Tractable Classes of Nonzero-Sum Open-Loop Nash Differential Games: Theory and Examples[1]

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Abstract. This paper identifies some classes of N-person nonzero-sum differential games that are tractable, in the sense that open-loop Nash strategies can be determined, either explicitly or qualitatively in terms of a phase-diagram portrait. The classes are characterized by conditions imposed on the Hamiltonians. Also, the underlying game structures needed to satisfy these conditions are characterized.

Key Words. Differential games, open-loop Nash equilibria, solvability, state separability, redundancy.

1. Introduction

This paper deals with the following problem. Which are the structural assumptions that could be made to obtain solutions to an N-person, nonzero-sum, open-loop Nash differential game? By solutions we mean, loosely speaking, that the controls, the state variables, and the adjoint variables can be explicitly specified as functions of time. Also, we consider the possibilities of obtaining insights into the qualitative behavior of the solution trajectories, for example in terms of phase diagrams. We assume that the Hamiltonians are nonlinear in the control variables.

Clemhout and Wan write (Ref. 1, p. 419):

"Much has been established both for the necessary conditions and the sufficient conditions pertaining to the optimal play in N-person, general

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sum differential games. The major difficulty blocking the application of such game models is that equilibrium strategies are extremely hard to determine or characterize. Specifically, the adjoint system arising from such games usually involves partial derivatives of unknown optimal strategies. Beside the linear-quadratic games and some special examples, the literature contained no other known class of solvable N-person general sum games until recently."

Clemhout and Wan (Ref. 1) define a class of differential games (hereafter DG's, for short), the trilinear games, which allows for qualitative insights. They write (Ref. 2, p. 19):

"Unlike linear-quadratic games, trilinear differential games can be easily scrutinized for qualitative insights. The simplicity of the game structure is due to the fact that the state variables are absent from the costate system and the choice of controls is independent of the value of x (the state variables)."

The term "trilinear" refers to the linearity of the Hamiltonians in the state and costate variables as well as in functions of the control variables (compare Refs. 1-6).

As will be demonstrated in Section 2, it is the absence of the state variables from the Hamiltonian maximizing conditions as well as from the adjoint equations that is a prerequisite for the possibility of getting solutions in a DG. In many cases[5] the linearity of the Hamiltonians in the state variables as well as the optimal controls' independence of the state variables is crucial for solvability. DG's that possess these properties will be termed *state-separable games*, since the determination of Nash optimal controls can be done separately from the determination of the state variables. For two-person DG's with scalar control variables, this creates a possibility of determining the qualitative behavior of the solutions by using phase-plane analysis.

After Refs. 1, 2, 4-6, not very much interest has been devoted to the subject of solvability of DG's; see, however, Ref. 3, Chapter 9; Ref. 9, pp. 16-27; and Ref. 10. What seems to have escaped observation is the fact that, besides the trilinear games, there are other classes of DG's, the solution of which can be explicitly found or qualitatively described. Examples of DG's that are not trilinear (but are state separable) are Refs. 11 and 12. Other games where a phase-diagram approach yields important insights into the structure of optimal strategies are, e.g., Refs. 5 and 7.

In Section 2, we give a formal presentation of the DG's under study, and the basic concepts are defined. In this section, we also state the main conclusions in the form of a series of propositions. In Section 3, with a

[5] Exceptions are, for instance, Refs. 7 and 8.

view to applications, we proceed to characterize, in terms of state equations and performance indices, the structure needed to guarantee various types of solvability. The results are illustrated in Section 4 by different management science applications. In Section 5, we give some results on the stability properties of interior equilibria. Finally, in Section 6, we summarize the results and state some unsolved problems.

