + O(\epsilon)$, Player 2's strategy is $\gamma_{2(1)} = \gamma_2^* + O(\epsilon^2)$. Now, by induction, one can obtain the desired result. □

The conclusions of Theorems 3.1 and 3.2 are important, because they establish the strategic stability of the asymptotic solution. As we mentioned at the beginning of Section 2, these results can readily be extended to the multiple-player case. There is one caveat, however, in the case of Theorem 3.2. When there are more than two players, it should be noted that only the player acting last, at each step of the Gauss–Seidel iteration, obtains an $O(\epsilon^{k+1})$ approximation to the actual solution, while the rest of the players obtain an $O(\epsilon^k)$ approximation to the actual solution. In other words, in the multiplayer (more than two players) case, the Gauss–Seidel iteration does not perform better than the Cournot iteration.

4. Existence, Uniqueness, and Convergence of the Solution

In this previous section, we showed that, under the assumption of asymptotic expandability of the Nash equilibrium solution, the original problem can be decomposed into a sequence of simpler, equivalent problems under all three information structures. In this section, we prove that the solution to each of these equivalent problems exists, and is unique. Further, we will show that the pair of strategies $\{\gamma_{1k}^*, \gamma_{2k}^*\}$, where

$$\gamma_{ik}^* = \sum_{l=0}^{k} \gamma_i^{(l)} \epsilon^l, \qquad i=1,2,\ k=1,2,\ldots,$$

are in an $O(\epsilon^{2k+2})$ Nash equilibrium.

We first make precise the notion of an $O(\epsilon^n)$ Nash equilibrium.

Definition 4.1. A pair of strategies $\{\gamma_1^*, \gamma_2^*\}$ constitutes an $O(\epsilon^n)$ Nash equilibrium if they satisfy the following pair of inequalities for all $\gamma_i \in \Gamma_i$, $i=1,2$:

$$J_1(\gamma_1^*, \gamma_2^*) \le J_1(\gamma_1, \gamma_2^*) + O(\epsilon^n),$$

$$J_2(\gamma_1^*, \gamma_2^*) \le J_2(\gamma_1^*, \gamma_2) + O(\epsilon^n).$$

The above definition reflects the fact that, for small values of ϵ, neither player has a significant incentive to deviate from the $O(\epsilon^n)$ equilibrium solution.

4.1. Open-Loop Information Structure. Before stating the main theorem of this subsection, we introduce the following conditions:

(A1) The zeroth order optimal control problems (24) admit unique continuous solutions.

Precise conditions for this can be found in texts on optimal control; see, e.g., Ref. 2.

(A2) The following inequalities hold:

$$H_{uu} > 0, \qquad H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} \ge 0, \qquad (\partial^2/\partial x_i^2) g_i(x(t_f)) \ge 0,$$

where

$$H_{uu} := (\partial^2/\partial u_i^2) g_i(x_i, u_i) + p_{ii}^{(0)'} (\partial^2/\partial u_i^2) f_i(x_i, u_i),$$

$$H_{xx} := (\partial^2/\partial x_i^2) g_i(x_i, u_i) + p_{ii}^{(0)'} (\partial^2/\partial x_i^2) f_i(x_i, u_i),$$

$$H_{ux} := (\partial^2/\partial u_i\, \partial x_i) g_i(x_i, u_i) + p_{ii}^{(0)'} (\partial^2/\partial u_i\, \partial x_i) f_i(x_i, u_i),$$

$$H_{xu} := (\partial^2/\partial x_i\, \partial u_i) g_i(x_i, u_i) + p_{ii}^{(0)'} (\partial^2/\partial x_i\, \partial u_i) f_i(x_i, u_i).$$

Theorem 4.1. Under Assumptions (A1)–(A2), there exists a unique solution to the kth order equivalent problem (25), $k=1,2,\ldots$. Further, if

Procedure 3.1 leads to the series

$$u_{ik}^*(t)=\sum_{l=0}^{k}\epsilon^l u_i^{(l)}(t),$$

then the open-loop policies $\{u_{1k}^*(t), u_{2k}^*(t)\}$ are in an $O(\epsilon^{2k+2})$ equilibrium.

