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Optimal Pricing in a Duopoly: A Noncooperative Differential Games Solution[1]

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Abstract. This paper deals with a differential games model of an oligopoly of *n* profit-maximizing firms competing for the same stock of customers. For the sale dynamics, it is assumed that the customers of each firm are driven away gradually by increasing product prices. Since the state variable is absent from the Hamiltonian maximizing conditions as well as from the adjoint equations, open-loop Nash solutions can be obtained. By using phase diagram analysis, for two players the behavior of the optimal pricing strategies can be characterized qualitatively. The main importance of the paper lies in the solution technique, rather than in the economic significance of the proposed model. Under the proposed assumptions, the two-point boundary-value problem resulting from the maximum principle is reduced to a terminal-value problem. It turns out that, for special salvage values of the market shares and if the planning horizon is not too short, nonmonotonic Nash-optimal price trajectories occur.

Key Words. Nash-optimal price policies, duopoly of profit-maximizing firms, two-person nonzero-sum differential games, phase portrait analysis of Nash solutions, state separability.

1. Introduction

In the past decade, nonzero-sum differential games theory was applied to a series of competitive dynamic economic models (see Refs. 1-3). In some of these applications, the players seek to determine optimal pricing strategies (see, e.g., Refs. 4-8).

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Clemhout and Wan (Ref. 9) define a class of differential games (hereafter DG's, for short), the trilinear games (see also Refs. 1, 2, 10, 11), which allow for qualitative insights into the behavior of their Nash solutions. The simplicity of this game structure is due to the fact that the state variables are absent from the adjoint equations, and the choice of controls is independent of the state variables.

The present paper has two purposes. First, we study the competitive behavior of n profit-maximizing firms. Assuming that the diffusion of customers is governed by the product prices charged by the firms, we are able to derive in the case of a duopoly a system of nonlinear differential equations for the Nash-optimal price policies which is used to obtain the qualitative traits of the optimal solutions. Second, we add a further example to the trilinear games, which is simple enough for the interactive mechanism between state and control variables of the players to be obvious. Admittedly, the model is perhaps not very realistic, since the situation has deliberately been simplified. Since the approach is explorative, the solutions of simple problems must be found before more realistic ones can be handled (see also Ref. 12).

The crucial point of the paper is that, under the assumptions made, the two-point boundary-value problem, resulting from the Pontryagin necessary conditions, is reduced to a terminal-value problem which can be treated in a much simpler way. Rather than claiming economic significance of the proposed model, the importance of the solution technique is stressed.

In Section 2, the dynamic pricing game is presented. In Section 3, the necessary and sufficient optimality conditions are derived from Pontryagin's maximum principle. Section 4 provides a basis for the stability analysis of the stationary value of the Nash pricing strategies. Since a more complete phase portrait analysis is not possible without specifying the price diffusion functions, in Section 5 constant elasticity functions are assumed. Finally, in Section 6, we summarize the results obtained and state some related problems for future research.

2. Model

Consider a differential games model of a oligopoly over a fixed period of time $[0, T]$. Assume that the rate of sale of a heterogeneous good is proportional to the number of customers buying the product (see Refs. 13 and 14). The numbers of customers buying at firm i, $i = 1, \ldots, n$, are easily transformed into market shares by dividing them by the total market potential. Let $x_i(t)$ denote the market share of firm i at time t. Let $p_i(t)$ be the product price charged by firm i, $i = 1, \ldots, n$, at time t. Assume that the gradual development of the market share of the firms is governed by the following diffusion process. For each firm i, there is a price level $\bar{p}_i$ so that, for $p_i > \bar{p}_i$, the customers leave firm i. The proportion for customers changing some firm i to firm $j \neq i$ increases more than proportionally with increasing difference $p_i - \bar{p}_i$ and linearly with the current market share of firm i. More specifically, we assume convex diffusion functions $g_i(p_i)$ with

$$g_i(p_i) > 0, \qquad g_i'(p_i) > 0, \qquad g_i''(p_i) > 0, \qquad p_i > \bar{p}_i, \tag{1a}$$

$$g_i(p_i) = 0, \qquad p_i \leq \bar{p}_i, \tag{1b}$$

$$g_i(\bar{p}_i) = 0, \qquad g_i'(\bar{p}_i) = 0, \tag{1c}$$

$$\lim_{p_i \to \infty} g_i'(p_i) = \infty, \tag{1d}$$

Note that (1d) is supposed for technical reasons. Economically more reasonable would be the assumption that there exists a price high enough to drive away all the customers.

The dynamics of the market share of firm i is the result of the loss of customers (due to its price policy) and of the increase of customers (due to the competitor's losses),

$$\dot{x}_i = -g_i(p_i)x_i + [1/(n-1)] \sum_{\substack{j=1 \\ j \neq i}}^{n} g_j(p_j)x_j, \qquad i = 1, \ldots, n. \tag{2}$$

In the system dynamics (2), it is assumed that the proportion of customers driven away from firm i, $g_i(p_i)x_i$, is allocated to its competitors with equal probabilities $(n-1)^{-1}$.

