JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 72, No. 3, MARCH 1992

On the Structure of Convex Piecewise Quadratic Functions[1]

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Communicated by F. Zirilli

Abstract. Convex piecewise quadratic functions (CPQF) play an important role in mathematical programming, and yet their structure has not been fully studied. In this paper, these functions are categorized into difference-definite and difference-indefinite types. We show that, for either type, the expressions of a CPQF on neighboring polyhedra in its domain can differ only by a quadratic function related to the common boundary of the polyhedra. Specifically, we prove that the monitoring function in extended linear-quadratic programming is difference-definite. We then study the case where the domain of the difference-definite CPQF is a union of boxes, which arises in many applications. We prove that any such function must be a sum of a convex quadratic function and a separable CPQF. Hence, their minimization problems can be reformulated as monotropic piecewise quadratic programs.

Key Words. Convex polyhedra, extended linear-quadratic programs, monotropic programming, piecewise quadratic functions, separability of functions.

1. Introduction

A convex function $f: R^n \mapsto R \cup \{\pm\infty\}$ is piecewise quadratic if its domain [i.e., the set $\operatorname{dom} f = \{x \in R^n \mid f(x) < \infty\}$] is a union of finitely many convex polyhedra, on each of which the function is given by a quadratic formula (including affine formula as a special case). Due to continuity of the convex piecewise quadratic function (CPQF) on its domain, the expressions of f on different polyhedra cannot be arbitrary. Our concern in this paper are

[1]This research was supported by Grant DDM-87-21709 of the National Science Foundation.
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the relationship between these expressions and the overall structure of a CPQF.

The interest in studying CPQF is stimulated by recent research of Rockafellar and Wets (Refs. 1-3) on stochastic programming and optimal control problems. In their theoretical framework, constraints are separated into two classes; one should be satisfied exactly and another may be violated. A monitoring term in the objective function is used to reduce the violation. Based on this formulation, a clear duality relationship can be derived and new algorithms are proposed for previously unsolvable problems. At the center of their model is a linear-quadratic minimax problem,

$$\text{(E)}\quad \text{minimax } L(x, y) = p^T x + q^T y + (x^T P x)/2 - (y^T Q y)/2 - y^T R x, \quad x \in U, y \in V,$$

where the superscript T designates the transpose of a vector. Problem (E) induces the primal-dual pair of extended linear-quadratic programs:

$$\text{(P)}\quad \min_{x \in U} f(x), \quad \text{where } f(x) = \sup_{y \in V} L(x, y),$$

$$\text{(D)}\quad \max_{y \in V} g(y), \quad \text{where } g(y) = \inf_{x \in U} L(x, y);$$

here, $U(\subset R^n)$ and $V(\subset R^m)$ are convex polyhedra, representing the constraints that should be satisfied exactly; p and q are fixed vectors; P (positive semidefinite), Q (positive semidefinite), and R are fixed matrices. Then, we have

$$f(x) = p^T x + (x^T P x)/2 + \rho_{VQ}(q - Rx),$$

$$g(y) = q^T y - (y^T Q y)/2 - \rho_{UP}(R^T y - p),$$

where

$$\rho_{VQ}(v) = \sup_{y \in V}\{y^T v - (y^T Q y)/2\},$$

$$\rho_{UP}(u) = \sup_{x \in U}\{x^T u - (x^T P x)/2\}.$$

The functions ρ_{VQ} and ρ_{UP} are the monitoring terms characterizing deviations of Rx from q and $R^T y$ from p, respectively. In Ref. 2, it is shown that both ρ_{VQ} and ρ_{UP} are CPQF in our sense. Of course, there are other applications of CPQFs, some of which are described in Refs. 4-8.

The structure problem of the CPQF has been studied from another angle. In Ref. 9, it is proved that a function is convex piecewise quadratic if and only if its subdifferential mapping is polyhedral in Robinson's (Ref. 10) sense. This property is used in analyzing the duality and parametric properties of convex piecewise quadratic programming. From the viewpoint of algorithmic development, however, it is convenient to know how the expressions of a CPQF are interrelated and under what conditions a CPQF can be decomposed into simpler functions. Especially, if a CPQF is separable [i.e., the function is of the form $f(x)=f_1(x_1)+\cdots+f_n(x_n)$, where x_i, $i=1,\ldots,n$, are components of x and $f_i(x_i)$ are one-dimensional convex piecewise quadratic functions], then, even if additional linear constraints exist, we may solve the corresponding minimization problem, the so-called monotropic piecewise quadratic program, quite efficiently (Ref. 11). Naturally, one wants to know the possibility of changing a general CPQF into a separable CPQF by a certain transformation of variables. Since the original function is linear-quadratic, affine transformation is preferable. This problem is tightly related to the structure problem to be investigated in this paper.

