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Nonscalarized Multiobjective Global Optimization[1]

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Abstract. A new approach to multiobjective optimization is presented which is made possible due to our ability to obtain full global optimal solutions. A distinctive feature of this approach is that a vector cost function is nonscalarized. The method provides a means for the solution of vector optimization problems with nonreconcilable objectives.

Key Words. Vector optimization, nonscalarized multiobjective programming.

1. Introduction

It is generally accepted that an optimization problem should have one sole objective to be mathematically solvable. If there are several conflicting objectives, they are to be somehow adjusted to provide for a substitute single objective problem with a scalar cost function. This approach led to different scalarization schemes: the utility (inverted utility) function method, Pareto optimality, the global criterion method, and the bounded objective function method (Refs. 1–13). The deficiency of such approach has long been recognized and different schemes with less straightforward scalarization have been developed, such as the lexicographic method (which fails, if at least one subproblem has one single minimizer), goal programming (minimizes a norm of vector deviation from partial minima or other goals related to component objective functions), and cone construction and perturbation methods (to determine the so-called efficient or weakly efficient points); see, e.g., Refs. 5 and 10–24.

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The concept of a scalar (or scalarized) objective function is believed to be self-evident due to our traditional thinking in terms of unique solutions under complete information. It stems from conventional point-to-point descent (ascent) methods, which deliver one or several pointwise optimizers that may contradict each other in the case of several conflicting objectives.

Recently developed full global optimization methods (Refs. 25–26), capable of providing the entire set of all global optimizers, allow us to work out a different approach under which vector optimization problems appear as natural as the scalar ones, and scalar optimization problems represent just a special case of balanced vector optimization. Scalarization becomes an irrelevant and needless procedure, if the objectives are really independent.

However, there are vector optimization problems that are poorly formulated and represent, in fact, scalar optimization with several reconcilable objectives. In this case, scalarization is natural and necessary. To illustrate the phenomenon, let us consider an example.

Example 1.1. Solve the problem

$$\max z_1 = x - 2y, \tag{1}$$

$$\min z_2 = 0.3x + y, \tag{2}$$

$$x + y \geq 1, \qquad x - y \leq 3, \qquad x \geq 0, \qquad y \geq 0. \tag{3}$$

The solution of subproblem (1), (3) is (3, 0) with $z_1(3, 0) = 3$. The solution of subproblem (2), (3) is (1, 0) with $z_2(1, 0) = 0.3$.

If one formally applies a linear utility function (we call it linear scalarization; cf. Pareto optimality), then, considering $\min(-z_1)$ instead of $\max z_1$ in (1) and denoting

$$z = \lambda_1(-z_1) + \lambda_2 z_2, \qquad \lambda_1 \geq 0, \qquad \lambda_2 \geq 0, \qquad \lambda_1 + \lambda_2 = 1,$$

or (which is simpler, with $\lambda_2 = \lambda$, $\lambda_1 = 1 - \lambda$)

$$z = -(1-\lambda)z_1 + \lambda z_2 = \lambda(1.3x - y) - x + 2y, \tag{4}$$

yields the problem of finding

$$\min z = \min_{x,y} [\lambda(1.3x - y) - x + 2y] \tag{5}$$

under the conditions (3). If $\lambda < 1/1.3 = 0.77$, then the optimal point is (3, 0) with

$$z_1^0 = 3, \qquad z_2^0 = 0.9 > 0.3, \qquad \Delta z_2 = 0.6.$$

If $\lambda > 0.77$, then the optimal point is (1, 0) with

$$z_2^0 = 0.3, \qquad z_1^0 = 1 < 3, \qquad \Delta z_1 = -2.$$

It is clear that both solutions just repeat the solution of subproblems above and are irrelevant to the solution of the original problem (1), (2), (3).

However, if $\lambda = 0.77$, then we have the optimal set $x \in [1, 3]$, $y = 0$, and one can find x^* that yields equal deviations from optimal values vis-a-vis z_1 and z_2 from the equation

$$|\Delta z_1| = 3 - x^* = 0.3x^* - 0.3 = \Delta z_2. \tag{6}$$

Equation (6) yields $x^* = 2.54$, with $z_1^* = 2.54 < 3$, $z_2^* = 0.76 > 0.3$, and $\Delta z_2 = |\Delta z_1| = 0.46$.

At first glance, it seems like goal programming, but it is not. Indeed, denote deviations from optimal values $z_1^0 = 3$, $z_2^0 = 0.3$ as follows:

$$d_1 = 3 - z_1 = 3 - x + 2y \geq 0, \tag{7}$$

$$d_2 = z_2 - 0.3 = 0.3x + y - 0.3 \geq 0. \tag{8}$$

Then, the goal programming requires minimization of a positive-definite function or a norm, for example,

$$\min_{x,y} (d_1^p + d_2^p)^{1/p}, \qquad p \geq 1, \tag{9}$$

under the constraints (3). It can be verified that (e.g., for $p = 1$ and $p = 2$) we get the same goal optimal point (3, 0) yielding the solution of subproblem (1), (3) with the cost function (2) not represented in the optimal solution. The objective (2) is dissolved in the procedure of the goal programming without any effect at all on the result, although it is present in the formulation of the goal programming cost function (9). We note also that minimizing a norm of deviations is a kind of scalarization that heavily depends on a definition of the norm and does not mean any equilibrium among component deviations.