Note that, in this model, it is the price level of a firm which drives away its customers. It is not assumed that a low price policy of a firm lures additional customers from its competitors, i.e., a pure push model is considered. Moreover, we suppose that the customers are rational in the sense that the price level $\bar{p}_i$ for which the customers' out-migration starts is not below the marginal production costs c_i of firm i,

$$\bar{p}_i \geq c_i, \qquad i = 1, \ldots, n. \tag{3}$$

Note that this assumption is made for technical reasons; the case $\bar{p}_i < c_i$ is briefly mentioned in Section 6.

Finally, let r_i be the constant discount rate of firm 1, and let S_i be the salvage value of the market share of firm i at the end of the planning horizon, $i = 1, \ldots, n$. S_i measures the value of one customer for the firm at the terminal time T.

The objective of each firm is to maximize the present value of its intertemporal profit

$$J_i = \int_0^T \exp(-r_i t)(p_i - c_i)x_i\, dt + \exp(-r_i T) S_i x_i(T), \qquad i = 1, \ldots, n. \tag{4}$$

Using the market share of firm i as state variable x_i and the prices p_i as control variables, we get a nonzero-sum differential game with the state equations (2), the initial conditions

$$x_i(0) = x_{io}, \qquad 0 \le x_{io} \le 1, \qquad i = 1, \ldots, n, \tag{5}$$

and the performance indices (4). Note that, for given initial market shares, the system dynamics (2) implies

$$0 \le x_i(t) \le 1, \qquad t \in [0, T].$$

Moreover, from

$$\sum_{i=1}^{n} x_i(0) = 1$$

and (2), it follows that the market shares add up to 1 for all t,

$$\sum_{i=1}^{n} x_i(t) = 1.$$

3. Optimality Conditions for Nash Solutions

Restricting our interest to open-loop Nash equilibrium solutions, we consider the current-value Hamiltonians

$$H^i = (p_i - c_i)x_i + \sum_{k=1}^{n} \lambda_k^i \left[-g_k(p_k)x_k + [1/(n-1)] \sum_{\substack{j=1 \\ j \ne k}}^{n} g_j(p_j)x_j \right]. \tag{6}$$

The necessary conditions in terms of the Hamiltonians are given, e.g., in Ref. 15-17. In Eq. (6), λ_k^i is the current-value adjoint variable measuring the value of an increment of the market share of firm k assessed by firm i. The shadow prices λ_k^i satisfy the adjoint equations

$$\dot\lambda_k^i = r_i \lambda_k^i - H_{x_k}^i = r_i \lambda_k^i + g_k(p_k)\left\{ \lambda_k^i - [1/(n-1)] \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j^i \right\} - (p_i - c_i)\delta_{ik}, \tag{7}$$

where $k=1,\ldots,n$, and

$$\delta_{ik}=\begin{cases}1, & i=k,\\ 0, & i\neq k.\end{cases}$$

The transversality conditions are given by

$$\lambda_k^i(T)=S_i\delta_{ik}. \tag{8}$$

Because of assumption (1c), the Hamiltonian maximizing condition is given by

$$H_{p_i}^i=0.$$

This yields

$$1+g_i'(p_i)\left\{-\lambda_i^i+[1/(n-1)]\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j^i\right\}=0. \tag{9}$$

Since the Hamiltonians H^i are jointly concave in x and p_i, the necessary conditions (2), (5), (7), (8), (9) are also *sufficient* for the optimality of the solutions.

Due to the structure of the Hamiltonians, the game is *state separable*, which means that neither the optimal price nor the adjoint equation depend on the state variable x.

For two players, it sufficies to consider only one state variable, e.g., the market share of the first firm, say $x(t)$. The state equation can now be written as

$$\dot{x}=-g_1(p_1)x+g_2(p_2)(1-x).$$

Defining new current-value adjoint variables λ_i by

$$\lambda_1=\lambda_1^1-\lambda_2^1, \qquad \lambda_2=\lambda_1^2-\lambda_2^2,$$

the adjoint equations (7) can be written as

$$\dot{\lambda}_1=\lambda_1[r_1+g_1(p_1)+g_2(p_2)]-(p_1-c_1), \tag{7a}$$

$$\dot{\lambda}_2=\lambda_2[r_2+g_1(p_1)+g_2(p_2)]+(p_2-c_2), \tag{7b}$$

with the transversality conditions

$$\lambda_1(T)=S_1, \qquad \lambda_2(T)=-S_2. \tag{8a}$$

Note that λ_i measures the current value of an increment of the market share of firm 1 assessed by firm i, $i=1, 2$. Furthermore, for $n=2$ the maximum condition is given by

$$\lambda_1 = 1/g_1'(p_1), \qquad \lambda_2 = -1/g_2'(p_2). \tag{9a}$$

Note that for $n \geq 3$, it is not possible to reduce the two-point boundary value problem to a two-dimensional terminal-value problem, since too many adjoint variables λ_j^i occur in the necessary optimality conditions. Moreover, a phase portrait analysis can be illustrated in a two-dimensional control space.

Since the Hamiltonians H^i are jointly concave in x and p_i, the necessary conditions (2), (5), (7), (8), (9) are also *sufficient* for the optimality of the solutions.