2. Basic Definitions and Main Results

We consider N-person nonzero-sum DG's, $N \geq 2$, and we seek open-loop Nash equilibria. The state of the game is given by the n-dimensional column vector

$$x = (x_1, \ldots, x_n)', \qquad n \geq 1.$$

The prime denotes the transposition of a vector. Each player controls a vector control variable $u^i \in \mathbb{R}^{m_i}$, where

$$m_i \geq 1, \qquad i = 1, \ldots, N.$$

For

$$u = (u^1, \ldots, u^N)',$$

the dynamics of the game are given by the system of ordinary differential equations,

$$\dot{x} = f(x, u, t), \qquad x(0) = x_0 \text{ fixed}, \tag{1}$$

where

$$f = (f_1, \ldots, f_n)'.$$

Admissible controls u^i belong to prescribed control regions $U^i \subseteq \mathbb{R}^{m_i}$. Performance indices are in Bolza form,

$$J^i(u) = \int_{t_0}^{T} \exp(-r^i t) L^i(x, u, t)\, dt + \exp(-r^i T) S^i(x(T), T), \tag{2}$$

where the horizon date T is fixed. Functions f_j, L^i, S^i are assumed to satisfy appropriate smoothness conditions, e.g., they are in C^2. The controls are open-loop, i.e.,

$$u^i = u^i(t, x_0).$$

The players are assumed to be J^i-maximizers.

A Nash equilibrium N-tuple

$$u^* = (u^{1^*}, \ldots, u^{N^*})$$

is defined as follows (see, e.g., Ref. 13). Let

$$u^* = (u^{1^*}, \ldots, u^{N^*})$$

and

$${}^i u = (u^{1^*}, \ldots, u^{i-1^*}, u^i, u^{i+1^*}, \ldots, u^{N^*}),$$

where u^i is arbitrarily chosen in U^i. Then,

$$J^i(u^*) \geq J^i({}^i u),$$

for all ${}^i u \in U^i$, for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, and for all $i = 1, \ldots, N$.

Let

$$\lambda(t) = [\lambda_j^i(t)]$$

be an $N \times n$ matrix of current-value adjoint variables (see Refs. 13-15); and let

$$\lambda^i = (\lambda_1^i, \ldots, \lambda_n^i)$$

denote the ith row of λ. Note that, here and in the sequel, we set

$$i = 1, \ldots, N, \qquad j = 1, \ldots, n.$$

Then,

$$H^i = H^i(x, u, \lambda^i, t) = L^i(x, u, t) + \sum_{j=1}^{n} \lambda_j^i(t) f_j(x, u, t) \tag{3}$$

is the current-value Hamiltonian of player i (see Refs. 13 and 14). The adjoint variable λ_j^i can be interpreted as the shadow price of the state variable x_j as assessed by player i.

Necessary conditions for an open-loop Nash equilibrium are given by the adjoint equations

$$\dot{\lambda}^i = r^i \lambda^i - H_x^i, \tag{4}$$

the terminal condition

$$\lambda^i(T) = S_x^i, \tag{5}$$

and the H^i-maximizing condition

$$H^i(x^*, u^*, \lambda^i, t) \geqq H^i(x^*, {}^i u, \lambda^i, t), \qquad {}^i u \in U^i. \tag{6}$$

In cases where U^i is open or when u^{i^*} belongs to the interior of a compact control region U^i, condition (6) can be replaced by

$$H_{u^i}^i(x^*, u^*, \lambda^i, t) = 0. \tag{7}$$

Note that, with H^i nonlinear in u and bounded controls, boundary solutions normally can be excluded by imposing appropriate assumptions on the functions defining the state equations and the integrands of the criteria.

To determine whether the N-tuple u^* is indeed optimal, a sufficiency condition has to be invoked, for instance, the sufficiency condition in Ref. 15 (Corollary 1). Essentially, the condition requires the concavity in x of $\max_{u^i} H^i$ or the concavity of H^i in (x, u^i), for all (λ^i, t). Concavity of the Hamiltonians in the state variables x and the optimal control's independence of x imply the concavity of $\max_{u^i} H^i$ with respect to x, because we restrict ourselves to open-loop solutions. This provides the sufficiency of the optimality conditions in almost all examples occurring in the following; exceptions are the research game treated in Ref. 7 and a pricing-investment game; see Refs. 8 and 17. Moreover, the function S^i must be concave in $x(T)$.