Proof. For $k\geq 1$, by Assumption (A2), the cost for the equivalent problem, given by (25b), is convex in (x_i, u_i) and is strictly convex in u_i. Hence, being linear-quadratic, the optimal control problem defined by (25) admits a unique continuous solution, provided that $M_{pk}(t)$ and $M_{uk}(t)$ are bounded and continuous for every k. To show the latter, note that $x_i^{(0)}(t)$, $p_i^{(0)}(t)$, $u_i^{(0)}(t)$, $i=1,2$, are continuous. Now, let us assume that $x_i^{(l)}(t)$, $p_i^{(l)}(t)$, $u_i^{(l)}(t)$, $i=1,2$, are all continuous for $l\leq k-1$. Then, being polynomial functions of $x_i^{(l)}(t)$, $p_i^{(l)}(t)$, $u_i^{(l)}(t)$, $i=1,2$, clearly $M_{uk}(t)$ and $M_{pk}(t)$ are also continuous and thereby bounded in the closed interval $[t_0, t_f]$. This establishes the existence and uniqueness of the solutions to the decomposed problems. To prove the remaining part of the theorem, we consider the following optimization problems:

$$\inf_{u_1} J_1(u_1, u_{2k}), \qquad \inf_{u_2} J_2(u_{1k}, u_2),$$

and simply note that the following relationships follow from Ref. 2, Chapter 3, Theorem 2.1:

$$J_1(u_{1k}^*, u_{2k}^*)=\inf_{u_1} J_1(u_1, u_{2k}^*)+O(\epsilon^{2k+2}),$$

$$J_2(u_{1k}^*, u_{2k}^*)=\inf_{u_2} J_2(u_{1k}^*, u_2)+O(\epsilon^{2k+2}).$$

□

We now specialize the above result to the class of linear-quadratic (LQ) differential games. Consider the following system equation and cost function J_i, for Player i, $i=1,2$:

$$\dot{x}_1=A_1x_1+\epsilon A_{12}x_2+B_1u_1, \tag{71a}$$

$$\dot{x}_2=A_2x_2+\epsilon A_{21}x_1+B_2u_2, \tag{71b}$$

$$J_i(u_i, u_j)=(1/2)x'(t_f)Q_f^ix(t_f)+(1/2)\int_{t_0}^{t_f}(x'Q^ix+u'R^iu)\,dt, \tag{71c}$$

where

$$x:=(x_1', x_2')',$$

$$R^1:=\text{block diag}\{R_{11}^1, \epsilon R_{22}^1\},$$

$$Q_f^1 := \text{block diag}\{Q_{11f}^1, \epsilon Q_{22f}^1\} \geq 0,$$

$$Q^1 := \text{block diag}\{Q_{11}^1, \epsilon Q_{22}^1\} \geq 0,$$

$$Q_f^2 := \text{block diag}\{\epsilon Q_{11f}^2, Q_{22f}^2\} \geq 0,$$

$$Q^2 := \text{block diag}\{\epsilon Q_{11}^2, Q_{22}^2\} \geq 0,$$

$$R^2 := \text{block diag}\{\epsilon R_{11}^2, R_{22}^2\}.$$

Further, assume that $R_{ii}^i > 0$, $i = 1, 2$.

Now, assume that there exists a unique solution set $P^i(t; \epsilon)$, $i = 1, 2$, to the coupled matrix Riccati differential equations

$$\dot{P}^i + P^i A + A P^i + Q^i - P^i \sum_{j=1,2} B^j (R_{jj}^j)^{-1} B^{j'} P^j = 0,$$

$$P^i(t_f) = Q_f^i, \tag{72}$$

where

$$A = \begin{bmatrix} A_1 & \epsilon A_{12} \\ \epsilon A_{21} & A_2 \end{bmatrix}, \qquad B^1 = [B_1', 0]', \qquad B^2 = [0, B_2']'. \tag{73}$$

