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Image Space Approach to Penalty Methods

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Abstract. In this paper, we introduce a unified framework for the study of penalty concepts by means of the separation functions in the image space (see Ref. 1). Moreover, we establish new results concerning a correspondence between the solutions of the constrained problem and the limit points of the unconstrained minima. Finally, we analyze some known classes of penalty functions and some known classical results about penalization, and we show that they can be derived from our results directly.

Key Words. Penalty functions, image space, separation functions, nonlinear programming, constrained optimization.

1. Introduction

Recently, a general approach to optimization problems and related questions has been proposed (see Ref. 1) by means of the image problem and the image-space concept. The purpose of this paper is to investigate an important part of optimization theory, that of penalty methods, by following the above-mentioned proposed general scheme, in order to establish more general penalty relationships.

The classical penalty method for a constrained extremum problem, with $\phi : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ as objective and constraint functions, respectively, leads us to study an unconstrained problem having $\phi + \pi(g; \omega)$ as the objective functions, with a suitable $\pi: \mathbb{R}^m \times \Omega \times \mathbb{R}^m$ and a suitable set Ω of parameters. In our scheme, the unconstrained problems will have $w(\phi, g; \omega)$ as an objective function, with a suitable choice of $w : \mathbb{R} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$, which will be called separation function.

Such an approach is clearly more general, and thus it permits us to achieve results of different types with respect to the classical ones. After the preliminaries and the definitions, in Sections 3 and 4 we shall study

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some relationships between the optima of unconstrained problems and the optimum of the constrained one. In Section 5, after having specified the family of functions w, we shall then establish a result concerning minima convergence of the unconstrained problems to the minimum of the given problem, which is a generalization of a well-known Courant theorem (see Ref. 2). In Section 6, we show that many classical results are a direct consequence of our results.

2. Preliminaries and Definitions

Let us suppose that we have the following problem[2]:

$$\min \phi(x), \qquad \text{s.t. } x \in R \triangleq \{x \in X : g(x) \geq 0\}, \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is an open set, $f: X \to \mathbb{R}$, $f \in C^0(X)$, and $g: \mathbb{R}^n \to \mathbb{R}^m$, $g \in C^0(\mathbb{R}^n)$.

The following definition will be useful.

Definition 2.1. A function $w: \mathbb{R}^{1+m} \to \mathbb{R}$ is called a weak separation function iff

$$\{(u, v) \in \mathbb{R} \times \mathbb{R}^m : w(u, v) > 0\} \supseteq \mathscr{H} \triangleq \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0, v \geq 0\};$$

it is called a strong separation function iff

$$\{(u, v) \in \mathbb{R} \times \mathbb{R}^m : w(u, v) > 0\} \subseteq \mathscr{H}.$$

Some relationships between separation functions (weak and strong) and penalty methods (exterior and interior) have recently been studied (see Ref. 1). Furthermore, given $\bar{x} \in R$, let us define $f(x) = \phi(\bar{x}) - \phi(x)$, and let us consider a family of separation functions $w(u, v; \omega)$, which depends on the parameter $\omega \in \Omega$.

The penalized problems will be the following ones:

$$\alpha(\omega) \triangleq \sup_{x \in X} w(f(x), g(x), \omega). \tag{2}$$

We will study $\limsup_{\omega \to +\infty} \alpha(\omega)$ and $\liminf_{\omega \to +\infty} \alpha(\omega)$ in order to investigate the optimality of $\bar{x}$; this will be done for both weak and strong separation functions in Sections 3 and 4, respectively. In Section 5, we will study the limit points of $x(\omega)$ which are the optima of (2), when they exist.

[2] If $x \in \mathbb{R}^n$, $x \geq 0$ means that $x_i \geq 0$ for every $i = 1, \ldots, n$.