Before clearing the situation, let us ask the question: is the problem (1), (2), (3) with two written objectives a double or a single objective problem?

Case 1. Suppose that z_1 is revenue and z_2 are expenses. Then, problem (1), (2), (3) is a poorly formulated single-objective problem with the profit function $z = z_1 - z_2 = 0.7x - 3y$ as its natural scalarization and the optimal value $z^0 = 2.1$ at the point (3, 0).

Case 2. Suppose that z_1 is profit of one division and z_2 is loss of another division of the same company. Then, the company can reconcile the

objectives with certain trade-off between two divisions which renders natural scalarization yielding again a single-objective problem.

Case 3. Consider $\min(-z_1) = -x + 2y$ in place of $\max z_1$ in (1), and let $-z_1$, z_2 be the expected collateral damage of two cities in two allied countries engaged in a conflict with a third country. Then, the magnitude of damage inflicted onto one city cannot be reconciled at the disfavor of another city, and the problem becomes essentially a double-objective problem with conflicting objectives. A sensible approach to the solution is to assure the minimal equal deviation from the minimal expected damage for each city. Denoting this common deviation by $\eta > 0$, we have under constraints (3), with the use of the solutions of subproblems (1), (3) and (2), (3),

$$3 - \eta \leq x - 2y \leq 3, \qquad \text{for } \max z_1, \tag{10}$$

$$0.3 \leq 0.3x + y \leq 0.3 + \eta, \qquad \text{for } \min z_2. \tag{11}$$

In Fig. 1, shaded areas denote regions where amount of damage has increased by η for both cities. Figure 1 is drawn in terms of $\max z_1$, $\min z_2$ to facilitate comparisons with previous solutions.

By inspection of Fig. 1, it is clear that, with increasing η, there will be the first point of intersection of augmented shaded areas and this first point will be located on the x-axis. Letting $y = 0$, omitting trivial inequalities in (10), (11), and dividing (11) by 0.3, we get

$$3 - \eta \leq x \leq 1 + \eta/0.3 = 1 + 3.33\eta. \tag{12}$$

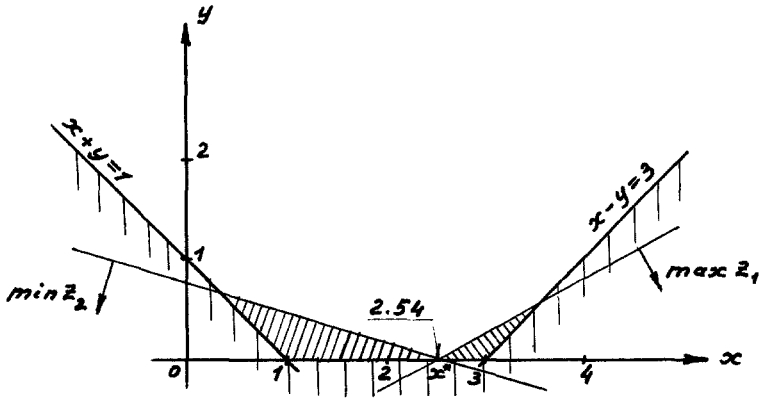

Fig. 1.

A solution for (12) exists if and only if

$$3-\eta \leq 1+3.33\eta, \tag{13}$$

that is, for $\eta \geq 0.46$; and for $\eta = 0.46$, we have the first single point of intersection of augmented shaded areas,

$$x^* = 3-0.46 = 2.54, \tag{14}$$

same as that obtained from (6). The minimal value of $\eta = 0.46$ is called the balance number, and the value $x^* = 2.54$ corresponds to the minimal equal increase of damage related to each city,

$$|\Delta z_1| = \Delta(-z_1) = \eta = 0.46 = \Delta z_2.$$

The above example shows that there are two kinds of multiobjective (vector) optimization problems:

(a) problems with reconcilable objectives where scalarization is natural and serves to rectify a poorly formulated vector optimization problem and turn it into an adequate single-objective problem;

(b) problems with nonreconcilable objectives; here, scalarization is irrelevant and may only distort the original problem, replacing it by a new single-objective problem that may have nothing to do with the original problem to the point that certain objectives may totally disappear from the results, as was the case (7)–(9) in the above example. This justifies the necessity of developing a new approach to the solution of vector optimization problems with nonreconcilable objectives.