Then, from Ref. 1, the LQ differential game admits a unique Nash equilibrium solution given by

$$u_i^*(t) = -[(R_{ii}^i(t))^{-1} B^{i'}(t) P^i(t; \epsilon)] x^*(t), \qquad i = 1, 2, \tag{74a}$$

$$x^*(t) = \Phi(t, t_0) x(t_0), \tag{74b}$$

$$(d/dt)\Phi(t, t_0)$$

$$= \left(A(t) - \sum_{i=1,2} B^i(t) (R_{ii}^i(t))^{-1} B^{i'}(t) P^i(t; \epsilon) x^*(t) \right) \Phi(t, t_0),$$

$$\Phi(t_0, t_0) = I. \tag{74c}$$

Suppose that P^i, $i = 1, 2$, admits an expansion in terms of ϵ as

$$P^i(t; \epsilon) = \sum_{l=0}^{\infty} \epsilon^l P^{i(l)}(t);$$

then,

$$P^{1(0)}(t) = \begin{bmatrix} P_{11}^{1(0)} & 0 \\ 0 & 0 \end{bmatrix}, \qquad P^{2(0)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & P_{22}^{2(0)}(t) \end{bmatrix}, \tag{75a}$$

$$\dot{P}_{ii}^{i(0)} + A_i' P_{ii}^{i(0)} + P_{ii}^{i(0)} A_i - P_{ii}^{i(0)} B_i' (R_{ii}^i)^{-1} B_i P_{ii}^{i(0)} + Q_{ii}^i = 0,$$

$$P_{ii}^{i(0)}(t_f) = Q_{iif}^i. \tag{75b}$$

Note that the above equation is the Riccati equation associated with the control problem obtained by setting $\epsilon = 0$. For $k \geq 1$, we have

$$\begin{aligned} &\dot{P}^{i(k)} + P^{i(k)}A^{(0)} + A^{(0)\prime}P^{i(k)} + A^{(1)\prime}P^{i(k-1)} + P^{i(k-1)}A^{(1)} \\ &\quad + Q^{(k)} - \sum_{l=0}^{k} P^{i(l)} \sum_{j=1,2} B^j (R^j_{jj})^{-1} B^{j\prime} P^{i(k-l)} = 0, \\ &P^{i(k)}(t_f) = Q_f^{i(k)}, \end{aligned} \tag{76}$$

where

$$\begin{aligned} &A^{(0)} = \operatorname{diag}\{A_1, A_2\}, && \epsilon A^{(1)} = A - A^{(0)}, \\ &Q_f^{1(0)} = \operatorname{diag}\{Q^1_{11f}, 0\}, && Q_f^{2(0)} = \operatorname{diag}\{0, Q^2_{22f}\}, \\ &\epsilon Q_f^{i(1)} = Q^i_f - Q_f^{i(0)}, && Q_f^{i(k)} = 0, \quad k \geq 2,\ i = 1, 2. \end{aligned}$$

Note that the higher-order terms given by (76) are linear equations, whereas the higher-order terms obtained using Procedure 3.1, specifically (26b), are Riccati equations. This apparent discrepancy can be explained by the fact that, in obtaining (76), we modified Procedure 3.1 whereby the control values of the DM's and the associated costate vectors were expanded in powers of ϵ, but the state vector was not. This modification is convenient in the case of LQ games as it leads to linear differential equations instead of Riccati equations. Now, we state the following theorem for the LQ differential game.

Theorem 4.2. There exists an $\epsilon_0 > 0$ such that the coupled set of Riccati equations (72) admits a unique solution for all $\epsilon \in [-\epsilon_0, \epsilon_0]$, and this solution is infinitely many times continuously differentiable in ϵ, at $\epsilon = 0$. Further, the pair of OL strategies $\{u_{1k}(t), u_{2k}(t)\}$, where $u_{ik}(t)$ is given by (74a), with $P^i(t; \epsilon)$ replaced by $\sum_{l=0}^{k} \epsilon^l P^{i(l)}(t)$, provides an $O(\epsilon^{2k+2})$ OL Nash equilibrium for the LQ differential game described by (71).

Proof. The first part of the theorem follows directly by applying the implicit function theorem stated in the Appendix to (72), and the rest follows from Theorem 4.1. □

4.2. Closed-Loop Perfect State Information Structure. In this subsection, we obtain the counterparts of Theorems 4.1 and 4.2 for the CLPS information structure. We again assume the validity of (A1) and (A2) stated in the previous subsection. Now we state the following theorem which is the counterpart of Theorem 4.1.