Finally, we want to observe that the following four examples of classic penalty functions (see Refs. 3–5) are included in the scheme of weak and strong separation functions when we consider $w(u, v; \omega) = u - \pi(v; \omega)$:

(i) $m = 1$, $$\pi(v; \omega) = \begin{cases} 0, & \text{if } v \geq 0, \\ \omega v / c(v + c), & \text{if } -c < v < 0, \\ +\infty, & \text{if } v \leq -c, \end{cases}$$

where the constraint of problem (1) is $g(x) \geq -c \in \mathbb{R}^-$;

(ii) $m = 1$, $$\pi(v; \omega) = \begin{cases} 0, & \text{if } v \geq 0, \\ \exp(-\omega v) - 1, & \text{if } v \leq 0; \end{cases}$$

(iii) $m = 1$, $\pi(v; \omega) = \omega\psi(v)$, where
$$\psi(v) = \begin{cases} 0, & \text{if } v \geq 0, \\ \geq 0, & \text{if } v < 0; \end{cases}$$

(iv) $m = 1$, $$\pi(v; \omega) = \begin{cases} \omega / v, & \text{if } v > 0, \\ +\infty, & \text{if } v \leq 0. \end{cases}$$

3. Exterior Penalty Method

We will suppose in this section that $\Omega \subseteq \mathbb{R}$, that the family $w(u, v; \omega)$ is a weak separation function, and that $w(0, 0; \omega) = 0$, $\forall \omega \in \Omega$. This last condition is really natural. It can be imposed without loss of generality, because it is possible to satisfy it by means of a translation and, moreover, it is verified by all the concrete examples of penalty functions existing in the current literature. Furthermore, we suppose that $w(u, v; \omega) \in C^0(\mathbb{R} \times \mathbb{R}^m \times \Omega)$. The following lemma holds.

Lemma 3.1. $\alpha(\omega) \geq 0$, $\forall \omega \in \Omega$.

Proof. Due to the fact that

$$\alpha(\omega) \geq w(f(\bar{x}); g(\bar{x}); \omega) = w(0, g(\bar{x}); \omega) \geq 0,$$

we obtain the desired thesis by observing that

$$\text{lev}_{>0}\, w(u, v; \omega) \triangleq \{(u, v) \times \mathbb{R}^{1+m}: w(u, v; \omega) > 0\} \supset \mathcal{H}$$

and that w is continuous. □

We call W_1 the class of functions $w(u, v; \omega)$ satisfying the following property:

$$\{(u, v) \in \mathbb{R}^{1+m}: \limsup_{\omega \to +\infty} w(u, v; \omega) = 0\} \cap \mathcal{H} = \varnothing; \tag{3}$$

and we call W_2 the class of functions $w(u, v; \omega)$ satisfying the following property:

$$\{(u, v) \in \mathbb{R}^{1+m}: \liminf_{\omega \to +\infty} w(u, v; \omega) = 0\} \cap \mathcal{H} = \emptyset. \tag{4}$$

We can now prove the following theorem.

Theorem 3.1. Let us suppose that the functions $w(u, v; \omega)$ belong to the class W_1 and $\Omega = \mathbb{R}^+$. Then, $\limsup_{\omega \to +\infty} \alpha(\omega) = 0$ implies that $\bar{x}$ is the minimum point of (1).

Proof. Let us suppose, *ab absurdo*, that $\bar{x}$ is not optimal. Then, there exists $\tilde{x} \in X$ such that, having defined $\tilde{u} = \phi(\bar{x}) - \phi(\tilde{x}) = f(\tilde{x})$ and $\tilde{v} = g(\tilde{x})$, we have $(\tilde{u}, \tilde{v}) \in \mathcal{H}$. If this were true, we should find that $\alpha(\omega) \geq w(\tilde{u}, \tilde{v}; \omega) > 0$, $\forall \omega \in \mathbb{R}$, because of the definitions of $\alpha(\omega)$ and weak separation function; consequently,

$$\limsup_{\omega \to +\infty} \alpha(\omega) \geq \limsup_{\omega \to +\infty} w(\tilde{u}; \tilde{v}; \omega) > 0,$$

where the last inequality follows from assumption (3). □

With only formal changes, the above proof can be used to prove the following theorem.

Theorem 3.2. Let us suppose that the function $w(u, v; \omega)$ belongs to the class W_2 and that $\Omega = \mathbb{R}^+$. Then, $\liminf_{\omega \to +\infty} \alpha(\omega) = 0$ implies that $\bar{x}$ is the minimum point of (1).