In the case of an infinite horizon, which is important in many economic and management science problems, the transversality conditions (5) are no longer necessary conditions, in general. However, provided the concavity condition holds, then the following limiting transversality conditions are sufficient for problems with free right-hand endpoints:

$$\begin{aligned}&\lim_{T\to\infty} \exp(-r^iT)\lambda_j^i(T)=0,\\ &\lim_{T\to\infty} \exp(-r^iT)\lambda_j^i(T)[x_j(T)-x_j^*(T)]\geq 0.\end{aligned} \tag{8}$$

In most applications, the system of necessary conditions does not admit an explicit solution, since the necessary conditions imply an intractable two-point boundary-value problem. Nevertheless, for qualitatively solvable DG's (see Definition 2.1), important insights into the structure of the Nash equilibrium can sometimes be obtained using phase-diagram and stability analysis: for such games, only a terminal value problem has to be solved.

Definition 2.1. Assume that u^i is differentiable with respect to time. A DG is qualitatively Nash solvable if the following system of differential equations can be derived from the necessary conditions:

$$\dot{u}^i = \phi^i(u, t). \tag{9}$$

Here and in the sequel, we suppress the star notation for optimal solutions. Note that, for dimensions higher than $N=2$, $m_i=1$, no qualitative insights are possible, since the theory of phase-diagram analysis in higher-dimensional spaces is not well developed. Also note that, even in the case $N=2$, $m_i=1$, the qualitative behavior in the (u^1, u^2)-plane of the equilibrium and the optimal trajectories may be very hard or impossible to characterize.

In some DG's, it turns out that one or more of the adjoint variables are of no importance in the determination of u^*. This motivates the following definition.

Definition 2.2. An adjoint variable λ_j^i is redundant if $H_{u^i}^i$ and $H_{x_k}^i$ do not contain λ_j^i, for all $k = 1, \ldots, n$, with $k \neq j$, where the partial derivatives are evaluated along the optimal path.

Thus, redundancy of an adjoint variable λ_j^i means that u^{i*} is independent of λ_j^i. In Section 3, we exhibit game structures that imply redundant adjoint variables.

As mentioned in the introduction, a special interest relates to state-separable DG's. The concept of state-separability is defined as follows.

Definition 2.3. A DG is state separable if

$$H_{u^i x}^i \big|_{H_{u^i}^i = 0} = 0, \tag{10a}$$

$$H_{xx}^i = 0. \tag{10b}$$

Condition (10a) states that $H_{u^i}^i$, maximized with respect to u^i, is independent of the state variables. Condition (10b) states that H^i is linear in the state variables. The implications of state separability are that u^{i*} is expressed only in terms of adjoint variables and that the adjoint equations do not contain the state variables. Thus, the essential feature is that the determination of the controls and the adjoint variables is separated from the determination of the state variables. The generalized trilinear games of Clemhout and Wan (Refs. 1, 2, 4–6) are state separable.

Note that, in the state-separable case, the open-loop Nash controls are independent of the initial state. By invoking sufficient conditions for feedback Nash equilibrium controls (Ref. 16), it can be shown that open-loop Nash equilibria are indeed feedback Nash solutions. As in Refs. 1, 4, 20 and related papers, we have the special situation in which there is a feedback control being constant with respect to the state and depending only on time.

The following proposition connects Definitions 2.1, 2.2, 2.3. We have to discern whether the total number of control variables

$$M := \sum_{i=1}^{N} m_i$$

is greater than or equal to the number of adjoint variables,

$$M \geq Nn, \tag{11}$$

or whether this is not the case, i.e.,

$$M < Nn. \tag{12}$$

Proposition 2.1. Consider a state-separable DG. If (11) is satisfied, then the game is qualitatively solvable. If (12) holds and at least $Nn-M$ adjoint variables are redundant, then the game is qualitatively solvable.

Proof. Assumption (10b) implies that the adjoint variables are independent of the state variables. Moreover, by assumption (10a), it holds that the M Hamiltonian maximizing conditions do not contain x. Applying the theorem of implicit functions, we obtain, under appropriate regularity conditions,

$$u^i = \kappa^i(\bar{u}^i, \lambda^1, \ldots, \lambda^N, t), \tag{13}$$

where $\bar{u}^i$ is defined as

$$\bar{u}^i = (u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^N). \tag{14}$$

With reasonable assumptions on L^i and f_j, then u^i is differentiable with respect to time. Differentiation in (13) with respect to time and substitution of the adjoint equations and the Hamiltonian maximizing conditions yields the first part of the proposition. Note that here assumption (11) has been used.