Theorem 4.3. Suppose that the strongly time consistent (feedback) Nash equilibrium solution $\gamma_i^*(t, x)$, $i = 1, 2$, is expandable in ϵ as

$\sum_{l=0}^{k} \gamma_i^{(l)}(t, x)\epsilon^l + O(\epsilon^{k+1})$. Then, $\gamma_i^{(l)}(t, x)$, $i=1, 2$, $0 \le l \le k$, are unique, under Assumptions (A1)–(A2), provided that the zeroth order strategies $\gamma_i^{(0)}(\cdot, \cdot)$, $i=1, 2$, have continuous first partial derivatives. Further, the pair of strategies $\{\gamma_{1k}^*(t, x), \gamma_{2k}^*(t, x)\}$, where

$$\gamma_{ik}^*(t, x) = \sum_{l=0}^{k} \gamma_i^{(l)}(t, x)\epsilon^l,$$

is in an $O(\epsilon^{2k+2})$ Nash equilibrium.

Proof. From (A1), we have that the zeroth order problem (which is the same as in the OL case, except that the class of permissible strategies have CLPS information) admits a unique solution. Further, by the hypothesis of the theorem, $\gamma_i^{(0)}(t, x)$ and $V_i^{(0)}(t, x)$ have continuous first and second partial derivatives. Suppose that $\gamma_i^{(l)}(t, x)$ and $V_i^{(l)}(t, x)$ have continuous first and second partial derivatives, for $l \le m$. Then, the equivalent problem defined by (45) corresponds to the partial differential equation (38), since $M_{vk}(t, x)$ admits continuous first partials in x and t. But clearly the evaluation of the cost subject to a state equation yields a unique function, which completes the uniqueness part of the theorem. To prove the remaining part of the theorem, we introduce the following optimization problems:

$$\inf_{\gamma_1 \in \Gamma_1} J_1(\gamma_1(t, x), \gamma_{2k}^*(t, x)), \qquad \inf_{\gamma_2 \in \Gamma_2} J_2(\gamma_{1k}^*(t, x), \gamma_2(t, x)).$$

Then, we simply note, from Ref. 2, Chapter 3, Theorem 5.1, the order relationships

$$J_1(\gamma_{1k}^*(t, x), \gamma_{2k}^*(t, x)) = \inf_{\gamma_1 \in \Gamma_1} J_1(\gamma_1(t, x), \gamma_{2k}^*(t, x)) + O(\epsilon^{2k+2}),$$

$$J_2(\gamma_{1k}^*(t, x), \gamma_{2k}^*(t, x)) = \inf_{\gamma_2 \in \Gamma_2} J_2(\gamma_{1k}^*(t, x), \gamma_2(t, x)) + O(\epsilon^{2k+2}),$$

which completes the proof. □

A a pecial case, let us consider the LQ differential game defined by (71). Then, from Ref. 1, we know that, if there exists a solution to the coupled matrix Riccati equations

$$\dot{Z}^i + Z^i\tilde{F} + \tilde{F}'Z^i + \epsilon \sum_{j=1,2} Z^j B^j (R_{jj}^j)^{-1} R_{ij}^i (R_{jj}^j)^{-1} B^{j'} Z^j + Q^i = 0,$$

$$Z^i(t_f) = Q_f^i, \tag{77}$$

where

$$\tilde{F}(t) := A(t) - \sum_{j=1,2} B^j (R_{jj}^j)^{-1} B^{j'} Z^j, \tag{78}$$

then under the CLPS information pattern there exists a Nash equilibrium solution in feedback strategies given by

$$\gamma_i^*(t, x) = -(R_{ii}^i)^{-1} B^i(t)' Z^i(t) x(t), \qquad i = 1, 2. \tag{79}$$

Procedure 3.2 for computing approximate solutions now reduces to computing approximate solutions to the coupled Riccati equations (78). Suppose that

$$Z^i(t) = \sum_{l=0}^{k} Z^{i(l)}(t)\epsilon^l + O(\epsilon^{k+1}).$$

Then, we have the following equations for computing $Z^{i(k)}(t)$, $k \geq 0$:

For $k = 0$,

$$Z^{1(0)}(t) = \begin{bmatrix} Z_{11}^{1(0)} & 0 \\ 0 & 0 \end{bmatrix}, \qquad Z^{2(0)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & Z_{22}^{2(0)} \end{bmatrix}, \tag{80a}$$

$$\dot{Z}_{ii}^{i(0)} + A_i' Z_{ii}^{i(0)} + Z_{ii}^{i(0)} A_i - Z_{ii}^{i(0)} B_i (R_{ii}^i)^{-1} B_i' Z_{ii}^{i(0)} + Q_{ii}^i = 0,$$

$$Z_{ii}^{i(0)}(t_f) = Q_{iif}^i; \tag{80b}$$

for $k \geq 1$,

$$Z^{i(k)} + Z^{i(k)} \tilde{F}^{(0)} + \tilde{F}^{(0)'} Z^{i(k)} + \sum_{l=0}^{k-1} Z^{i(l)} \tilde{F}^{(k-l)} + \sum_{l=0}^{k-1} \tilde{F}^{(k-l)'} Z^{i(l)}$$

$$+ \sum_{l=0}^{k-1} \sum_{j=1,2} Z^{j(l)} B^j (R_{jj}^j)^{-1} R_{jj}^i (R_{jj}^j)^{-1} B^{j'} Z^{j(k-l-1)} + Q^{i(k)} = 0,$$

$$Z^{i(k)}(t_f) = 0. \tag{81}$$

The above set of equations was first obtained in Ref. 8, but we have derived it here directly by using Procedure 3.2 without the need for the solution of the perturbed problem. To show this, note that, for $k = 1$ and $i = 1$, (34) is given by

$$-(\partial/\partial t) V_1^{(1)}(t, x_1, x_2) = x_1' Z_{11}^{1(0)} A_{12} x_2 + x_2' Q_{22}^1 x_2$$

$$+ x_2' Z_{22}^{2(0)} B_2 (R_{22}^1)^{-1} B_2' Z_{22}^{2(0)} x_2 + ((\partial/\partial x_1) V)'(A_1 - B_1 B_1' Z_{11}^{1(0)}) x_1$$

$$+ ((\partial/\partial x_2) V)'(A_2 - B_2 B_2' Z_{22}^{2(0)}) x_2,$$

$$V_1^{(1)}(t_f, x_1, x_2) = x_2' Q_{22f}^1 x_2. \tag{82}$$

Assuming that $V_1^{(1)}(t, x_1, x_2) = x' Z_1^{(1)} x$, one can verify, by substituting this form in (82), that $Z_1^{(1)}(t)$ indeed satisfies (81).

Now, we state the counterpart of Theorem 4.2 for the CLPS case.

Theorem 4.4. There exists an $\epsilon_0 > 0$ such that the set of coupled Riccati equations (77) admits a unique solution for all $\epsilon \in [-\epsilon_0, \epsilon_0]$, and this solution is infinitely many times continuously differentiable in ϵ, at $\epsilon=0$. Further, the pair of feedback strategies $\{\gamma_{1k}(t,x), \gamma_{2k}(t,x)\}$, where $\gamma_{ik}(t,x)$ is given by (79), with $Z^i(t)$ replaced by $\sum_{l=0}^{k} \epsilon^l Z^{i(l)}(t)$, is in an $O(\epsilon^{2k+2})$ equilibrium.

Proof. The first part of the theorem follows directly by applying to (77) the implicit function theorem given in the Appendix; the rest follows from Theorem 4.1. □

4.3. Mixed Information Structure. We first state the following counterpart of Theorems 4.1 and 4.3 for the OL-CLPS information structure.