2. Notions of Balance Number, Balance Point, and Balance Set

The notion of balance numbers for conflicting objectives was first introduced in Ref. 25, pp. 138–139, and we briefly reproduce here some pertinent points. Consider continuous m-vector function $f = \{f_1, \ldots, f_m\}$ defined on a compact robust set $\bar{X} \subset R^n$. Each component of f defines a separate optimization problem,

$$\min f_i(x), \quad x \in \bar{X}, \qquad i = 1, \ldots, m.$$

If we apply to those problems a set-to-set full global optimization method, such as the beta algorithm (Ref. 25, pp. 92–101) or the integral global optimization method (Ref. 26, pp. 17–20), then we obtain exact

(in the limit) or η-precise setwise solutions,

$$(c_i^o, \bar{X}_i^o) \qquad c_i^o = \min_{x \in \bar{X}} f_i(x),$$

$$\bar{X}_i^o = \{x \in \bar{X} \mid f_i(x) = c_i^o\}, \qquad i = 1, \ldots, m, \tag{15}$$

$$(c_{\eta i}^o, \bar{X}_{\eta i}^o) \qquad c_{\eta i}^o - c_i^o \leq \eta,$$

$$\bar{X}_{\eta i}^o = \{x \in \bar{X} \mid f_i(x) - c_i^o \leq \eta\}, \qquad i = 1, \ldots, m, \tag{16}$$

where

$$c_{\eta i}^o = \max_{x \in \bar{X}_{\eta i}^o} f_i(x).$$

It may happen that the set $\bar{X}_i^o$ is a singleton, $\bar{X}_i^o = \{x_i^o\}$. It is important however that, for robust $\bar{X}$, the set $\bar{X}_{\eta i}^o$ is also robust. Indeed, for the cubic algorithm, $\bar{X}_{\eta i}^o$ is a quasi-cubic set $\bar{K}^*$ (Ref. 25, pp. 43–47), which is obviously robust. For the beta algorithm, we can take as a candidate for $\bar{X}_{\eta i}^o$ an appropriate quasi-cubic set $\bar{B}_k$ (robust) for sufficiently high index k. Such a set may contain infeasible points that can be eventually excluded by taking the intersection $\bar{X}_{\eta i}^o = \bar{B}_k \cap \bar{X}$.

Consider a vector global optimization problem, denoted briefly as $\min(f, \bar{X})$, with its corresponding partial solutions (15) and (16).

Definition 2.1. A vector problem $\min(f, \bar{X})$ is said to be balanced, if there is a nonempty intersection,

$$\bar{X}^o = \bigcap_{i=1}^{m} X_i^o \neq \emptyset; \tag{17}$$

otherwise, it is called unbalanced.

Definition 2.2. The set $\bar{X}^o$ of (17) is called the global optimal solution of a balanced problem $\min(f, \bar{X})$.

Definition 2.3. Given $\eta > 0$, a vector problem $\min(f, \bar{X})$ is said to be η-balanced, if $\bar{X}^o = \emptyset$, but there is a nonempty intersection,

$$\bar{X}_\eta^o = \bigcap_{i=1}^{m} \bar{X}_{\eta i}^o \neq \emptyset; \tag{18}$$

otherwise, it is called η-unbalanced.

Definition 2.4. The set X_η^o of (18) is called the η-optimal solution of an η-balanced problem $\min(f, \bar{X})$.

Clearly, every balanced problem is also η-balanced for any $\eta \geq 0$, but not vice versa. For an η-balanced problem $\min(f, \bar{X})$, the values $c^o_{\eta i}$, $i = 1, 2, \ldots, m$, of (16) are upper bounds for η-optimal values of each subproblem $\min(f_i, \bar{X})$ and may define the vector of global minimum values $c^o_\eta = \{c^o_{\eta i}\}$, $\eta \geq 0$ given, for the vector problem $\min(f, \bar{X})$ depending on points contained in $\bar{X}^o_\eta$.

If we take

$$\eta \geq \max_i \max_{x \in \bar{X}} f_i(x) - \min_i \min_{x \in \bar{X}} f_i(x), \tag{19}$$

then $\bar{X}^o_\eta = \bar{X}$, so that for such η every problem $\min(f, \bar{X})$ is η-balanced. This justifies the following definition.

Definition 2.5. The quantity

$$\eta_0 = \min \eta |_{\bar{X}^o_\eta \neq \varnothing} \geq 0 \tag{20}$$

is called the balance number of the vector problem $\min(f, \bar{X})$.

The balance number represents a measure of conflict between components of the vector cost function f. For nonconflicting f_i, the problem $\min(f, \bar{X})$ is balanced, $\eta = 0$, which fact in general cannot be detected by point-to-point methods for a multiextremal problem. The importance of the notion of the balance number for multiobjective economic and engineering problems is clear.

Example 2.1. Consider the multiobjective linear programming problem

$$\min x, \min y, \min z, \tag{21}$$

$$x + y + z = 1, \qquad x \geq 0, \qquad y \geq 0, \qquad z \geq 0. \tag{22}$$

Here, $f = \{x, y, z\}$ and $\bar{X} \subset R^3_+$ is the triangle in (22). Each single-objective problem has the solution

$$s^o_1 = 0, \qquad \bar{X}^o_1 = \{(x, y, z) \in R^3_+ \mid x = 0, y + z = 1\} \subset \partial X, \qquad (1, 2, 3), \tag{23}$$

where $(1, 2, 3)$ means the circular permutation of $(1, 2, 3)$ and (x, y, z) simultaneously.