Example 3.1. We now show that Theorem 3.1 is not true if condition (3) does not hold. Let us suppose that $\phi(x) = |x|$, $g(x) = 0$, $X = \mathbb{R}$, $\bar{x} = 1$, and

$$w(u, v; \omega) = \begin{cases} \omega u + v, & \text{if } u < 0, \\ (u/\omega) + v, & \text{if } u \geq 0. \end{cases}$$

Remark 3.1. It is clear that the assumption $\Omega = \mathbb{R}^+$ can be changed with another that ensures us that $+\infty$ is an accumulation point of Ω.

Remark 3.2. We want to observe that $W_2 \supset W_1$ and that the classical penalty families in the literature are in W_1.

Remark 3.3. A family of weak separation functions for problem (1) can be obtained if we consider a classical sequence $\{\pi_r\}_{r \in \mathbb{N}}$ of functions $\pi_r: \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\pi_r \in C^0(\mathbb{R}^n)$;

(ii) $\pi_r(x)=0 \Leftrightarrow x \in R$;
(iii) $\pi_r(x)>0 \Leftrightarrow x \notin R$;
(iv) $\pi_{r+1}(x)>\pi_r(x)$, $\forall x \notin R$;
(v) $\lim_{r\to+\infty} \pi_r(x)=+\infty$, $\forall x \notin R$.

Then, we can define $w(u, v; \omega)=u-\pi_{\lfloor\omega\rfloor}(v)$, taking into account that the class $\{\pi_r\}$ must realize the continuity of w.[3] We observe that such a family belongs to class W_2.

The study of $\limsup_{\omega\to+\infty} \alpha(\omega)$ gives us a complete outline. In fact, we can prove the following theorem.

Theorem 3.3. Let us suppose that X is compact, that $\Omega=\mathbb{R}^+$, and that the functions $w(u, v; \omega)$ satisfy the following properties:
(a) $\{(u, v)\in\mathbb{R}^{1+m}: \limsup_{\omega\to+\infty} w(u, v; \omega)\geq 0\}=\bar{\mathscr{H}}$[4];
(b) $\limsup_{\omega\to+\infty} w(0, v; \omega)=0$, if $v\geq 0$;
(c) $\limsup_{\omega\to+\infty} w(u, 0; \omega)=0$, if $u\geq 0$.

Then, if $\limsup_{\omega\to+\infty} \alpha(\omega)>0$, $\bar{x}$ is not a minimum point for (1).

Proof. If

$$\limsup_{\omega\to+\infty} \alpha(\omega)=\bar{\alpha}>0,$$

for every $\epsilon>0$, $\epsilon\ll 1$, there exists a sequence $\{\omega_r\}_{r\in\mathbb{N}}$, with

$$\lim_{r\to+\infty} \omega_r=+\infty,$$

such that

$$0<\bar{\alpha}-\epsilon\leq\alpha(\omega_r)\leq\bar{\alpha}+\epsilon.$$

Because of the definition of $\alpha(\omega)$,

$$\forall r, \exists x_r\in X: \quad 0<\bar{\alpha}-2\epsilon<w(f(x_r), g(x_r); \omega_r)\leq\bar{\alpha}+2\epsilon.$$

Let $\tilde{x}$ be a cluster point of the sequence $\{x_r\}_{r\in\mathbb{N}}$; since X is compact, $\tilde{x}\in X$; and let us suppose that $\{x_{r_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_r\}_{r\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} x_{r_k}=\tilde{x};$$

since w is continuous, we have that

$$0<\bar{\alpha}-2\epsilon\leq\lim_{\omega\to+\infty} w(f(\tilde{x}), g(\tilde{x}); \omega)\leq\bar{\alpha}+2\epsilon;$$

[3] $\lfloor\omega\rfloor$ denotes the lower integer part of ω, namely the maximum integer less than or equal to ω.
[4] $\bar{\mathscr{H}}$ denotes the closure of the set $\mathscr{H}$.