Theorem 4.5. Suppose that the robust Nash equilibrium policies $\gamma_1^*(t)$ and $\gamma_2^*(t,x)$ are expandable in ϵ as

$$\sum_{l=0}^{k} \gamma_1^{(l)}(t)\epsilon^l + O(\epsilon^{k+1}) \quad \text{and} \quad \sum_{l=0}^{k} \gamma_2^{(l)}(t,x)\epsilon^l + O(\epsilon^{k+1}),$$

respectively. Then, $\gamma_i^{(l)}(\cdot)$, $i=1,2$, $0\leq l\leq k$, are unique under Assumptions (A1)–(A3). Further, the pair of strategies $\{\gamma_{1k}^*(t), \gamma_{2k}^*(t,x)\}$, where

$$\gamma_{ik}^*(\cdot) = \sum_{l=0}^{k} \gamma_i^{(l)}(\cdot)\epsilon^l,$$

provides an $O(\epsilon^{2k+2})$ Nash equilibrium.

Proof. It is similar to the proofs of Theorems 4.1 and 4.3. □

Theorem 4.6. There exists an $\epsilon_0>0$ such that the LQ problem admits a unique, robust OL-CLPS equilibrium solution for all $\epsilon\in[-\epsilon_0, \epsilon_0]$, and this solution is infinitely many times continuously differentiable in ϵ, at $\epsilon=0$.

Proof. From Ref. 11, there exists a unique, robust Nash equilibrium solution to the LQ OL-CLPS problem if and only if there exists a unique solution to a certain class of linear differential equations with mixed boundary conditions. Applying the theorem of Appendix (see Remark 6.2) to this set of differential equations yields the desired result. □

We will not give the general expressions for $\gamma_i^{(k)}(\cdot)$ for the LQ case as they are complicated. But it should be noted that $\gamma_1^{(0)}(t)$, $\gamma_1^{(1)}(t)$ are the same as in the OL case, $\gamma_2^{(0)}(t,x)$ is the same as in the CLPS case, whereas $\gamma_2^{(1)}(t,x)$ is of the form $Dx+d$, which is an affine function of x unlike the CLPS case.

4.4. Comparison of OL, CLPS, and OL-CLPS Nash Equilibrium Solutions. In the case of intrinsic nonzero-sum two-person differential games, it is well known (Ref. 1) that the state trajectories generated by a pair of strategies that are in Nash equilibrium are generally different depending on whether the problem has an OL information structure or a CLPS information structure. This is definitely also true in the weakly coupled nonlinear differential game problem considered in this paper. Our objective in this subsection is to establish this directly. Toward this goal, what we will show is that, if the players use $O(\epsilon^{k+1})$ approximate Nash strategies, i.e.,

$$\gamma^i=\sum_{l=0}^{k}\gamma_i^{(l)}\epsilon^l, \qquad k\geq 0,\ i=1,2,$$

then only the zeroth order trajectories are identical under the OL and CLPS information structures, and the higher-order state trajectories are generally different. We will verify this explicitly for $k=1$; for the case $k\geq 1$, the argument is very similar.

As we indicated earlier, the zeroth order problems are the same under both information structures. Hence, from optimal control theory, we know that

$$p_{ii}^{(0)}(t)=(\partial/\partial x_i)V_i^{(0)}(t,x_1,x_2),$$

and the zeroth order state trajectory $x^{(0)}$ is the same under the two information structures and is given by (10). The first-order trajectory $x_i^{(1)}(t)$, $i=1,2$, in the case of the OL information structure is given by (11). Substituting for $u_i^{(1)}(t)$ in the equation for $x_i^{(1)}(t)$ yields

$$\begin{aligned}\dot{x}_i^{(1)}=&[(\partial/\partial x_i)f_i(x_i^{(0)},u_i^{(0)})\\&-(\partial/\partial u_i)f_i(x_i^{(0)},u_i^{(0)})(H_{uu})^{-1}(\partial^2/\partial x_iu_i)g(x_i^{(0)},u_i^{(0)})]x_i^{(1)}\\&-(\partial/\partial x_i)f(x_i^{(0)},u_i^{(0)})(\partial/\partial x_i)f'(x_i^{(0)},u_i^{(0)})p_{ii}^{(0)}+f_{ij}(x_j^{(0)}),\\x_i^{(1)}(t_f)=&0.\end{aligned}\tag{83}$$