The vector problem is unbalanced since

$$\bar{X}^o = \bigcap \bar{X}^o_i = \varnothing.$$

However, each two-criteria problem is balanced and has the solution

$$s_3^o = 0, \qquad \bar{X}_3^o = \{(0, 0, 1)\} = \{x = y = 0, z = 1\}, \qquad (1, 2, 3). \tag{24}$$

The balance number of the vector problem is $\eta_0 = 1/3$ and its $1/3$-optimal global solution is

$$\bar{X}_{1/3}^o = \{(1/3, 1/3, 1/3)\}, \qquad \text{with } s_{1/3,i}^o = 1/3, \qquad i = 1, 2, 3. \tag{25}$$

If we take in (21)

$$\min w = \lambda_1 x + \lambda_2 y + \lambda_3 z,$$

then for $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$ we would have, due to (22), $w = 1/3 = \text{const}$, so that $\bar{X}^o(\lambda) = \bar{X}$, the entire triangle, and the point (25) cannot be located. If we take another choice of $\lambda_1, \lambda_2, \lambda_3$, then $\bar{X}^o(\lambda)$ would be a piece of boundary $\bar{X}^o(\lambda) \subset \partial X$ of the set $\bar{X}$ in (22), whereas the point (25) is in the interior, $\bar{X}_{1/3}^o \subset \text{int } X$.

This is a general phenomenon. A balanced multicriteria linear programming problem does not need scalarization and has its solution on the boundary of the feasible set. An unbalanced multicriteria linear programming problem may have its η-optimal solution in the interior of the feasible set. In general, it is not scalarizable, since any linear scalarization will have its optimal solution on the boundary of the feasible set.

The common η in (16), (18) reflects the same deviation from the global minimum value allowed for all partial single-objective problems leading to the uniform suboptimal global solution (18) for the vector problem. Generalization for different η_i is straightforward. Denote

$$\eta = \eta_1 + \cdots + \eta_m, \qquad \alpha_i = \eta_i / \eta,$$

and replace η in (16) by $\eta_i = \alpha_i \eta$, allowing different deviations for each subproblem. Definitions 2.3, 2.4, 2.5 remain the same with some fixed proportion in allowable deviations given by the numbers $\alpha_i \geq 0$, $\Sigma \alpha_i = 1$. If certain $\eta_{i_0} = \alpha_{i_0} \eta = 0$ (always $\eta > 0$), this means that, for those cost functions, the exact global minimum value is required.

A fixed proportion given by the numbers $\alpha_i \geq 0$ can also be removed. However in this case, the notion of the balance number introduced in Ref. 25, p. 139, is insufficient and should be replaced by the notions of balance points and of the balance set.

Definition 2.6. An m-tuple $\tilde{\eta} = \{\eta_1, \ldots, \eta_m\}$, $\eta_i \geq 0$ is called a balance point, if the intersection

$$\bar{X}_{\tilde{\eta}}^o = \bigcap_{i=1}^{m} \bar{X}_{\eta_i}^o \neq \varnothing \tag{26}$$

is nonempty and every intersection (26) with one η_i replaced by a smaller positive number is empty.

It is clear that substitution of more than one η_i by smaller numbers will also lead to empty intersection. However, certain η_i can be made smaller at the expense of some other η_i being greater. This leads to the notion of the balance set.

Definition 2.7. A set of all m-tuples $\tilde{\eta}$ satisfying (26), i.e., a set of all balance points, is called the balance set.

To compute the balance set, let us fix a balance point $\tilde{\eta} = \{\eta_1, \eta_2, \ldots, \eta_m\}$ and a point $x^* \in \bar{X}^o_{\tilde{\eta}}$. Due to (26),

$$x^* \in \bar{X}^o_{\eta_i}, \qquad \text{for all } i = 1, \ldots, m;$$

and since $\bar{X}^o_{\eta_i}$ are some partial sets as $\bar{X}^o_{\eta i}$ of (16) but corresponding to different $\eta_i \neq \eta = \text{const}$ as in (16), at x^* we have the inequalities

$$f_i(x^*) - c_i^o \leq \eta_i, \qquad i = 1, \ldots, m. \tag{27}$$

If at least one of inequalities (27) were strict (nonsaturated), then we could decrease the corresponding η_i meaning that, in this case, the point $\tilde{\eta}$ would not be a balance point; see Definition 2.6. Thus, for a balance point $\tilde{\eta}$, all relations in (27) are equalities,

$$f_i(x^*) - c_i^o = \eta_i, \qquad i = 1, \ldots, m.$$

Suppose that $\tilde{\eta}$ is not an isolated point, and consider small admissible variations $\delta\tilde{\eta} = \{\delta\eta_1, \ldots, \delta\eta_m\}$ within the balance set. Suppose that variations $\delta\tilde{\eta}$ will produce small variations of $\bar{X}^o_{\tilde{\eta}}$ within $\bar{X}$ leading to another point

$$\tilde{x} = x^* + \delta x \in \bar{X}^o_{\tilde{\eta} + \delta\tilde{\eta}} \subset \bar{X}.$$

Suppose finally that every f_i is differentiable at $x^* \in \bar{X}$. Then, in a neighborhood

$$\mathcal{N}_\gamma(x^*) \subset \bigcup_{\tilde{\eta}} \bar{X}^o_{\tilde{\eta}},$$

we can write, for equalities in (27),

$$\nabla f_i(x^*)\delta x = \sum_{j=1}^{n} [\partial f_i(x^*)/\partial x_j]\delta x_j = \delta\eta_i, \qquad i = 1, \ldots, m. \tag{28}$$

It may happen that, according to the problem under consideration, all δx_j, x_j^* can be excluded from the m equations (28) resulting in one or more differential relations in $\tilde{\eta}$-space yielding the balance set. If those relations are integrable, then we obtain the balance set or balance surface in conventional analytic form.