Differentiation of (9a) with respect to time yields

$$\dot{\lambda}_1 = -\dot{p}_1 g_1''(p_1)/g_1'(p_1)^2, \qquad \dot{\lambda}_2 = \dot{p}_2 g_2''(p_2)/g_2'(p_2)^2. \tag{10}$$

Using (9a) and (10), the adjoint equations can be transformed into two differential equations in the control variables,

$$\dot{p}_i = [g_i'(p_i)/g_i''(p_i)][g_i'(p_i)(p_i - c_i) - g_i(p_i) - g_j(p_j) - r_i], \tag{11}$$

where here and in the following we assume that $i, j = 1, 2$, and $j \neq i$. The terminal conditions for the system (11) resulting from (8a) and (9a) are given by

$$g_i'(p_i(T)) = 1/S_i, \qquad i = 1, 2. \tag{12}$$

From (1d) follows the unique solvability of (12) for each nonnegative salvage value S_i, $i=1, 2$. For

$$S_i = 0,$$

we have

$$p_i(T) = \infty.$$

Defining the elasticities of the marginal diffusion functions as

$$e_i = [g_i''(p_i)/g_i'(p_i)]p_i, \tag{13}$$

the system (11) may also be written as

$$\dot{p}_i = (p_i/e_i)[g_i'(p_i)(p_i - c_i) - g_i(p_i) - g_j(p_j) - r_i]. \tag{11a}$$

The following form of (11) is useful to determine the stability properties of the equilibrium of the system:

$$\dot{p}_i = \{[g_i'(p_i)]^2/g_i''(p_i)\}\{p_i - c_i - [1/g_i'(p_i)][g_i(p_i) + g_j(p_j) + r_i]\}. \tag{11b}$$

4. Stability Analysis of Nash Equilibrium Solutions

The system (11) has four isoclines, namely [according to (1c)], at the boundary,

$$p_i = \bar{p}_i, \tag{14}$$

and in the interior,

$$G^i(p_1, p_2) = g_i(p_i) - g_i'(p_i)(p_i - c_i) + g_j(p_j) + r_i = 0. \tag{15}$$

In the sequel, let us denote the interior isoclines defined implicitly by (15) as

$$\dot{p}_i = 0.$$

Because of (1a) and (3), the $\dot{p}_i = 0$ curves (15), $i = 1, 2$, have positive slopes:

$$dp_2/dp_1|_{\dot{p}_1=0} = g_1''(p_1)(p_1 - c_1)/g_2'(p_2) > 0, \tag{16a}$$

$$dp_2/dp_1|_{\dot{p}_2=0} = g_1'(p_1)/(g_2''(p_2)(p_2 - c_2)) > 0. \tag{16b}$$

For

$$g_i'''(p_i) > 0, \tag{17}$$

it holds that

$$g_i''(p_i)(p_i - c_i) > g_i'(p_i).$$

Thus, according to (16), the slope for $\dot{p}_1 = 0$ is steeper than that for $\dot{p}_2 = 0$,

$$dp_2/dp_1|_{\dot{p}_1=0} - dp_2/dp_1|_{\dot{p}_2=0} > 0. \tag{18}$$

Since our aim is to gain qualitative insights into the behavior of the Nash-optimal paths, we need information on the stationary points of the autonomous system (11) of nonlinear differential equations. We now state a *sufficient condition* for the uniqueness and the existence of an interior equilibrium of the system (11).

Proposition 4.1. Let there be at least one $r_i > 0$. If (17) is satisfied, then at most one equilibrium $(\hat{p}_1, \hat{p}_2)$ exists in the interior of the control domain, i.e., $\hat{p}_i > \bar{p}_i$. If, additionally, (21) holds, the existence $(\hat{p}_1, \hat{p}_2)$ is guaranteed. Moreover, if the Jacobian determinant (24) evaluated at the stationary point $(\hat{p}_1, \hat{p}_2)$ is positive, $(\hat{p}_1, \hat{p}_2)$ is an unstable node.

Proof. Define $\tilde{p}_1$ as the solution of

$$L^1(p_1) \triangleq G^1(p_1, \bar{p}_2) = -g_1'(p_1)(p_1 - c_1) + g_1(p_1) + r_1 = 0. \tag{19}$$

We have

$$dL^1/dp_1 = -g_1''(p_1)(p_1 - c_1) < 0,$$

$$L^1(\bar{p}_1) = r_1 \geq 0.$$

Because of the convex shape (1) of $g_1(p_1)$, there exists values of p_1 such that

$$L^1(p_1) < 0.$$

Thus, we get a unique crossing point $(\tilde{p}_1, \bar{p}_2)$ of

$$\dot{p}_1 = 0 \quad \text{and} \quad p_2 = \bar{p}_2.$$

Note that

$$\tilde{p}_1 > \bar{p}_1, \qquad \text{iff } r_1 > 0.$$

Analogously,

$$\dot{p}_2 = 0$$

crosses

$$p_1 = \bar{p}_1$$

at a point $(\bar{p}_1, \tilde{p}_2)$ with

$$\tilde{p}_2 > \bar{p}_2 \qquad \text{iff } r_1 > 0.$$

To prove the unique existence of an interior stationary point, we substitute

$$p_2 = p_2(p_1),$$

defined implicitly by

$$G^1(p_1, p_2) = 0$$

of (15), into

$$G^2(p_1, p_2) = 0.$$

This yields the following equation for $p_1 = \hat{p}_1$:

$$F(p_1) = G^2(p_1, p_2(p_1))$$
$$= g_2(p_2(p_1)) - g_2'(p_2(p_1))[p_2(p_1) - c_2] + g_1(p_1) + r_2 = 0. \qquad (20)$$