By (10), we see that the Hamiltonian maximizing conditions are a system of linear equations in the nonredundant adjoint variables. Denote these variables by $\lambda_1, \ldots, \lambda_K$, where

$$K \leq M, \quad \lambda_i \neq \lambda_j, \qquad i, j = 1, \ldots, K.$$

By using the second assumption of the proposition, we can calculate

$$\lambda_i = \psi_i(u, t). \tag{15}$$

Differentiation in (15) with respect to time yields

$$\dot{\lambda}_i = \sum_{h=1}^{N} (\partial\psi_i/\partial u^h)\dot{u}^h + (\partial\psi_i/\partial t). \tag{16}$$

Substituting (15) and (16) into (4) yields the system (9). □

In the *autonomous case* (i.e., when the functions f, L^i, S^i are not explicitly dependent of time), then the differential equations (9) can, at least in principle, be analyzed in the u-space using phase-diagram analysis. See, for example, Refs. 7, 18, 19. The terminal conditions are

$$\lambda_i^i(T) = \psi^i(u(T)) = S^i_{x_i}(x(T)). \tag{17}$$

(17) can only be evaluated if S^i is linear in x_i.

Note that a state-separable DG with scalar state variable is qualitatively solvable. Also note that the conditions of Proposition 2.1 are sufficient, but not necessary; i.e., there exist qualitatively solvable DG's that are, for instance, not state separable. Such a game is studied in Ref. 7 (see also Refs. 8 and 17). Finally, note that redundancy in itself does not guarantee qualitative solvability; see, e.g., the advertising game in Ref. 3.

A slightly more restrictive assumption than state separability is obtained by deleting in (10a) the qualification

$$H^i_{u^i} = 0,$$

i.e., assuming

$$H^i_{u^i x} = 0, \qquad H^i_{xx} = 0. \tag{18}$$

The advertising game studied in Ref. 20 has this property.

The next proposition states a sufficient condition for a DG to be explicitly solvable. By this, we mean that the controls can be explicitly specified as functions of time. Of course, this assumes that all relevant functions are specified.

Proposition 2.2. If a DG has the property

$$H^i_{ux} = H^i_{xx} = 0, \tag{19}$$

then it is explicitly solvable.

Proof. (19) implies that the adjoint equations are independent of x. Also, the candidates for Nash optimality do not depend on x; i.e., u^i is given by (13). The adjoint equations are a system of linear differential equations which can be analytically solved. This yields

$$\lambda^i_j = \lambda^i_j(t).$$

Substituting these adjoint variables into κ^i yields

$$u^i = \eta^i(\bar{u}^i, t),$$

which can be solved by invoking the theorem of implicit functions. □

Note that there exists DG's that are explicitly solvable even if (19) does not hold. Examples are linear-quadratic games and the games studied in Refs. 11 and 17 (see also Ref. 21).

If, in addition to (19), we assume

$$H^i_{u^i \bar{u}^i} = 0,$$

then the DG turns into N independently solvable optimal control problems. Examples of this are found in Refs. 22 and 23. Note that the condition

$$H^i_{xx} = 0$$

may be slightly weakened; $H^i_{x_jx_j}$ need not be equal to zero. However, it could be maintained that such problems are in a sense degenerate games, since player i (under optimal play) determines his control independently of $\bar{u}^{i^*}$, the state x_j, and the costates λ^i_j. Note that there also exist DG's where one player solves an independent optimal control problem, but the optimal strategy of the other player depends on that of his competitor (see, e.g., Refs. 18 and 21).

In Proposition 2.1, it was shown that state separability plus redundancy imply qualitative solvability. In Proposition 2.2, we demonstrated that (19) implies explicit solvability, and therefore also qualitative solvability.

3. Game Structures and Solvability

Consider the following five sets of assumptions concerning the Hamiltonians, each reflecting a particular structure of the DG's model:

(A1) $H^i_{ux} = H^i_{xx} = H^i_{uu} = 0$;

(A2) $H^i_{ux} = H^i_{xx} = H^i_{u^i\bar{u}^i} = 0$;

(A3) $H^i_{ux} = H^i_{xx} = 0$;

(A4) $H^i_{u^ix} = H^i_{xx} = 0$;

(A5) $H^i_{u^ix}|_{H^i_{u^i}=0} = 0$; $H^i_{xx} = 0$.