Remark 2.1. If we treat (28) as a linear system with a fixed Jacobian matrix, then we obtain not the balance set but a small piece of its linear approximation in a neighborhood $\mathcal{N}_\epsilon(\tilde{\eta})$ of a balance point $\tilde{\eta}$.

To illustrate the procedure and possible pitfalls, consider again the above examples.

In Example 1.1, we have, for (27),

$$3-z_1=3-x+2y=\eta_1, \qquad \text{for max } z_1, \tag{29a}$$

$$z_2-0.3=0.3x+y-0.3=\eta_2, \qquad \text{for min } z_2. \tag{29b}$$

If we formally take variations, we see that the Jacobian matrix is nonsingular so that δx, δy cannot be excluded, leading to nonexistence of a balance set. However, such solution is incorrect, since (28) is valid not for $\bar{X}$ but for

$$\mathcal{N}_\gamma(x^*)\subset\bigcup_{\tilde{\eta}}\bar{X}^o_{\tilde{\eta}}.$$

In our example,

$$\bar{X}^o_{\tilde{\eta}}\subset\{x, y \mid 1\leq x\leq 3, y=0\},$$

see Fig. 1, so that we have first to set $y=0$ in (29) and then take variations, yielding

$$-\delta x=\delta\eta_1 \quad \text{and} \quad 0.3\delta x=\delta\eta_2. \tag{30}$$

Excluding δx, we get

$$0.3\delta\eta_1+\delta\eta_2=0,$$

yielding the integral

$$0.3\eta_1+\eta_2=b=\text{const}. \tag{31}$$

To determine the constant b, we note that $\eta_1=0$ corresponds to the point (3, 0) of global max z_1. At this point, we have $z_2(3, 0)=0.9$; thus,

$$\eta_2=z_2(3, 0)-\min z_2=0.9-0.3=0.6.$$

From (31) with $\eta_1=0$, we have $b=\eta_2=0.6$, so that

$$0.3\eta_1+\eta_2=0.6. \tag{32}$$

This calculation is valid for any $\bar{X}^o_{\tilde{\eta}}$ from the segment [1, 3]; hence, we obtained the entire balance set,

$$3\eta_1+10\eta_2=6, \qquad \eta_1\geq 0, \qquad \eta_2\geq 0. \tag{33}$$

In Example 2.1, all partial minima are equal to zero, so that we have, from (21),

$$x-0=\eta_1, \qquad y-0=\eta_2, \qquad z-0=\eta_3, \tag{34}$$

yielding

$$\delta x=\delta\eta_1, \qquad \delta y=\delta\eta_2, \qquad \delta z=\delta\eta_3. \tag{35}$$

The Jacobian is again nonzero; however, from (22), we get

$$\delta\eta_1+\delta\eta_2+\delta\eta_3=\delta x+\delta y+\delta z=0,$$

with the integral

$$\eta_1+\eta_2+\eta_3=b=\text{const.} \tag{36}$$

We can determine b from (24), where

$$\eta_1=\eta_2=0 \quad \text{and} \quad \eta_3=1-0=1;$$

thus, $b=1$ and we obtain the balance set,

$$\eta_1+\eta_2+\eta_3=1, \qquad \eta_{1,2,3}\geq 0. \tag{37}$$

If the balance set is known, then balance numbers can be trivially determined. To calculate the uniform balance number η_0 of (20), we set $\eta_i=\eta_0$, $i=1,\ldots,m$, and solve (33), (37) for η_0. In Example 1.1, we get from (33)

$$\eta_0=6/13=0.46,$$

as above. In Example 2.1, we get from (37)

$$\eta_0=1/3,$$

as above. To calculate the apportioned balance number, we set $\eta_i=\alpha_i\eta$, $\alpha_i\geq 0$ given, $\Sigma\alpha_i=1$, and find η from the equation of the balance set. For example, from (37) we get

$$\eta=1/\Sigma\alpha_i=1,$$

and solutions (24), (25) follow for corresponding α_i.

The balance set presents an important characterization of a multiobjective problem. Of course, precisions $\eta_i>0$ are assigned on the basis of economic or technical considerations and do not have to belong to the balance set which is usually unknown and difficult to determine. However, theoretically the balance set represents a boundary; on one side of the boundary, the margins η_i are insufficient and the problem is unbalanced, $\bar{X}^o_\eta=\varnothing$; on the other side, η_i are excessive, the problem is balanced, $\bar{X}^o_\eta\neq\varnothing$, and every solution is inside certain margins η_i.

3. Solution of Nonscalarized Vector Optimization Problems

Suppose that, for a given precision $\eta > 0$, the problem is balanced. The computation of $\bar{X}^o_\eta$ by taking the intersection, as in (18), is too time consuming. Instead, certain modifications of the beta algorithm can be worked out to solve such problems. Note that the set $\bar{X}^o_{\eta i}$ is not a singleton; thus, an η-optimal point $x \in \bar{X}^o_{\eta i}$ found by a point-to-point descent method may not be η-optimal with respect to f_j, $j \neq i$, implying $x \notin \bar{X}^o_\eta$.