that is,

$$\exists \bar{\omega}, \forall \omega \geq \bar{\omega}: \quad 0 < \bar{\alpha} - 3\epsilon \leq w(f(\bar{x}), g(\bar{x}); \omega) \leq \bar{\alpha} + 3\epsilon.$$

Now, thanks to (a), we have only three possible cases:

(i) $f(\bar{x}) = 0$ and $g(\bar{x}) \geq 0$; in this case, assumption (b) is violated;

(ii) $f(\bar{x}) \geq 0$ and $g(\bar{x}) = 0$; in this case, assumption (c) is violated;

(iii) $f(\bar{x}) > 0$ and $g(\bar{x}) \geq 0$; in this case, $\bar{x}$ is not optimal. So, we have the thesis. □

Remark 3.4. Assumption (a) of Theorem 3.3 is verified, for example, by all the families of weak separation functions $w(u, v; \omega)$ such that

$$\forall \omega_1 \geq \omega_2, \quad \mathrm{lev}_{\geq 0}\, w(u, v; \omega_1) \subseteq \mathrm{lev}_{\geq 0}\, w(u, v; \omega_2)$$

and

$$\bigcap_{\omega \in \mathbb{R}^+} \{\mathrm{lev}_{\geq 0}\, w(u, v; \omega)\} \equiv \bar{\mathcal{H}}.$$

Example 3.2. We want to show that Theorem 3.3 does not hold if one of the assumptions (a), (b), (c) is not true.

(i) Let us consider $\phi(x) = |x|$, $g(x) = 1$, $X = \mathbb{R}$, $\bar{x} = 0$, and

$$w(u, v; \omega) = \begin{cases} \omega u + v, & \text{if } u \leq 0, \\ (1/\omega)u + v, & \text{if } u \geq 0. \end{cases}$$

It is obvious that $\limsup_{\omega \to +\infty} \alpha(\omega) = 1$ and that $\bar{x}$ is optimal, but assumption (b) is not verified.

(ii) Let us consider $\phi(x) = x$, $g(x) = x(1 - x)$, $\bar{x} = 0$, $X = \mathbb{R}$, and

$$w(u, v; \omega) = \begin{cases} \omega u + v, & \text{if } u < 0, \\ u + \omega v, & \text{if } u \geq 0. \end{cases}$$

It is obvious that $\limsup_{\omega \to +\infty} \alpha(\omega) = 1$ and that $\bar{x}$ is optimal, but assumption (c) is not verified.

(iii) Let us consider $\phi(x) = x$, $g(x) \equiv 1$, $\bar{x} = 0$, $X = (0, +\infty)$, and

$$w(u, v; \omega) = \begin{cases} u + v, & \text{if } u < 0, \\ u + \omega v, & \text{if } u \geq 0. \end{cases}$$

It is obvious that $\limsup_{\omega \to +\infty} \alpha(\omega) > 0$ and that $\bar{x}$ is optimal, but assumption (a) is not verified.

Remark 3.5. With the position $F = (f, g)$, the optimality for (1) is obviously equivalent to the impossibility of the generalized system

$$F(x) \in \mathcal{H}, \qquad x \in X, \tag{5}$$

which has been studied in Ref. 1. Then, a possible idea for generalizing the concept of penalty method is to conceive the penalization of a generalized system. In particular, given $\tilde{w}:\mathbb{R}^{\nu}\times\mathbb{R}\to\mathbb{R}$, $\mathcal{H}\subset\mathbb{R}^{\nu}$, $F:X\subseteq\mathbb{R}^{\nu}\to\mathbb{R}^{\nu}$, assuming that $\tilde{w}$ is a generalized weak separation function in the sense that $\{h\in\mathbb{R}^{\nu}: \tilde{w}(h)>0\}\supseteq\mathcal{H}$, and defining

$$\tilde{a}(\omega)\triangleq\sup_{x\in X}\tilde{w}(F(x);\omega),$$

we find the following proposition.

Proposition 3.1. If $\liminf_{\omega\to+\infty}\tilde{a}(\omega)>0$, then the system (5) is impossible.