In the next section, we categorize the CPQF into two types (difference-definite and difference-indefinite) and derive the relationship between expressions on neighboring polyhedra for both types. The result implies that the difference-indefinite type is not separable under any nonsingular affine transformation. On the other hand, we prove that the monitoring functions in problems (P) and (D) are difference-definite. In Section 3, we analyze the difference-definite CPQF whose domain is a union of boxes and show that any such function must be a sum of a convex quadratic function and a separable CPQF. Consequently, we point out that the problem of minimizing this function is equivalent to solving a monotropic piecewise quadratic program.

2. Structure of General Convex Piecewise Quadratic Functions

In the following derivations, without loss of generality, we assume that the CPQF under discussion satisfies the following conditions:

(C1) The dimension of dom f is n. Otherwise, we could discuss the problem in a lower-dimensional space by changing the coordinate system.

(C2) dom $f=P_1\cup\cdots\cup P_m$, where all P_i, $i=1,\ldots,m$, are convex polyhedra of dimension n and $\operatorname{int} P_i\cap\operatorname{int} P_j=\varnothing$, $i\neq j$. This assumption is reasonable because all lower-dimensional polyhedra in dom f must be contained in some of the n-dimensional P_i's; therefore, the removal of those lower-dimensional polyhedra from dom f does not change f due to continuity of f in dom f. Furthermore, if two polyhedra have a common internal point, then the quadratic expressions on them must be identical and it is a trivial case pertaining to our purposes.

Definition 2.1. Let P_i, P_j $(i \neq j) \subset \text{dom}\, f$. We say that P_i and P_j are neighboring with each other if the affine hull of $P_i \cap P_j$ is of dimension $n-1$. The affine hull of $P_i \cap P_j$ (a hyperplane) is called their common boundary.

Definition 2.2. A CPQF is said to be of difference-definite type if all of the differences between its expressions on neighboring polyhedra have positive or negative semidefinite Hessian. Otherwise, it is said to be of difference-indefinite type.

Proposition 2.1. Let $f(x)$ be a CPQF. Let P_1 and P_2 be two neighboring polyhedra in $\text{dom}\, f$ with common boundary $\{x \mid a^T x = b\}$. Let $f_1(x)$ and $f_2(x)$ be the quadratic expressions of f on P_1 and P_2, respectively. Then, there exist a vector $\bar{a}$ and a constant $\bar{b}$ such that

$$f_2(x) = f_1(x) + [a^T x - b][\bar{a}^T x - \bar{b}]. \tag{1}$$

Moreover, a and $\bar{a}$ are linearly dependent if f is difference-definite, whereas if f is difference-indefinite, there exists at least a pair of P_1 and P_2, such that $\bar{a}$ is linearly independent of a.

Proof. Let $x^0 \in P_1 \cap P_2$, and let Q be an orthogonal matrix such that $Q^T R Q$ is diagonal, where R is the Hessian of $f_2 - f_1$. Under the affine transformation

$$x = Qy + x^0,$$

the function

$$f(x) = f(Qy + x^0)$$

has expressions $f_1(Qy + x^0)$ and $f_2(Qy + x^0)$, respectively, on

$$\mathscr{P}_1 = Q^{-1}(P_1 - x^0) \quad \text{and} \quad \mathscr{P}_2 = Q^{-1}(P_2 - x^0),$$

and the common boundary of $\mathscr{P}_1$ and $\mathscr{P}_2$ passes zero. Without loss of generality, let

$$H \equiv \{y \mid h(y) \equiv y_1 - c_2 y_2 - \cdots - c_n y_n = 0\}$$

be this common boundary, and let

$$\begin{aligned} f_3(y) &\equiv f_2(Qy + x^0) - f_1(Qy + x^0) \\ &= (d_1 y_1^2 + s_1 y_1) + \cdots + (d_n y_n^2 + s_n y_n). \end{aligned}$$

By continuity of f in $\text{dom}\, f$, $f_3(y) \equiv 0$ on H. Then, one of the following must be true:

(i) $f_3(y) = s_1 h(y)$;
(ii) $f_3(y) = d_1 h(y)^2 + s_1 h(y)$ and $h(y) = y_1$;
(iii) $f_3(y) = h(y)[d_1(y_1 + c_j y_j) + s_1]$, $d_1 \neq 0$ and $h(y) = y_1 - c_j y_j$, $c_j \neq 0$.