3.1. Sequential Procedure. Take the first cost function $f_1(x)$ and apply the beta algorithm to obtain the first η-optimal solution $\bar{X}^o_{\eta 1}$ in the form of a quasi-cubic set $\bar{B}_k$ which can be determined by the method described in Ref. 25, pp. 43–47, with additional exclusions by a distinction operator acting independently in the framework of the beta algorithm. Replace $f_1(x)$ by $f_2(x)$ and again apply the beta algorithm, now to the set $\bar{B}_k$, which we shall denote $\bar{B}_{\eta 1}$, to emphasize its relation to $\bar{X}^o_{\eta 1}$,

$$\bar{X}^o_{\eta 1} \supseteq \bar{B}_k \cap \bar{X} = \bar{B}_{\eta 1} \cap \bar{X}.$$

The set $\bar{B}_{\eta 1}$ may contain infeasible subsets which do not hinder the process and can be later excluded by deletion and distinction operators. The second run, with $f_2(x)$ as the cost function, yields a quasi-cubic set $\bar{B}_{\eta 2} \subseteq \bar{B}_{\eta 1}$. Apply again the beta algorithm to the set $\bar{B}_{\eta 2}$, with $f_3(x)$ as the cost function, to get the set $\bar{B}_{\eta 3}$, etc., until the last cost function $f_m(x)$ has been applied to obtain the set $\bar{B}_{\eta m}$. By construction, all those sets are nested,

$$\bar{B}_{\eta 1} \supseteq \bar{B}_{\eta 2} \supseteq \cdots \supseteq \bar{B}_{\eta m}.$$

It is clear that

$$\bar{X}^o_\eta \supseteq \bar{B}_{\eta m} \cap \bar{X}. \tag{38}$$

The inclusion (not the equality) in (38) is natural, due to the action of the beta algorithm, specifically, of its deletion operator. Now, it remains to clear the intersection, which can be done by further application of distinction operator; see Ref. 25, pp. 83–92. In reality, if we need just one point $x^o \in \bar{X}^o_\eta$, such point can be located by picking points in $\bar{B}_{\eta m}$ at random and checking their membership in $\bar{X}$ until one point fits in.

3.2. Rotational Procedure. It may be advantageous to install a function rotator within the beta algorithm in the following way. In the first iteration, make deletions by applying all functions $f_1(x), \ldots, f_m(x)$ one by one, then proceed with further iterations, applying all cost functions in turn at each iteration. This constitutes a version of the multicriteria beta algorithm as this procedure can be applied to obtain the exact solution $\bar{X}^o$ of

(17), in the limit, for $\eta = 0$. Note that, if for at least one i in (18) the set $\bar{X}_i^o$ is a singleton, then it trivially yields the solution of a balanced vector optimization problem which can be found also by point-to-point methods.

It is worth noting that the sequential procedure, in general, cannot yield $\bar{X}^o$, not only computationally (impossibility to obtain $\bar{X}_1^o$ in iterations) but also theoretically, as the first exact set $\bar{X}_1^o$ of global minimizers may prove to be nonrobust, preventing further application of the beta algorithm to minimize over $\bar{X}_1^o$ with respect to $f_2(x), \ldots, f_m(x)$.

3.3. Windmill Procedure. The functions $f_1(x), \ldots, f_m(x)$ may be different in the amount of computations needed to obtain their values and also in the estimates of their respective constants $L_1, \ldots, L_m$. This makes it more effective to use an adaptive controller instead of a simple function rotator. The structure of such a controller depends on given cost functions $f_i(x)$, $i = 1, \ldots, m$, and cannot be fixed in advance. The controller makes automatic replacements of cost functions in the iteration process according to some acquired experience, providing a vast area of creativity for the user.

3.4. Parallel Procedure for Unbalanced Problems. Suppose now that, for given precisions $\eta_i > 0$, the vector optimization problem is unbalanced. Then there arise two problems, one of finding the balance number $\eta_0 \geq \eta_i$, $i = 1, \ldots, m$, and another one of finding the set $\bar{X}_{\eta_0}^o$ which may prove to be a singleton. Solving these problems presents much difficulty, since pictures like Fig. 1 are generally unavailable. Even the notion of a solution may vary in this case.

One possible approach is as follows. Apply the beta algorithm to each of m subproblems in parallel, but with different $\eta_i > 0$ and the same translated grid generator. An η_i-solution for each subproblem will be obtained in a finite number of iterations (see Ref. 25, pp. 43–47 and 92–95) and will represent a quasi-cubic set. Due to the common grid generator, the task of determining the intersection

$$\bar{X}^* = \bigcap_{i=1}^{m} \bar{X}_{\eta_i}^o, \tag{39}$$

analogous but not identical to (18), is trivial and $\bar{X}^*$ will consist of none, one, or several subcubes singled out from solution sets $\bar{X}_{\eta_i}^o$. If $\bar{X}^* \neq \varnothing$, then the problem is actually η^*-balanced; this is not known in advance and $\eta^* = \{\eta_1, \ldots, \eta_m\}$ is not necessarily a balance point.