Proof. Let us suppose, *ab absurdo*, that there exists $\tilde{x}\in X$ such that $F(\tilde{x})\in\mathcal{H}$; hence, $\tilde{w}(F(\tilde{x});\omega)>0$, for every $\omega\in\mathbb{R}$, and $\sup_{x\in X}\tilde{w}(F(x);\omega)>0$, for every $\omega\in\mathbb{R}$; therefore,

$$\liminf_{\omega\to+\infty}\alpha(\omega)\geq 0. \qquad \square$$

4. Interior Penalty Method

Let us suppose in the rest of this section that the family $w(u, v;\omega)$ is a strong separation family, that $w(u, v;\omega)$ is continuous, and that $\Omega=\mathbb{R}^{+}$.

Remark 4.1. In this case, $\alpha(\omega)$ can also be negative.

Let us prove the following proposition.

Proposition 4.1. If there exists ω such that $\alpha(\omega)>0$, then $\bar{x}$ is not optimal.

Proof. The assumption means that there exists $\tilde{x}\in X$ such that $w(f(\tilde{x}), g(\tilde{x});\omega)>0$, and hence $(f(\tilde{x}), g(\tilde{x}))\in\mathcal{H}$. The thesis follows. $\square$

Moreover, the following theorem holds.

Theorem 4.1. Let us suppose that the functions $w(u, v;\omega)$ satisfy the following properties:

(a) $\forall(\tilde{u},\tilde{v})\in\operatorname{int}\mathcal{H}$, we have that

$$\forall\bar{\omega}, \exists\omega\geq\bar{\omega}:\quad (\tilde{u},\tilde{v})\in\operatorname{lev}_{>0}w(u, v;\omega);$$

(b) $\limsup_{\omega\to+\infty}w(u, 0;\omega)=0$, if $u\geq 0$.

Then, if $\limsup_{\omega\to+\infty}\alpha(\omega)<0$, $\bar{x}$ is global optimum for (1).

Proof. Let us suppose, *ab absurdo*, that $\bar{x}$ is not optimal. Then, there exist $\tilde{x} \in X$ such that $(\tilde{u}, \tilde{v}) \triangleq (f(\tilde{x}), g(\tilde{x})) \in \mathscr{H}$. Thus, we have only two possible situations:

$$\text{(i)} \quad g(\tilde{x}) > 0; \qquad \text{(ii)} \quad g(\tilde{x}) = 0.$$

If (i) holds, then $(\tilde{u}, \tilde{v}) \in \operatorname{int} \mathscr{H}$, and hence $w(\tilde{u}, \tilde{v}; \omega) > 0$ frequently[5] when $\omega \to +\infty$ because of assumption (a), and consequently $\limsup_{\omega \to +\infty} \alpha(\omega) \geq 0$; if (ii) holds, then from $\alpha(\omega) \geq w(u, 0; \omega)$, we deduce that

$$\limsup_{\omega \to +\infty} \alpha(\omega) \geq \limsup_{\omega \to +\infty} w(\tilde{u}, 0; \omega) = 0,$$

where the last equality derives from assumption (b). □

Example 4.1. Let us show by means of two examples that, without assumptions (a) and (b), Theorem 4.1 does not hold.

(i) Let us suppose that $u > 0$, $v > 0$, $w(u, v; \omega) = uv - 1$, $\bar{x} = 1/2$, $\phi(x) = x$, $g(x) = x$. We see that $\limsup_{\omega \to +\infty} \alpha(\omega) < 0$ and that $\bar{x}$ is not optimal, but assumption (a) is not verified.

(ii) Let us suppose that $u > 0$, $v > 0$, $w(u, v; \omega) = \omega uv - 1$, $\bar{x} = 1$, $\phi(x) = x^2$, $g(x) \equiv 0$. We see that $\limsup_{\omega \to +\infty} \alpha(\omega) < 0$ and that $\bar{x}$ is not optimal, but assumption (b) is not verified.

Now, we analyze the case in which

$$\limsup_{\omega \to +\infty} \alpha(\omega) = 0.$$

It is useful to define the following function[6]:

$$\phi(\epsilon) \triangleq \limsup_{\omega \to +\infty} \left(\sup_{x \in X} w(f(x), g(x) + \underline{\epsilon}; \omega) \right),$$

where $\underline{\epsilon} = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$. Then, the following theorem holds.