To prove this, denote $\{2, \ldots, n\}$ by J and notice that $f_3(y) \equiv 0$ on H implies that

$$0 \equiv \sum_{j \in J} (d_j y_j^2 + s_j y_j) + d_1\left(\sum_{j \in J} c_j y_j\right)^2 + s_1 \sum_{j \in J} c_j y_j.$$

A quadratic form identically equals zero only if all of its coefficients are zero. Therefore, we get

$$d_j + d_1 c_j^2 = 0, \qquad \forall j \in J, \tag{2}$$

$$d_1 c_j c_k = 0, \qquad \forall j, k \in J, j \neq k, \tag{3}$$

$$s_j + s_1 c_j = 0, \qquad \forall j \in J. \tag{4}$$

We consider the following three cases separately:

Case 1. $d_1 = 0$;
Case 2. $d_1 \neq 0$ and for all $j \in J$, $c_j = 0$;
Case 3. $d_1 \neq 0$ and there exists a $j \in J$ such that $c_j \neq 0$.

Case 1. We have $d_1 = 0$. Then, $d_j = 0$, $\forall j \in J$, because of (2). From (4), we get $s_j = -s_1 c_j$, $\forall j \in J$. Thus, $f_3(y) = s_1 h(y)$. This is (i).

Case 2. We have $d_1 \neq 0$ and $c_j = 0$ for all $j \in J$. Then, from (2) and (4), we get $d_j = s_j = 0$, for all $j \in J$. This implies (ii).

Case 3. From (2)-(4), we get $d_k = s_k = c_k = 0$, $\forall k \in J - \{j\}$. Hence, we have

$$f_3(y) = d_1 y_1^2 + d_j y_j^2 + s_1 y_1 + s_j y_j, \qquad h(y) = y_1 - c_j y_j,$$

and

$$d_j + d_1 c_j^2 = 0, \qquad s_j + s_1 c_j = 0.$$

Thus,

$$f_3(y) = d_1(y_1^2 - c_j^2 y_j^2) + s_1(y_1 - c_j y_j) = h(y)[d_1(y_1 + c_j y_j) + s_1]. \tag{5}$$

This is (iii). Notice that, in this case, f_3 is indefinite.

Now, the inverse transformation

$$y = Q^{-1}(x - x^0)$$

will change $h(y)$ back to a multiple of $a^T(x - x^0) = a^T x - b$ and $f_3(y)$ to $f_2(x) - f_1(x)$. If f is difference-definite, only (i) and (ii) can happen; then, we have

$$f_2(x) = f_1(x) + \alpha(a^T x - b)^2 + \beta(a^T x - b),$$

where α and β are certain constants. Thus, (1) is valid with $\bar{a}$ being a multiple of a. If f is difference-indefinite, either (i), (ii), or (iii) could happen, but there is at least a pair of P_1 and P_2 such that (iii) is valid. In addition, from (5), the normal vectors of

$$h(y) = y_1 - c_j y_j = 0 \quad \text{and} \quad d_1(y_1 + c_j y_j) + s_1 = 0$$

should be linearly independent because $c_j \neq 0$. This implies (1) and the linear independence of a and $\bar{a}$. □

Remark 2.1. When a and $\bar{a}$ are independent, the images of a and $\bar{a}$ under a nonsingular linear transformation of variables should be still independent. Thus, the term $[a^T x - b][\bar{a}^T x - \bar{b}]$ will not be separable under the transformation. Therefore, the difference-indefinite CPQF is not separable under any nonsingular affine transformation of variables. Moreover, the following corollary says that, if f is difference-indefinite, then there exists two neighboring polyhedra P_1 and P_2 such that the restriction of f on $P_1 \cup P_2$ naturally belongs to a nonconvex function.