Since the first-order solution corresponds to an LQ control problem, we have that

$$p_{ii}(t)=S_{ii}(t)x_i^{(1)}(t)+s_{ii}(t),$$

where $S_{ii}(t)$ satisfies a Riccati equation. Using this in (83), we have

$$\begin{aligned}\dot{x}_i^{(1)}=&[(\partial/\partial x_i)f_i(x_i^{(0)},u_i^{(0)})\\&-(\partial/\partial u_i)f_i(x_i^{(0)},u_i^{(0)})(H_{uu})^{-1}(\partial^2/\partial x_iu_i)g(x_i^{(0)},u_i^{(0)})\\&-(\partial/\partial x_i)f(x_i^{(0)},u_i^{(0)})(\partial/\partial x_i)f'(x_i^{(0)},u_i^{(0)})S_{ii}(t)]x_i^{(1)}\\&-(\partial/\partial x_i)f(x_i^{(0)},u_i^{(0)})(\partial/\partial x_i)f'(x_i^{(0)},u_i^{(0)})s_{ii}(t)+f_{ij}(x_j^{(0)}),\\x_i^{(1)}(t_f)=&0.\end{aligned}\tag{84}$$

Now, let us look at the state equation for $x_i(t)$, when the CLPS Nash equilibrium strategy is used:

$$\dot{x}_i = f_i(x_i, \gamma_i^*(t, x)) + \epsilon f_{ij}(x_j). \tag{85}$$

Expanding the above equation in terms of ϵ, and retaining only the first-order terms yields,

$$\begin{aligned} \dot{x}_i^{(1)} &= [(\partial/\partial x_i) f(x_i^{(0)}, \gamma_i^{(0)}(t, x_i^{(0)})) + (\partial/\partial u_i) f(x_i^{(0)}, \gamma_i^{(0)}(t, x_i^{(0)}))] \\ &\quad \times [(\partial/\partial x_i)\gamma_i^{(0)}(t, x_i^{(0)})] x_i^{(1)} + \gamma_i^{(1)}(t, x_i^{(0)}), \\ x_i^{(1)}(t_f) &= 0, \end{aligned} \tag{86}$$

where we have assumed that the zeroth order solution has continuous first partial derivatives. Clearly, (84) and (86) need not yield the same solution for $x_i^{(1)}(t)$. This is more obvious, if we look at (76) and (81), for $k=1$, which are the first-order gain matrices in the LQ case for the OL and the CLPS information patterns, respectively. The matrix R_{jj}^i appears only in (81); hence, by changing R_{jj}^i, we can make $Z^{i(1)}(t)$ different from $P^{i(1)}(t)$. Also, both the OL and CLPS trajectories are different from the trajectory obtained in the OL-CLPS case, because as we noted at the end of the last section, the first-order strategy for Player 2 (the one using CLPS information) is an affine function of the state, unlike the case when both players use the same information. This is in contrast with the result in regularly perturbed optimal control problems (Ref. 2), where the state trajectories are identical, to all orders of ϵ, regardless of whether the solution is derived using Pontryagin's minimum principle or dynamic programming.

5. Conclusions

In this paper, we have studied a class of nonzero-sum, nonlinear, two-person differential nes where the players are weakly coupled through the state equation and their performance indices. We have obtained conditions under which unique $O(\epsilon^{2k+2})$ Nash equilibrium strategies exist, under the following information structures and with the further refinement of strong time consistency and/or robustness: both players use either open-loop or closed-loop information; or one player uses open-loop information and the other player uses closed-loop information. Further, we have developed an iterative procedure to obtain the Nash equilibrium. It should be noted that the iterative procedure can be interpreted as a policy iteration scheme in the following manner. If one player uses a policy that is $O(\epsilon^k)$ close to his Nash equilibrium policy, then the other player's response is $O(\epsilon^{k+1})$ close to his Nash equilibrium policy. This result is intuitive if we note that, to obtain an $O(\epsilon^{k+1})$ approximation to the Nash equilibrium solution of one player, we need only terms up to the kth order in the ϵ expansion of the policy of the other player.

We have also established certain similarities and differences between the weakly coupled game problem and the regularly perturbed single decision maker optimal control problem. We have shown that, under all three information structures, the equivalent problems associated with each stage of the iteration are similar. However, while in the optimal control problem the state trajectory is the same irrespective of whether a feedback policy or open-loop policy is used (Ref. 2), in the genuine game case only the zeroth order trajectories are the same, and all the higher-order trajectories are different for different types of information structures.