Here, Assumption (Ap) implies Assumption (A9) for $p < q$ and $p, q = 1, \ldots, 5$. Note that each of Assumptions (A1)-(A3) is sufficient, but not necessary, for qualitative solvability.

With a view to applications, we express Assumptions (A1)-(A5) in terms of assumptions regarding the state equations and the performance indices. In this way, it can easily be ascertained whether a particular DG's model is qualitatively or explicitly solvable. We also give examples of DG's that possess these structural properties.

The condition (10b) is present in all five assumptions and is also needed in Propositions 2.1 and 2.2. Note, however, that there exist qualitatively solvable DG's in which H^i is nonlinear in the state. To guarantee satisfaction of condition (10b), the state equations and the integrand of the performance indices must be linear in the state, i.e.,

$$f(x, u, t) = g(u, t)x + h(u, t),$$
$$L^i(x, u, t) = M^i(u, t)x + N^i(u, t). \tag{20}$$

Now, we give the following definition.

Definition 3.1. A DG is *state-control separated with respect to dynamics and objectives* (SCSDO) if

$$f(x, u, t) = g(x, t) + h(u, t),$$
$$L^i(x, u, t) = M^i(x, t) + N^i(u, t); \tag{21}$$

see also Ref. 9, p. 16.

In (A1)-(A3), we have the condition

$$H^i_{ux} = 0.$$

This condition will hold if the game has the SCSDO-property, and vice versa. Note that, if the game has the SCSDO-property and the functions g and M^i are linear, then (10b) also holds.

In (A4), the condition

$$H^i_{u^i x} = 0$$

will hold whenever functions f_j and L^i have the following structure:

$$f_j(x, u, t) = g_j(x, \bar{u}^i, t) + h_j(u, t),$$
$$L^i(x, u, t) = M^i(x, \bar{u}^i, t) + N^i(u, t). \tag{22}$$

The structure given by (22) is, however, not necessary for (10a) to hold.

The condition $H^i_{uu} = 0$ in (A1) means that H^i must be linear in u; i.e.,

$$f(x, u, t) = \sum_{i=1}^{N} g^i(x, t) u^i,$$
$$L^i(x, u, t) = \sum_{i=1}^{N} M^i(x, t) u^i. \tag{23}$$

Here, we depart from the assumption that the Hamiltonians are nonlinear in the controls.

The weaker condition $H^i_{u^i \bar{u}^i} = 0$ in (A2) will be satisfied whenever the game has the following structure:

$$f(x, u, t) = \sum_{i=1}^{N} g^i(x, u^i, t),$$
$$L^i(x, u, t) = \sum_{i=1}^{N} M^i(x, u^i, t); \tag{24}$$

i.e., the controls are separated in the dynamics as well as in the criteria.

Consider the special case $n = N$.

Definition 3.2. A DG is said to have *noninteracting dynamics with respect to state and controls* (NIDSC) if

$$f(x, u, t) = (f_1(x_1, u^1, t), \ldots, f_n(x_n, u^n, t)). \tag{25}$$

Games with this property are treated in Refs. 24 and 25; see also Ref. 9, p. 21.

In Definition 3.2 we may think of x_i as the state variable of player i. This is in fact the case in many economic and management science problems. For example, x_i may represent the market share or sale rate of firm i.

Proposition 3.1. If a DG has the NIDSC-property, then all $N(n-1)$ adjoint variables λ^i_j, $i \neq j$, are redundant.

Proof. By assumption, for $i \neq j$, we have

$$f_{jx_i} = f_{ju^i} = 0,$$

which implies that

$$H^i_{u^i} = 0$$

does not contain the adjoint variables λ^i_j. Also,

$$\dot{\lambda}^i_i = r^i \lambda^i_i - H^i_{x_i}$$

is independent of λ^i_j. In conclusion, the strategies u^i can be determined independently of λ^i_j. □

Definition 3.3. A DG is said to have *noninteracting dynamics and objectives with respect to state* (NIDOS) if

$$\begin{aligned} f(x, u, t) &= (f_1(x_1, u, t), \ldots, f_n(x_n, u, t)), \\ L^i(x, u, t) &= L^i(x_i, u, t). \end{aligned} \tag{26}$$

Proposition 3.2. An adjoint variable λ^i_j vanishes identically for all t, if the game has the NIDOS-property and if $\lambda^i_j(T) = 0$.