By differentiation, we obtain

$$F'(p_1) = -g_2''(p_2(p_1))(dp_2/dp_1)[p_2(p_1) - c_2] + g_1'(p_1)$$
$$= -[1/g_2'(p_2(p_1))][g_1''(p_1)(p_1 - c_1)g_2''(p_2(p_1))(p_2(p_1) - c_2)$$
$$- g_1'(p_1)g_2'(p_2(p_1))] < 0,$$

where (16a) and (17) have been used. Substituting $\tilde{p}_1$ into (20), because of

$$p_2(\tilde{p}_1)=\bar{p}_2 \quad \text{and} \quad r_1+r_2>0,$$

we get

$$F(\tilde{p}_1)=g_2(\bar{p}_2)-g_2'(\bar{p}_2)(\bar{p}_2-c_2)+g_1(\tilde{p}_1)+r_2=g_1(\tilde{p}_1)+r_2>0.$$

If there would exist a p_1 such that

$$F(p_1)\leq 0$$

the existence of a solution of (20), say $\hat{p}_1$, would be guaranteed. Sufficient for this would be, according to (20), that

$$g_2'(p_2(p_1))[p_2(p_1)-c_2]-g_2(p_2(p_1))$$

goes faster to infinity than $g_1(p_1)$ or, because of (15),

$$\lim_{p_2\to\infty}\{g_2'(p_2(p_1))[p_2(p_1)-c_2]-g_1'(p_1)(p_1-c_1)\}\geq 0. \tag{21}$$

The existence of $\hat{p}_2$ can be proven under an assumption analogous to (21).

To derive the stability property of a possibly existing equilibrium point, we calculate the Jacobian determinant

$$D=\begin{vmatrix}\partial\dot{p}_1/\partial p_1 & \partial\dot{p}_1/\partial p_2\\ \partial\dot{p}_2/\partial p_1 & \partial\dot{p}_2/\partial p_2\end{vmatrix}. \tag{22}$$

From (11b), we obtain

$$\partial\dot{p}_i/\partial p_i|_{\dot{p}_i=0}=g_i(p_i)+g_j(p_j)+r_i>0, \tag{23a}$$

$$\partial\dot{p}_i/\partial p_j|_{\dot{p}_i=0}=-[g_i'(p_i)/g_i''(p_i)]g_j'(p_j)<0. \tag{23b}$$

This yields, for D evaluated in the equilibrium,

$$D=[g_1(p_1)+g_2(p_2)+r_1][g_1(p_1)+g_2(p_2)+r_2]$$
$$-[g_1'(p_1)g_2'(p_2)]^2/g_1''(p_1)g_2''(p_2). \tag{24}$$

If D evaluated at $(\hat{p}_1,\hat{p}_2)$ is positive, the interior equilibrium is an unstable node, because the main diagonal elements (23a) of (22) are also positive. □

Since condition (21) is not very manageable and (22) cannot be evaluated for the general convex functions (1), we restrict our interest in the following section to diffusion functions with constant elasticity, i.e., to power functions.

5. Phase Portrait Analysis for Constant Elasticity Functions

To obtain results on the existence and the stability property of the equilibrium $(\hat{p}_1, \hat{p}_2)$, we specify the diffusion functions as power functions,

$$g_i(p_o) = \begin{cases} \gamma_i(p_i - \bar{p}_i)^{\alpha_i}, & p_i \geq \bar{p}_i, \\ 0, & p_i < \bar{p}_i, \end{cases} \tag{25}$$

with

$$\gamma_i > 0, \qquad \alpha_i > 1.$$

Note that the elasticity of (25) is constant and positive,

$$e_i = \alpha_i - 1.$$

(11) can be written as

$$\dot{p}_i = [(p_i - \bar{p}_i)/(\alpha_i - 1)] \times [\alpha_i \gamma_i (p_i - \bar{p}_i)^{\alpha_i - 1}(p_i - c_i) - \gamma_i(p_i - \bar{p}_i)^{\alpha_i} - \gamma_j(p_j - \bar{p}_j)^{\alpha_j} - r_i]. \tag{26}$$

The terminal conditions are specified as

$$p_i(T) = \bar{p}_i + (S_i \alpha_i \gamma_i)^{-1/(\alpha_i - 1)}. \tag{27}$$

The following corollary follows from Proposition 4.1.

Corollary 5.1. Let be at least one $r_i > 0$ and

$$(\alpha_1 - 1)(\alpha_2 - 1) \geq 1. \tag{28}$$

Then, there exists a unique equilibrium $(\hat{p}_1, \hat{p}_2)$ in the interior of the admissible control demain, i.e., with $\hat{p}_i > \bar{p}_i$. If, additionally,

$$(\alpha_1 - 1)(\alpha_2 - 1) \geqslant \alpha_1 + \alpha_2 + 1, \tag{29}$$

then the stationary point $(\hat{p}_1, \hat{p}_2)$ is an unstable node.