As mentioned in Section 2, there are no conceptual difficulties in extending the results of this paper to obtain $O(\epsilon^{2k+2})$ Nash strategies when there are more than two players. Again, we expect the zeroth term in the expansion of the Nash equilibrium solution to be the solution of the zeroth order problems and the higher-order terms to be interpreted as the solutions to simpler optimal control problems and/or the solutions to static optimization problems. Further, a policy-iteration interpretation of these solutions is also possible as mentioned in Section 3.4. Direct extensions to zero-sum games also seem to be possible, where in fact stronger results could be obtained. It is well known (Ref. 1) that, in zero-sum differential games, the optimal state trajectory is the same irrespective of the type of information pattern (even though the existence of a saddle point will depend on the particular information structure used). Hence, it would be an interesting exercise to connect our approach in this paper with the results of Ref. 2 for the optimal control problem, in the context of zero-sum differential games with weakly coupled players. Another area for future work in this direction would be the study of stochastic games with weakly coupled decision makers. Some results have been reported in this context in Ref. 6, when the available information is common to both players.

6. Appendix: Implicit Function Theorem for Ordinary Differential Equations

The theorems stated below deal with the existence and other properties of solutions to the following perturbed differential equation (Ref. 7):

$$(d/dt)y = f(y, t, \epsilon), \qquad y(0, \epsilon) = y_0. \tag{87}$$

Theorem 6.1. Suppose that, in the domain $G = \{0 \le t \le T,\ |y| < b,$

$|\epsilon| \leq \bar{\epsilon}\}$, the function $f(y, t, \epsilon)$ is continuous with respect to the set of its variables and satisfies the Lipschitz condition

$$|f(y_1, t, \epsilon) - f(y_2, t, \epsilon)| \leq N|y_1 - y_2|,$$

where N is the same constant for all ϵ on the segment $|\epsilon| \leq \bar{\epsilon}$. Suppose that the solution to the scalar differential equation

$$(d/dt)\bar{y} = f(\bar{y}, t, 0), \qquad \bar{y}(0) = y_0,$$

exists, is unique on $[0, T]$, and belongs to $D = \{0 \leq t \leq T, |y| < b\}$. Then, for sufficiently small ϵ, the solution $y(t, \epsilon)$ of (87) also exists and is unique on $[0, T]$; it belongs to D, and we have the following limit uniformly with respect to t:

$$\lim_{\epsilon \to 0} y(t, \epsilon) = \bar{y}(t).$$

Further, if $f(y, t, \epsilon)$ possesses continuous and uniformly bounded partial derivatives with respect to y and ϵ to the order $k+1$ inclusive, in the domain G, then the solution $y(t, \epsilon)$ to (87) has the following asymptotic representation in the interval $[0, T]$:

$$y(t, \epsilon) = \bar{y}(t) + \epsilon(\partial/\partial\epsilon)y(t, 0)$$
$$+ \cdots + (\epsilon^k/k!)(\partial^k/\partial\epsilon^k)y(t, 0) + \epsilon_{k+1}(t, \epsilon),$$

where$\epsilon_{k+1}(t, \epsilon) = O(\epsilon^{k+1})$.

Remark 6.1. Theorem 6.1 remains essentially intact even if y is a vector and/or the initial condition is also perturbed, i.e., $y(0, \epsilon) = y_0 + w(\epsilon)$, where $w(\epsilon)$ is $O(\epsilon)$.

Remark 6.2. Suppose that y is a vector, some of the components of y are specified at time $t = 0$, and the rest of the components are specified at time $t = T$. Again, the results of Theorem 6.1 are valid, because of the following reason. Assume that there exists a unique solution to the differential equation with $\epsilon = 0$. Then, there exists a unique initial condition y_0 that leads to satisfaction of the final condition, when $\epsilon = 0$. Now, as in Remark 6.1, by assuming an initial condition of the form $y(t_0) = y_0 + w(\epsilon)$, we have a unique solution to the differential equation. Hence, we can choose a function $w(\cdot)$ to lead to satisfaction of the final condition.

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