Proof. The adjoint equation is

$$\begin{aligned} \dot{\lambda}^i_j &= r^i \lambda^i_j - H^i_{x_j} = r^i \lambda^i_j - L^i_{x_j} - \sum_{k=1}^{n} \lambda^i_k f_{kx_j} \\ &= \lambda^i_j [r^i - f_{jx_j}(x_j, u, t)]. \end{aligned}$$

The solution of this equation is

$$\lambda^i_j(t) = \lambda^i_j(T) \exp\left\{ \int_t^T f_{jx_j}\, dt - r^i (T - t) \right\}. \tag{27}$$

Since

$$\lambda_j^i(T) = 0$$

was assumed, $\lambda_j^i(t)$ must vanish identically for all t. □

4. Examples

In the preceding part, we have considered the general case,

$$N \geq 2, \qquad n \geq 1, \qquad m_i \geq 1.$$

The essential feature of qualitative solvability is the derivation of the differential equations system (9). However, from an applications point of view, this system may not be very useful. It cannot be analyzed (in control space), unless we restrict ourselves to the case

$$N = 2, \qquad m_i = 1,$$

i.e., two-player games with scalar controls. Note that the number of state variables does not matter in this connection. The reason for this intractability is that the qualitative theory of differential equations in higher-dimensional spaces is not very well understood; see, e.g., Ref. 26.

Therefore, in this part, we consider the case

$$\dot{u}^i = \phi^i(u_1, u_2), \qquad i = 1, 2, \tag{28}$$

where u^i is scalar, assuming also that system (9) is autonomous.

In the following, we present some simple examples illustrating each of the Assumptions (A1)–(A5) listed above. Here, we consider the case

$$N = 2, \qquad n \leq 2, \qquad m_i = 1. \tag{29}$$

The dynamics as well as the objectives have been slightly simplified.

Example 4.1. (*Ref. 27*). Here, $n = 1$,

$$f(x, u^1, u^2) = u^1 - u^2,$$

$$L^i(x, u^1, u^2) = x_i - u^i,$$

where

$$x_j = 1 - x_i.$$

Example 4.2. (*Ref. 22*). Here, $n = 1$,

$$f(x, u^1, u^2) = \log u^1 - \log u^2,$$

$$L^i(x, u^1, u^2) = x_i - u^i,$$

where

$$x_j = 1 - x_i.$$

Example 4.3. (*Ref. 28*). Here, $n = 2$,

$$f = (f_1, f_2) = (u^1 - (u^1)^2 - u^2u^1 - x_1,\ u^2 - (u^2)^2 - u^1 - x_2),$$

$$L^i(x, u_1, u_2) = -x_i - u_i.$$

Example 4.4. (*Ref. 20*). Here, $n = 2$,

$$f = (f^1, f^2), \qquad \text{where } f^i = u^i - (u^i)^2 - x_i - x_i u^j,$$

$$L^i(x, u^1, u^2) = x_i - u^i.$$

Example 4.5. (*Ref. 11*). Here, $n = 1$,

$$f(x, u^1, u^2) = u^1u^2(1 - x),$$

$$L^i(x, u^1, u^2) = c^i(u^i)(1 - x) - u^1u^2(1 - x) - x,$$

where the function c^i is concave.

5. Stability Properties of Interior Equilibria

In this section, we assume that the functions f, L^i, S^i do not depend explicitly on time, and we restrict ourselves to the case (29). In the following three propositions, we assume, for $n = 2$, that

$$\lambda^i_j \text{ are redundant for } i = 1, 2,\ j = 1, 2,\ i \neq j. \tag{30}$$

Proposition 5.1. Assume that (A2) and (30) hold, and that

$$r^i - f_{ix_i} > 0. \tag{31}$$

If in

$$\lambda^i_i = \psi^i(u^i) \tag{32}$$

the function ψ^i is monotonic and

$$(\psi^i)^{-1}(L^i_{x_i}/(r^i - f_{ix_i})) \in U^i, \tag{33}$$

then there exists a unique equilibrium in the interior of the admissible domain U^i being an unstable node. Moreover, the isoclines are parallel to the coordinate axes.