Proof. (17) is equivalent to

$$\alpha_i > 2. \tag{30}$$

Under this assumption, (21) holds providing the existence of the interior equilibrium $(\hat{p}_1, \hat{p}_2)$. However, this result can be proved also under the weaker condition (28) (see Fig. 1). Let

$$d_i \triangleq \bar{p}_i - c_i.$$

The isocline

$$\dot{p}_1 = 0$$

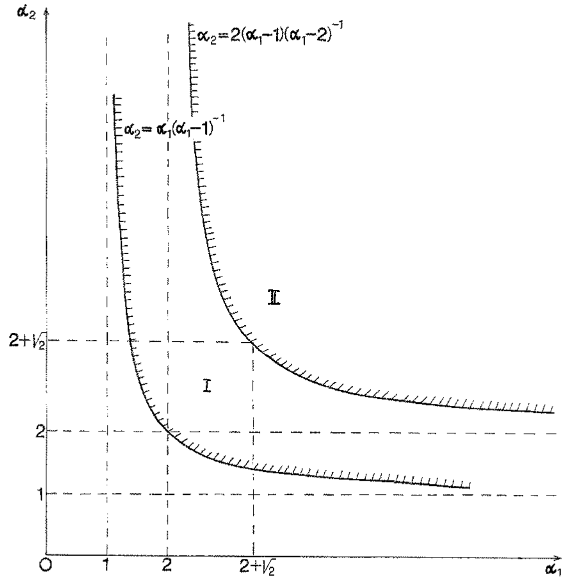

Fig. 1. Pairs of exponents α_1, α_2, for which an interior equilibrium does exist [region I described by (28)] and for which this stationary point is an unstable node [region II characterized by (29)).

can be written as

$$p_2=\bar{p}_2+[G(p_1)/\gamma_2]^{1/\alpha_2},$$

with

$$G(p_1)=(\alpha_1-1)\gamma_1(p_1-\bar{p}_1)^{\alpha_1}+\alpha_1\gamma_1 d_1(p_1-\bar{p}_1)^{\alpha_1-1}-r_1. \tag{31}$$

$F(p_1)$ defined in (20) has now the form

$$F(p_1)=(\alpha_2-1)G(p_1)+\alpha_2\gamma_2\alpha_2[G(p_1)/\gamma_2]^{(\alpha_2-1)/\alpha_2}-\gamma_1(p_1-\bar{p}_1)^{\alpha_1}-r_2. \tag{32}$$

It holds that

$$F(\tilde{p}_1)<0.$$

From (31) and (32), we obtain, together with (28),

$$\lim_{p_1\to\infty} F(p_1)=\infty.$$

Moreover, by differentiation of (32), we see that, for (28), it holds that

$$F'(p_1)>0.$$

This provides the existence and the uniqueness of an interior stationary point. For (29) and $p_i > \bar{p}_i$, we obtain from (24)

$$D > \gamma_1\gamma_2(p_1-\bar{p}_1)^{\alpha_1}(p_2-\bar{p}_2)^{\alpha_2}[2-\alpha_1\alpha_2/(\alpha_1-1)(\alpha_2-1)] \geq 0. \tag{33}$$

Since, according to (23a), also the elements in the main diagonal are positive, the stationary point $(\hat{p}_1, \hat{p}_2)$ is an exploding node.

Figure 1 illustrates the pairs of exponents (α_1, α_2) with (28) or (29), respectively. (29) means that the product of both elasticities e_1 and e_2 is greater or equal to 1. Note that (29) implies (28). Notice further that, from (29) and $\alpha_i > 1$, (30) follows. The boundaries of (28) and (29), respectively, are the hyperbola branches

$$\alpha_2 = \alpha_1(\alpha_1-1)^{-1}, \qquad \alpha_1 > 1,$$

$$\alpha_2 = 2(\alpha_1-1)(\alpha_1-2)^{-1}, \qquad \alpha_1 > 2,$$

respectively. Finally, it should be mentioned that the conditions (28) and (29) are sufficient but not necessary for the existence of a unique interior stable node. □

The equilibrium behavior at the boundary $p_i = \bar{p}_i$ of the admissible control region is summarized in the following proposition.

Proposition 5.1. For $r_1 > 0$, $r_2 > 0$, there exist three equilibria at the boundary, whereby $(\bar{p}_1, \bar{p}_2)$ is a stable node and $(\bar{p}_1, \tilde{p}_2)$, $(\tilde{p}_1, \bar{p}_2)$ are saddle points. For $r_1 > 0$, but $r_2 = 0$, there are two boundary equilibria, $(\bar{p}_1, \bar{p}_2)$ and $(\tilde{p}_1, \bar{p}_2)$, where $(\tilde{p}_1, \bar{p}_2)$ is a saddle point. For $r_1 = r_2 = 0$, there occurs only one equilibrium, namely $(\bar{p}_1, \bar{p}_2)$, provided that (18) is satisfied.