Theorem 4.2. Let us suppose that $\limsup_{\omega \to +\infty} \alpha(\omega) = 0$ and that assumption (a) of Theorem 4.1 holds and that the following property is true:

$$\exists \hat{\epsilon} > 0 \colon \phi(\underline{\epsilon}) \equiv 0, \qquad \forall \underline{\epsilon} \in [0, \hat{\epsilon}]. \tag{6}$$

Then, $\bar{x}$ is a minimum point of (1).

Proof. Let us suppose, *ab absurdo*, that $\bar{x}$ is not optimal. Then, there exists $\tilde{x} \in R$ such that $(\tilde{u}, \tilde{v}) \triangleq (f(\tilde{x}), g(\tilde{x})) \in \mathscr{H}$. Two cases are possible:

$$\text{(i)} \quad (\tilde{u}, \tilde{v}) \in \mathring{\mathscr{H}}; \qquad \text{(ii)} \quad (\tilde{u}, \tilde{v}) \in \mathscr{H} / \mathring{\mathscr{H}}.$$

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[5] In the sense that $\forall \bar{\omega}, \exists \omega \geq \bar{\omega} \colon w(u, v; \omega) > 0$.

[6] *If* $a, b \in \mathbb{R}^n$, $a + b$ means $(a_1 + b_1, \ldots, a_n + b_n)$.

In case (i), because of assumption (a), we have $\alpha(\omega) \geq k > 0$ frequently[7] when $\omega \to +\infty$, and consequently

$$\limsup_{\omega \to +\infty} \alpha(\omega) > 0,$$

and this is against the hypothesis. In case (ii), we can consider $(\tilde{u}, \tilde{v} + \eta)$, where η is a vector which is zero in the components where $\tilde{v} > 0$ and otherwise is $\hat{\epsilon}/2$. Now, $(\tilde{u}, \tilde{v} + \eta) \in \mathring{\mathscr{H}}$, and we can conclude analogously after using (6). □

Remark 4.2. Let us suppose that

$$\limsup_{\omega \to +\infty} \alpha(\omega) = 0$$

and that condition (6) does not hold, but condition (a) of Theorem 4.1 holds. The idea for generalizing Theorem 4.2 might consist in supposing the existence of a function $E: \mathbb{R}^n \to \mathbb{R}^m$ with the following properties:

(a) $\lim_{\|x\| \to +\infty} E(x) = 0$;

(b) $\limsup_{\omega \to +\infty} [\sup_{x \in X} w(f(x), g(x) + E(x); \omega)] = 0$.

The penalty scheme of Sections 3 and 4 represents a theoretical approach, in the sense that every time we have a particular problem we must choose our class of separation functions. A possibility in this sense will be exploited in the next section. Nevertheless, we want to observe that our scheme is in the spirit of penalty methods, because we study an unconstrained problem instead of the constrained one.

5. Convergence of Penalized Optima

Now, we will study a particular family of weak separation functions, and we will obtain, directly from Theorem 3.1, a result ensuring us that the limit points of the optimal solutions of the penalized problem are optimal for (1). So doing, we show the first possibility to apply the theoretical scheme of Sections 3 and 4 to have a result useful from the computational viewpoint.

Let us suppose in the rest of the section that

$$w(u, v; r) \triangleq u - \pi(v; r),$$

with $\pi: \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ (i.e., $\Omega = \mathbb{N}$) satisfying the following properties:

$$\pi(\cdot\,; r) \in C^0(\mathbb{R}^m), \qquad \forall r \in \mathbb{N}, \tag{7a}$$

$$\pi(v; r) = 0, \qquad \text{iff } v \geq 0, \tag{7b}$$

$$\pi(v; r) \geq 0, \qquad \forall v, \forall r, \tag{7c}$$

$$\lim_{r \to +\infty} \pi(v; r) = +\infty, \qquad \forall v \ngeq 0. \tag{7d}$$

[7] See Footnote 5.