Proof. From (7), (A2), (30), we obtain (32). According to (A2), the game is qualitatively solvable, and (9) may be written as

$$\dot{u}^i = [\psi^i_{u^i}(u^i)]^{-1}[\psi^i(u^i)(r^i - f_{ix_i}) - L^i_{x_i}]. \tag{34}$$

Note that f_{ix_i} as well as $L^i_{x_i}$ are constant. The isoclines

$$\dot{u}^i = 0$$

are given by

$$\psi^i(u^i) = L^i_{x_i}(r^i - f_{ix_i}). \tag{35}$$

From (35) and (31), we get

$$\partial \dot{u}^i/\partial u^i|_{\dot{u}^i=0} = r^i - f_{ix_i} > 0,$$

$$\partial \dot{u}^i/\partial u^j|_{\dot{u}^i=0} = 0, \qquad i \neq j.$$

Thus, the isoclines are parallel to the axes and the stationary point is an unstable node.

If ψ^i is monotonic in u^i and

$$L^i_{x_i}(r^i - f_{ix_i})^{-1} \in \psi^i(U^i),$$

then the solution of (35) is unique. □

Proposition 5.1 is illustrated by Example 3.2 and the following example.

Example 5.1. (*Ref. 29*). Here, $n = 1$,

$$f(x, u^1, u^2) = g^1(u^1) - g^2(u^2),$$

$$L^i(x, u^i) = x - u^i,$$

where the functions g^i are concave. We have taken a modification of the objective functional.

If we replace (A2) in Proposition 5.1 by (A3), then the following proposition can be proved.

Proposition 5.2. Under the Assumptions (A3), (30), (31), and if

$$r_1 = r_2 = r, \tag{36}$$

$$f_{1x_1} = f_{2x_2}, \tag{37}$$

each stationary point in the interior of the admissible domain is an unstable node.

Proof. Using the necessary optimality conditions (4) and (7), we can derive a differential equations system in the controls,

$$\dot{u}^i = \Delta^i/\Delta, \tag{38}$$

where

$$\Delta = \psi^1_{u^1}\psi^2_{u^2} - \psi^1_{u^2}\psi^2_{u^1},$$

with

$$\lambda^i_i = \psi^i(u^1, u^2),$$

from (7), and

$$\Delta^i = \psi^j_{u^i}[\psi^i(r^i - f_{ix_i}) - L^i_{x_i}] - \psi^i_{u^j}[\psi^j(r^j - f_{jx_j}) - L^j_{x_j}].$$

From (38), we obtain

$$\partial\dot{u}^i/\partial u^i|_{\dot{u}^i=0} = (\partial\Delta^i/\partial u_i)/\Delta, \tag{39}$$

$$\partial\dot{u}^i/\partial u^j|_{\dot{u}^i=0} = (\partial\Delta^i/\partial u^j)/\Delta, \qquad j \neq i. \tag{40}$$

Evaluating (39) and (40) at the stationary point and using the assumptions (36), (37), (20), we get

$$\partial\dot{u}^i/\partial u^i|_{\dot{u}^i=0,\dot{u}^j=0} = r - f_{ix_i} > 0, \tag{41}$$

$$\partial\dot{u}^i/\partial u^j|_{\dot{u}^i=0,\dot{u}^j=0} = 0, \qquad j \neq i. \tag{42}$$

Thus, the Jacobian determinant of the system (38) is

$$(r - \partial\dot{x}_i/\partial x_i)^2 > 0.$$

(41) and (42) provide the result of the proposition. □

Note that we have proven that each stationary point is an unstable node, but not the existence or uniqueness of an interior stationary point.

Examples of games satisfying the assumptions in Proposition 5.2 are Example 3.3 and the following example.

Example 5.2. (*Ref. 23*). Here, $n = 1$,

$$f(x, u^1, u^2) = g(u^1, u^2),$$

$$L^i(x, u^1, u^2) = x_i - u^i,$$

where $g_{u^iu^j} \neq 0$.

It is interesting to note that, for the games considered above, only unstable nodes are possible as stationary solutions. If we consider the class of games which are state separable and where the adjoint variables λ^i_j are redundant for $i \neq j$, then there are two possible types of interior stationary solutions: either unstable nodes or saddle points.