Proof. To derive the stability properties of the stationary points, we determine the Jacobian determinant of (22) and evaluate it at these points. From (22), we obtain

$$\begin{aligned}\partial\dot{p}_i/\partial p_i = {} & [1/(\alpha_i-1)][\gamma_i\alpha_i^2(p_i-\bar{p}_i)^{\alpha_i-1}(p_i-c_i) \\ & -\gamma_i(p_i-\bar{p}_i)^{\alpha_i}-\gamma_j(p_j-\bar{p}_j)^{\alpha_j}-r_i],\end{aligned} \tag{34}$$

$$\partial\dot{p}_i/\partial p_j = -[\gamma_j\alpha_j/(\alpha_i-1)](p_i-\bar{p}_i)(p_j-\bar{p}_j)^{\alpha_j-1}. \tag{35}$$

If we evaluate the Jacobian determinant at $(\bar{p}_1, \bar{p}_2)$ we get

$$D(\bar{p}_1, \bar{p}_2) = \begin{vmatrix} -r_1/(\alpha_1-1) & 0 \\ 0 & -r_2/(\alpha_2-1) \end{vmatrix},$$

providing the stable node property of $(\bar{p}_1, \bar{p}_2)$ for r_1 and r_2 both positive. Note that

$$D(\bar{p}_1, \bar{p}_2) = 0$$

if at least one of the discount rates vanishes. Note that the isocline

$$\dot{p}_i = 0$$

goes through $(\bar{p}_1, \bar{p}_2)$ if

$$r_i = 0.$$

Let us now evaluate D at $(\bar{p}_1, \tilde{p}_2)$. It holds that

$$\partial \dot{p}_1/\partial p_1|_{(\bar{p}_1,\tilde{p}_2)} = -[1/(\alpha_1 - 1)][\gamma_2(\tilde{p}_2 - \bar{p}_2)^{\alpha_2} + r_1], \tag{36}$$

$$\partial \dot{p}_2/\partial p_2|_{(\bar{p}_1,\tilde{p}_2)} = [1/(\alpha_2 - 1)] \times [\gamma_2 \alpha_2^2 (\tilde{p}_2 - \bar{p}_2)^{\alpha_2 - 1}(\tilde{p}_2 - c_2) - \gamma_2(\tilde{p}_2 - \bar{p}_2)^{\alpha_2} - r_2], \tag{37}$$

$$\partial \dot{p}_1/\partial p_2|_{(\bar{p}_1,\tilde{p}_2)} = [\partial \dot{p}_2/\partial p_1]_{(\bar{p}_1,\tilde{p}_2)} = 0.$$

Since, for $r_1 \geq 0$ but $r_2 > 0$, (36) is positive and (37) is negative, $(\bar{p}_1, \tilde{p}_2)$ is a saddle point. The remaining cases are shown analogously. Note that the functional determinant vanishes in a stationary point, if it is determined by three or more isoclines. Thus, in these cases, the stability property remains undetermined.

To show that, for $r_1 = r_2 = 0$, $(\bar{p}_1, \bar{p}_2)$ is the unique equilibrium, we prove that

$$\dot{p}_1 = 0$$

has always a steeper slope than

$$\dot{p}_2 = 0,$$

i.e., that (18) holds. Note that, for $\alpha_i \geq 2$, $i = 1, 2$, (17) is satisfied, which implies (18). Moreover, it is easy to show that (18) holds for

$$\alpha_1 \alpha_2 > \alpha_1 + \alpha_2,$$

i.e., if in (28) strict inequality holds. □

Before we draw some conclusions from this phase portrait analysis, for the behavior of the optimal pricing strategies we consider the case

$$\bar{p}_i = c_i. \tag{38}$$

It turns out that, under this assumption, the analysis becomes simpler and that the coordinates of the interior stationary point can be determined explicitly as solution of a system of two linear equations.

Proposition 5.2. Assume that (38) holds and that at least one r_i is positive. It is necessary and sufficient for the existence of a unique interior equilibrium $(\hat{p}_1, \hat{p}_2)$ that (28) holds as strict inequality. Moreover, $(\hat{p}_1, \hat{p}_2)$ is given explicitly by (44). For $r_i = 0$ and

$$\alpha_1 \alpha_2 = \alpha_1 + \alpha_2, \tag{39}$$

the interior isoclines coincide, providing an infinite set of equilibria on the curve (45).

Proof. For (38), the isoclines of the system (26) simplify to

$$\begin{aligned} &\gamma_1(\alpha_1-1)(p_1-\bar{p}_1)^{\alpha_1}-\gamma_2(p_2-\bar{p}_2)^{\alpha_2}=r_1,\\ &-\gamma_1(p_1-\bar{p}_1)^{\alpha_1}+\gamma_2(\alpha_2-1)(p_2-\bar{p}_2)^{\alpha_2}=r_2. \end{aligned} \tag{40}$$

(40) is a system of two equations, linear in $(p_i-\bar{p}_i)^{\alpha_i}$. Its solution is provided by

$$(p_i-\bar{p}_i)^{\alpha_i}=\Delta_i\Delta^{-1}, \tag{41}$$

with

$$\Delta=\gamma_1\gamma_2[\alpha_1\alpha_2-(\alpha_1+\alpha_2)], \tag{42}$$

$$\Delta_i=\gamma_j[r_i(\alpha_j-1)+r_j]. \tag{43}$$

Necessary and sufficient that, for the solution of (40), $p_i>\bar{p}_i$ holds is the condition

$$\alpha_1\alpha_2>\alpha_1+\alpha_2$$

(region I in Fig. 1). From (41), (42), (43), it follows that

$$\hat{p}_i=\bar{p}_i+[[r_i(\alpha_j-1)+r_j]/\gamma_i(\alpha_1\alpha_2-\alpha_1-\alpha_2)]. \tag{44}$$