We want to observe that assumptions (7) are just those of the well-known Courant theorem. The following theorem ensures that such a theorem is now a consequence of Theorem 3.1, because the family satisfying (7) belongs to W_1.

Theorem 5.1. Let us suppose that assumptions (7) hold and also that there exists a sequence $\{x_r\}_{r\in\mathbb{N}}$ of global maximum points of the problems (2). Then, every limit point of the sequence $\{x_r\}_{r\in\mathbb{N}}$ is a global minimum point for problem (1).

Proof. Let x^0 be a cluster point of $\{x_r\}_{r\in\mathbb{N}}$, and let us suppose that $x_r \to x^0$ (otherwise, we can consider a subsequence). It is well known (see Ref. 2) that $x^0 \in R$; therefore, we can consider

$$\alpha(r) = w(\phi(x^0) - \phi(x_r), g(x_r); r) = \phi(x^0) - \phi(x_r) - \pi(g(x_r); r).$$

Let us observe that our class of penalty functions belongs to W_1 and then we can apply Theorem 3.1,

$$0 \le \limsup_{r\to+\infty} \alpha(r) = \limsup_{r\to+\infty} [\phi(x^0) - \phi(x_r) - \pi(g(x_r); r)] = 0,$$

where the last equality comes from the continuity of ϕ, g, and π. Taking into account Theorem 3.1, we achieve the desired thesis. □

6. Comparison with Classical Results

We want to briefly observe that some known results and some known classes of penalty functions are enclosed in that satisfying conditions (7) and then they can be derived from our results. We quote Theorem 1 of Ref. 6, p. 196:

"Let $H: \mathbb{R}^n \to \mathbb{R}$ be a penalty exterior function for problem (1), in the sense that $H(x) = \sum_{i=1}^m h(g_i(x))$, with $h: \mathbb{R} \to \bar{\mathbb{R}}$ such that $h(y) = 0$, if $y \ge 0$, and $h(y) = +\infty$, if $y < 0$. Let us suppose that H verifies the following conditions:

(i) $H(x) \ge 0, \forall x$,
(ii) $H(x) = 0 \Leftrightarrow x \in X = \{x \mid g_i(x) \le 0, \forall i\}$,
(iii) H is continuous.

Moreover, let us suppose that f is continuous, X is closed, and that one of the following two conditions is verified:

(a) $f(x) \to +\infty$, when $\|x\| \to +\infty$,
(b) X is bounded and $H(x) \to +\infty$ when $\|x\| \to +\infty$.

Then, when $r \to +\infty$, we have that the sequence

$$\bar{x}(r) \triangleq \inf_{x \in \mathbb{R}^n} \{f(x) + rH(x)\}$$

admits a cluster point that is optimum for (1) and $H(\bar{x}(r)) \to 0$."

We observe that conditions (7) are verified, because (iii)$\Leftrightarrow$(7a), (ii)$\Leftrightarrow$(7b), (i)$\Leftrightarrow$(7c), (i)$\Rightarrow$(7d), and then we can apply our Theorem 5.1. Then, Theorem 1 of Ref. 6 can be derived from our results. Analogously, this happens for the penalty function $P(x, r)$ of Ref. 7, for the augmented Lagrangian of Ref. 8 (p. 96), for the nondifferentiable exact penalty function P of Ref. 8 (p. 180), for the penalty class of Ref. 9, for that of Ref. 10, (p. 90) and Ref. 10 (p. 93), and for many others in the literature.

7. Conclusions

In this paper, we have shown that the theory of the penalty methods can be studied in a more general way by means of the analysis of the image problem (Ref. 1). We can derive the classical results and the classical penalty function families. Naturally, the absolute generality of the separation functions, which give us the theoretical scheme of Sections 2, 3, 4 with the relative results, does not permit us to obtain theoretical and numerical results on the convergence of minima. But the natural specification of the family w, which has been treated in Section 5, shows how it is possible to generalize some classical results, like the Courant theorem, also in the field of minima convergence.

Furthermore, we want to observe that our results can be extended, with obvious changes, to extremum constrained problems in spaces more general than $\mathbb{R}^n$, such as the complete metric spaces.

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