If $\bar{X}^* = \varnothing$, then the problem is η^*-unbalanced. For this eventuality, we have to retain in the memory several previous iterations (otherwise, we would have to repeat the solution again), which contain quasi-cubic sets

corresponding to greater η_i; see Ref. 25, pp. 43–47. Checking the intersection (39) in backward order will always solve the problem. The first nonempty intersection

$$\bar{X}_{\bar{\eta}} = \bigcap_{i=1}^{m} \bar{K}_{\eta_i} \neq \varnothing \tag{40}$$

yields the solution set and a point

$$\bar{\eta} = \{\alpha_1 \eta_1, \ldots, \alpha_m \eta_m\}, \qquad \alpha_i \geq 1, \qquad i = 1, \ldots, m,$$

for which the problem is $\bar{\eta}$-balanced. This semiautomatic procedure augments all η_i according to a fixed common grid generator; see Ref. 25, pp. 43–47.

Obviously, the set $\bar{X}_{\bar{\eta}}$ can be further refined to diminish each $\alpha_i \eta_i$ or a controller can be introduced to modify the iteration process and obtain some smaller η_i, $i = 1, 2, \ldots, k$, $k < m$, for certain objective functions f_i deemed to be more important than others. A version of such selective procedure is described in Ref. 25, pp. 143–145.

4. Discussion and Conclusions

A new approach to multiobjective optimization is presented. Scalarization is retained for cases of poorly formulated problems in order to rectify the formulation by the introduction of a proper single objective instead of several objectives unthoughtfully put forward at the first glance consideration.

For cases with essential and independent objectives, it is shown that scalarization does not reconcile the objectives and may have nothing to do with the vector problem. In such cases, scalarization serves only to provide for the possibility of mathematical solution by outdated point-to-point optimization methods.

In contrast, with the application of set-to-set full global optimization methods, scalarization is needless and vector problems have natural and economically sensible solutions which correspond to a measure of satisfaction provided for each of the conflicting objectives. This measure of satisfaction can be specified in advance through the new notions of a balance point and of the balance set introduced in the paper.

Numerical methods are presented for the solution of vector optimization problems without scalarization. The solution thus obtained represents a point or a complete set of equioptimal points corresponding to the balance number or to a prespecified balance point $\bar{\eta}$.

A scalarized objective function may also imply certain balance point $\tilde{\eta}$, though unknown in advance, and it might seem that the introduction of balance points amounts to a sort of scalarization. That this is not the case was shown in Example 2.1 where no linear scalarization existed for the balance number $\eta_0 = 1/3$.

For a given balance point $\tilde{\eta}$, a corresponding scalarized objective may not exist within a certain class of functions. Even if it exists, it may be more difficult to find one than to solve a vector optimization problem. The application of a full global optimization method (Refs. 25, 26) in parallel to all component scalar subproblems renders the global optimal solution of the vector problem automatically for any balance point $\tilde{\eta}$ specified in advance. Each coordinate η_i of the balance point $\tilde{\eta} = \{\eta_1, \ldots, \eta_m\}$ represents a measure of achievement of a certain objective within the equilibrium defined by the balance set. This set corresponds to the best possible solutions of the vector problem with conflicting objectives. If given margins η_i are not equal to coordinates of a balance point, then it is either impossible to find a solution within given η_i, or certain η_i can actually be improved. This demonstrates the importance of the balance set which, if presented to decision makers, gives a clear picture of what is possible to achieve in the situation with several conflicting objectives.

The balance set is difficult to determine, even for multiobjective linear programming problems. Fortunately, we do not need the balance set for the solution of a multicriteria optimization problem. Depending on the complexity of constituent subproblems $\min(f_i, \bar{X})$, $i = 1, \ldots, m$, there may be two cases.

Case 1. If subproblems $\min(f_i, \bar{X})$ are easy to solve, they can be solved first and partial optima c_i^o become known. Then, if $\bar{X}^o$ of (17) is empty, certain margins η_i^* can be specified in advance for each objective, no matter whether or not they are equal to coordinates of a balance point $\tilde{\eta} = \{\eta_1, \ldots, \eta_m\}$. If the vector problem happens to be balanced for given $\{\eta_i^*\}$, which is the case if and only if there is $\tilde{\eta} = \{\eta_i\}$ such that $\eta_i^* \leq \eta_i$, for all $i = 1, \ldots, m$, then a solution is found by the methods described above. If the problem is unbalanced for those $\{\eta_i^*\}$, then the first three procedures (designed for balanced problems) will not be finite, and after some iteration there will be no visible improvement in the values of some partial minima. This signals that the problem is unbalanced and appropriate η_i^* should be increased. Nevertheless, if actually attained η_i are acceptable, then the problem is solved and those $\{\eta_i\}$ represent an approximation to a balance point $\tilde{\eta}$. This balance point presents actual margin distribution among different objectives and is generated by a particular function controller applied in the algorithm. If such balance point is unacceptable, then the computations

should be restarted with the fourth algorithm (parallel procedure for unbalanced problems).