For

$$r_1=r_2=0,$$

the system (40) is a homogeneous one, which is solvable if and only if

$$\Delta=0,$$

i.e., if (39) holds. The solutions satisfy

$$p_2=\bar{p}_2+[(\alpha_1-1)\gamma_1(p_1-\bar{p}_1)^{\alpha_1}/\gamma_2]^{1/\alpha_2}. \tag{45}$$

For

$$\alpha_1=\alpha_2=2,$$

e.g., (39) is satisfied, and (45) is the straight line

$$p_2=\bar{p}_2+(p_1-\bar{p}_1)\sqrt{(\gamma_1/\gamma_2)}. \tag{45a}$$

□

Note that the crossing points of $\dot{p}_i=0$ with $p_j=\bar{p}_j$, $(\bar{p}_2, \tilde{p}_1)$ and $(\bar{p}_1, \tilde{p}_2)$, respectively, can now be determined explicitly. It holds that

$$\tilde{p}_i=\bar{p}_i+[r_i/\gamma_i(\alpha_i-1)]^{1/\alpha_i}. \tag{46}$$

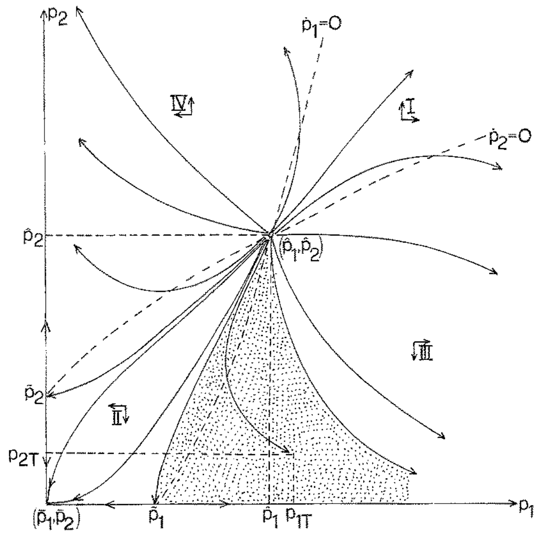

Fig. 2. Phase portrait of Nash optimal pricing strategies for positive discount rates.

By the results derived, we can gain qualitative insights into the structure of Nash-optimal solutions. The direction of the trajectories follows, e.g., from (34) and is depicted by arrowheads in Fig. 2. The behavior of the trajectories near $p_i = \bar{p}_i$ can be seen from (26); note that

$$dp_2/dp_1 = \dot{p}_2/\dot{p}_1.$$

The phase diagram analysis is summarized in Fig. 2.

To each terminal point (p_{1T}, p_{2T}), there exists a unique solution path emanating from the unstable equilibrium $(\hat{p}_1, \hat{p}_2)$. The initial point can be determined by backward integration of (11) over the planning period of length T.

The shape of the Nash solution path depends on the relation between the coordinates of the points $(\hat{p}_1, \hat{p}_2)$ and (p_{1T}, p_{2T}). Let us consider the trajectories for various salvage values S_i. According to (12), small values of S_1 and S_2 imply a terminal point in region I, i.e., monotonously increasing pricing strategies. If both S_i are large, then the prices of both firms decrease monotonously. For S_1 small enough, but S_2 large enough, we obtain a terminal point (p_{1T}, p_{2T}) with a relatively large abscissa, but small p_{2T}. In such a case, it seems economically reasonable that firm 1 increases its price, whereas firm 2 lowers it gradually (see region III in Fig. 2). For S_1 large, but S_2 small, the behavior is just the opposite (region IV). However, there exist also nonmonotonous Nash-optimal pricing strategies. Consider for instance a salvage value S_1 such that p_{1T} is not too far from $\hat{p}_1$, S_2 is small, and T is large enough. Then, $p_2(t)$ is upward sloping, whereas $p_1(t)$ increases first, but decreases later. Since, due to its increasing price policy, firm 1 is increasingly left by its customers, firm 2 can also initially afford an increasing price to obtain a larger profit, in trade-off for a loss of customers. Only in the second part of the planning interval is a decreasing price $p_1(t)$ charged to reach p_{2T} corresponding to S_1. If S_1 is again of medium size but S_2 is large, then $p_2(t)$ always decreases. If player 1 would increase its price, it would lose too many customers. On the contrary, it is optimal for firm 1 to accommodate its behavior to the price policy of its competitor. The region where policies of this shape are possible is marked in Fig. 2. Note that the resulting shape of the pricing strategies is economically reasonable in the sense of a Nash equilibrium. Thus, it is Nash optimal to reduce first the price so as not to drive away too many customers, but in the sequel to increase the price to meet the terminal condition. For values of p_{2T} near $\hat{p}_2$ and for S_1 small or large, the situation is symmetric.