Proposition 5.3. Under the Assumptions (A5), (30), (31), and

$$H^i_{u^i u^j} = 0, \qquad \text{for } i \neq j,$$

the interior stationary solutions are, according to the sign of the Jacobian determinant, either unstable nodes or saddle points.

Proof. By assumption, we get from Proposition 2.1 that the differential game is qualitatively solvable, and we can derive the following set of differential equations in the controls:

$$\dot{u}^i = [\psi^i_{u^i}(u^i)]^{-1}[r^i \psi^i(u^i) - H^i_{x_i}], \tag{43}$$

where we have used (32). The isoclines

$$\dot{u}^i = 0$$

have the form

$$r^i \psi^i(u^i) - H^i_{x_i} = 0.$$

The elements of the Jacobian determinant evaluated at a stationary point are

$$\partial \dot{u}^i / \partial u^i |_{\dot{u}^i = 0} = [\psi^i_{u^i}(u^i)]^{-1}[r^i \psi^i_{u^i}(u^i) - H^i_{x_i \lambda^i_i}(\partial \lambda^i_i / \partial u^i)]$$
$$= r^i - f_{ix_i}, \tag{44}$$

$$\partial \dot{u}^i / \partial u^j |_{\dot{u}^i = 0} = -H^i_{x_i u^j} / \psi^i_{u^i}(u^i). \tag{45}$$

From (44) and (45), we see that the functional determinant of the system (43) can either be positive or negative. This, together with (44), shows the proposition. □

Note that neither existence nor uniqueness of a stationary point was proved.

Proposition 5.3 is illustrated by the following examples.

Example 5.3. (*Ref. 30*). Here, $n = 2$,

$$f(x, u^1, u^2) = (g^1(u^1) - x_1 - \alpha^1(u^2)x_1, g^2(u^2) - x_2 - \alpha^2(u^1)x_2),$$

$$L^i(x, u^1, u^2) = x_i - u^i,$$

where g^i is concave and α^i increases monotonically.

Remark 5.1. If

$$H^i_{x_i u^j} = 0$$

holds for one i, then the Jacobian determinant has a positive sign, implying that the stationary point is an unstable node. This is illustrated by the following example.

Example 5.4. (*Ref. 18*). Here, $n = 1$,

$$f(x, u^1, u^2) = h^1(u^1) - h^2(u^2)x,$$

$$L^1(x, u^1, u^2) = q^1x - u^1,$$

$$L^2(x, u^1, u^2) = u^2x - q^2x,$$

where h^1 is concave, h^2 is convex, and q^1, q^2 are constant.

6. Concluding Remarks

The main results on solvability are stated in the propositions of Section 2 and 3. Properties of interior equilibria were treated in Section 5. We have identified a class of differential games with nice structural properties. For this class, which includes the trilinear games, a system of differential equations for the open-loop Nash solutions can be derived. In some cases, this system can be solved explicitly whereas, for other differential games, qualitative insight into the properties of the solution may be gained.

The concepts of state separability and redundancy play an important role in the solution of a large class of differential games. To our knowledge, these concepts have not been explicitly defined before (see, however, Ref. 20, p. 648, and Refs. 1-2). We believe that the results on solvability, apart from a theoretical value, should prove useful in management science and other applications of differential games theory. Thus, a proposed DG's model can easily be scrutinized with a view to ascertaining whether a solution can be obtained. However, this remark needs some modifications.

Much work still remains to be done with respect to solvability in general. First, we note that, in most cases, the conditions stated are sufficient, but not necessary, implying that a solution may be obtainable even if the conditions fail to apply. Second, given redundancy, is state separability the basic condition, in the sense that it is the only one which guarantees qualitative solvability? Third, games with Hamiltonians linear in the controls and with constrained controls should also be investigated. Here, it might be conjectured, difficulties probably will appear, calling for revisions of definitions and reformulation of propositions. Note that the general theory of such games still appears to be rather incomplete. Finally, note that vast areas have not been covered in the present exposition, for instance, cooperative games, noncooperative games with other solution concepts than the

Nash equilibrium, and games with closed-loop strategies. Thus, there are still a lot of promising avenues for further research on solvability of differential games.

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