Case 2. Preliminary solution of subproblems $\min(f_i, \bar{X})$ may be costly, and it is not required for the solution of a vector problem. Without the solution of subproblems $\min(f_i, \bar{X})$, the partial minima c_i^o are unknown and margins η_i^* cannot be assigned. In this case, it is risky to apply procedures with a specified function controller (which are good for balanced problems). The parallel procedure should be applied, and the decision to accept actually produced c_{η_i} for a nonempty $\bar{X}^*$ of (39) should be made in the process of iterations. Once all c_{η_i} are accepted, the process is terminated and the problem is solved. In this case, theoretical partial minima c_i^o remain unknown as well as precisions η_i of the values c_{η_i}, $c_{\eta_i} - c_i^o \leq \eta_i$. However, the set $\bar{X}^*$ and the values c_{η_i} are produced by the algorithm yielding a global optimal solution $(c_{\eta_i}, \bar{X}^*)$ of the vector problem without solving component subproblems. If $\bar{X}^*$ is not a singleton, then the solution can be continued and the values c_{η_i} can be further improved.

References

1. PARETO, V., *Cours d'Économie Politique*, Lausanne Rouge, Lausanne, Switzerland, 1896.
2. HWANG, C. L., MASUD, A. S. M., PAIDY, S. R., and YOON, K., *Multiple-Objective Decision Making: Methods and Applications*, Springer-Verlag, Berlin, Germany, 1979.
3. SAWARAGI, Y., NAKAYAMA, H., and TANINO, T., *Theory of Multiobjective Optimization*, Academic Press, Orlando, Florida, 1985.
4. BORM, P. E. H., TIJS, S. H., and VAN DEN AARSSEN, J. C. M., *Pareto Equilibria in Multiobjective Games*, Methods of Operations Research, Anton Hain, Bonn, Germany, pp. 303–312, 1990.
5. GEOFFRION, A. M., *Proper Efficiency and Theory of Vector Maximization*, Journal of Mathematical Analysis and Applications, Vol. 22, pp. 618–630, 1968.
6. COHON, J. L., *Multiobjective Programming and Planning*, Academic Press, New York, New York, 1978.
7. ZELENY, M., *Linear Multiobjective Programming*, Springer-Verlag, Berlin, Germany, 1974.
8. ROY, B., *Méthodologie Multicritère d'Aide à la Décision*, Economica, Paris, France, 1985.
9. STEUER, R. E., *Multiple-Criteria Optimization: Theory, Computation, and Applications*, Wiley, New York, New York, 1986.
10. RAO, S. S., *Optimization Theory and Applications*, 2nd Edition, Wiley, New York, New York, 1984.

11. WEIDNER, P., *An Approach to Different Scalarizations in Vector Optimization*, Wissenschaftliche Zeitschrift der Technische Hochschule Ilmenau, Vol. 36, pp. 103–110, 1990.
12. LOOTSMA, F. A., *Optimization with Multiple Objectives*, Mathematical Programming, Recent Developments and Applications, Edited by M. Iri and K. Tanabe. Kluwer, Dordrecht, Holland, pp. 333–364, 1989.
13. ZELENY, M., *Multiple-Criteria Decision Making*, McGraw-Hill, New York, New York, 1982.
14. KEENEY, R., and RAIFFA, H., *Decisions with Multiple Objectives: Preferences and Value Trade-offs*, Wiley, New York, New York, 1976.
15. SAATY, T. L., *The Analytical Hierarchy Process, Planning, Priority Setting, Resource Allocation*, McGraw-Hill, New York, New York, 1980.
16. KACPRZYK, J., and ROUBENS, M., Editors, *Nonconventional Preference Relations in Decision Making*, Springer-Verlag, Berlin, Germany, 1987.
17. IGNIZIO, J. P., *Goal Programming and Extensions*, Lexington Books, Lexington, Massachusetts, 1976.
18. YU, P. L., and ZELENY, M., *The Set of All Nondominated Solutions in Linear Cases and a Multicriteria Simplex Method*, Journal of Mathematical Analysis and Applications, Vol. 49, pp. 430–468, 1975.
19. YU, P. L., *Multiple-Criteria Decision Making: Concepts, Techniques, and Extensions*, Plenum Press, New York, New York, 1985.
20. YU, P. L., *Cone Convexity, Cone Extreme Points, and Nondominated Solutions in Decision Problems with Multiobjectives*, Journal of Optimization Theory and Applications, Vol. 14, pp. 319–377, 1974.
21. HARTLEY, R., *On Cone-Efficiency, Cone-Convexity, and Cone-Compactness*, SIAM Journal on Applied Mathematics, Vol. 34, pp. 211–222, 1978.
22. PASCOLETTI, A., and SERAFINI, P., *Scalarizing Vector Optimization Problems*, Journal of Optimization Theory and Applications, Vol. 42, pp. 499–524, 1984.
23. HELBIG, S., *An Algorithm for a Vector Optimization Problem over a Disjunctive Feasible Set*, Methods of Operations Research, Anton Hain, Bonn, Germany, Vol. 60, pp. 397–405, 1990.
24. HELBIG, S., *Scalarization in Vector Optimization by Cone Perturbation*, Preprint, 1990.
25. GALPERIN, E. A., *The Cubic Algorithm for Optimization and Control*, NP Research Publication, Montreal, Quebec, Canada, 1990.
26. GALPERIN, E. A., and ZHENG, Q., *Global Solutions in Optimal Control and Games*, NP Research Publication, Montreal, Quebec, Canada, 1991.