Finally, let us take a look at the case

$$\bar{p}_i = c_i, \qquad r_1 = r_2 = 0.$$

As we have shown before, the two interior isoclines coincide, generating an infinite set of equilibria lying all at the curve provided by (45). In this case, we see that regions I and II (Fig. 2) are both empty. Hence, it is impossible that the pricing paths of the firms are both increasing or decreasing. If the terminal point (p_{1T}, p_{2T}) is left of the curve (45), $p_1(t)$ decreases, whereas $p_2(t)$ increases. If (p_{1T}, p_{2T}) is right of (45), the situation is the opposite. For simplicity, let us assume that

$$\alpha_1 = \alpha_2 = 2 \quad \text{and} \quad \gamma_1 = \gamma_2.$$

Then, (45) is given by

$$p_2 = p_1 + \bar{p}_2 - \bar{p}_1. \tag{45b}$$

If

$$S_1 > S_2,$$

then

$$p_{1T} < p_{2T},$$

i.e., the Nash optimal price strategy of firm 1 is downward sloping, whereas $p_2(t)$ increases. Note that this behavior makes economic sense. The general case (45) has additionally taken into consideration γ_i and α_i.

6. Concluding Remarks

The main feature of the system dynamics of the pricing game analyzed above is the change in the customers' stock of firm 1 as a net result of the losses due to its price policy and of the increase due to the competitors' pricing. It is assumed that the customers of firm i endure a maximal price level $\bar{p}_i$, but that they begin to leave the firm i as soon as the price charged by this firm exceeds the critical level $\bar{p}_i$. This diffusion process is set in motion gradually because of loyality of the customers to their firm, lower distances to the warehouses of the firms, etc. If the customers would be informed on the production costs c_i, then it would be reasonable to set the critical price level at which diffusion takes place above a level not smaller than c_i, i.e., to make assumption (3). For $\bar{p}_i < c_i$, no results on the existence and uniqueness of an interior equilibrium are available. However, if there is such a stationary point and (29) is satisfied, the equilibrium is an unstable node.

Two reasons are responsible for the fact that we restrict ourselves to a dynamic duopoly. First, a phase diagram analysis can be illustrated generally only in two dimensions. Second and more important, the reduction of the optimal control problem to the solution of an initial-value problem for a system of differential equations for the control variables with given terminal conditions can be carried out only for $n = 2$ players.

The Nash-optimal pricing strategies derived by phase portrait analysis have been discussed at the end of the preceding section. The most interesting cases are the nonmonotonous optimal price policies isolated for player i, if S_i has medium values and the time horizon is appropriate. More precisely, there exist four regions in the (p_1, p_2)-phase plane, so that, for a planning horizon large enough, either $p_1(t)$ or $p_2(t)$ has initially the same behavior as the monotonous pricing strategy of the competitor, but changes this trend in the second part of the planning interval. Whereas the terminal pricing depends on S_1 or S_2, respectively, the initial parallel behavior is due to the Nash solution concept.

From (12), we see that an increase in the intensity of diffusion $g_i(\cdot)$ implies, *ceteris paribus*, a decrease of the terminal price p_{iT}. This behavior makes economic sense: the higher the risk of a loss in market share, the more prudent the optimal pricing policy will be.

For differential games with infinite time horizon, $T = \infty$, the equilibrium solution

$$p_i(t) \equiv \hat{p}_i, \qquad i = 1, 2, \tag{47}$$

is optimal, since the sufficient conditions are satisfied (see, e.g., Refs. 17 and 18).

The main purpose of our investigation was to illustrate the solution technique, rather than to claim economic significance of the proposed model.

The most unrealistic feature of the model analyzed is the assumption that the flow of customers from firm i to firm j depends only on the price level of firm i. This assumption has been made to guarantee state separability and qualitative solvability of the differential game. A more realistic model must take into consideration the push-and-pull character of the price level. Thus, it would be reasonable to suppose that the customers' flow depends on the difference of the prices charged by both competitors. Assume two convex diffusion functions $h_i(p_2-p_1)$, such that

$$h_1(p_2-p_1)=\begin{cases}0 & p_2\geq p_1,\\ \text{monotonously decreasing and convex,} & p_2<p_1,\end{cases}$$

$$h_2(p_2-p_1)=\begin{cases}\text{monotonously increasing and convex,} & p_2>p_1,\\ 0, & p_2\leq p_1.\end{cases}$$

The dynamics of the market share of firm 1 is governed by the differential equation

$$\dot{x}=-h_1(p_2-p_1)x+h_2(p_2-p_1)(1-x). \tag{48}$$

The first term on the right-hand side of (48) describes the diffusion of customers from firm 1 to firm 2 due to the price difference p_2-p_1. The second term refers to the diffusion from firm 2 to firm 1. Note that, for $p_i>p_j$, there is no flow from firm j to i.

From (48), we see that

$$\dot{x}>0, \qquad p_2>p_1,$$

$$\dot{x}<0, \qquad p_1>p_2.$$

Thus, in equilibrium it holds that

$$\hat{p}_1=\hat{p}_2.$$

The crucial assumption for the qualitative solvability of the model is the absence of the state variable from the Hamiltonian maximizing conditions as well as from the adjoint equations. It is this property of the differential game which makes it possible to reduce the boundary-value problem resulting from the necessary condition of Pontryagin's maximum principle to a terminal value problem which can be solved in a much simpler way. This trick has been used also in some other examples which are of interest in economics and management science (see Refs. 19 and 20).

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