Corollary 2.1. (Extensible Convexity) A CPQF is difference-definite if and only if, for any neighboring P_1, $P_2 \subset \text{dom } f$, the function

$$\bar{f}(x) \equiv \begin{cases} f_1(x), & \text{if } x \text{ is on } P_1\text{'s side of the common boundary,} \\ f_2(x), & \text{if } x \text{ is on } P_2\text{'s side of the common boundary,} \end{cases}$$

is convex, where f_1 and f_2 are the quadratic formulas of f on P_1 and P_2, respectively.

Proof. $\bar{f}$ is convex if and only if, for any $x \in \text{dom } f$, $w \in R^n$, the function $\phi(\alpha) = \bar{f}(x + \alpha w)$ is convex, where $\alpha \in R$. The latter is true if and only if $\phi^-(\alpha) \leq \phi^+(\alpha)$, where ϕ^- and ϕ^+ are the ordinary left and right derivatives of ϕ. It is obvious that we only have to consider such α that $x + \alpha w$ is on the common boundary. Since affine transformation does not change convexity, it suffices to show that, for any $y \in H = \{y \mid h(y) = 0\}$ and z pointing to the $\mathscr{P}_2$'s side of the hyperplane H, the directional derivatives of $f_3'(y, z) \geq 0$. If z is the opposite direction, we can show that $f_3'(y, z) \leq 0$ similarly. Here, $f_3(y)$ and $h(y)$ are the same as in the proof of Proposition 2.1. Note that

$$f_3'(y, z) = \nabla f_3(y)^T z.$$

For case (i), $\nabla f_3(y)^T z$ is independent of y. Thus, if for some $y^0 \in H$ we have $f_3'(y^0, z) \geq 0$, then there holds

$$f_3'(y, z) \geq 0, \qquad \text{for all } y \in H.$$

According to the convexity of f, such y^0 certainly exists. For (ii), $\nabla f_3(y)$ only depends on y_1. However, $y_1 = 0$ for all $y \in H$, so $\nabla f_3(y)^T z$ is also independent of y on the boundary. In summary, if f is difference-definite, the local convexity of $\bar{f}$ around some $x^0 \in P_1 \cup P_2$ implies the global convexity of $\bar{f}$.

On the other hand, if f is difference-indefinite, then there exist P_1 and P_2 such that (iii) in the proof of Proposition 2.1 is valid. Hence, we have

$$f_3(y) = h(y)[d_1(y_1 + c_j y_j) + s_1] = h(y)(2d_1 y_1 + s_1),$$

$$\text{on } H = \{y \mid y_1 - c_j y_j = 0,\ c_j \neq 0\}.$$

Therefore,

$$\nabla f_3(y)^T z = (z_1 - c_j z_j)(2d_1 y_1 + s_1), \qquad \text{for } y \in H.$$

However, because $z \notin H$, we have $z_1 - c_j z_j \neq 0$. Thus, $\nabla f_3(y)^T z$ cannot keep the same sign for all $y \in H$ due to $d_1 \neq 0$. Thus, $\bar{f}$ is not convex. □

A simple example of the difference-indefinite CPQF is

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2^2, & \text{if } x_1 \leq 0,\ x_2 \geq 0, \\ x_1^2 + x_1 x_2 + x_2^2, & \text{if } x_1 \geq 0,\ x_2 \geq 0, \\ +\infty, & \text{if } x_2 < 0. \end{cases}$$

If we extend the formula $x_1^2 + x_2^2$ to the second and third quadrants and the formula $x_1^2 + x_1 x_2 + x_2^2$ to the first and fourth quadrants, the resulting function $\bar{f}$ is convex on the upper half of the plane, but not on the lower half of the plane.

An important CPQF is the monitoring function $\rho_{VQ}(u)$ in problem (P) [similarly, $\rho_{UP}(v)$ in (D)]. We now show that this function is difference-definite. For briefness, we only discuss the case of Q being positive definite. The discussion on semidefinite Q can be reduced to this case; see Ref. 2.

Proposition 2.2. The function

$$\rho_{VQ}(x) = \sup_{y \in V} \{y^T x - (y^T Q y)/2\}$$

is a CPQF of difference-definite type, where V is a convex polyhedron and Q is a positive-definite symmetric matrix.

Proof. Since nonsingular linear transformation of variables does not change difference-definiteness, without loss of generality, we assume that

$$y \in R^n, \qquad y^T Q y = y_1^2 + \cdots + y_n^2.$$

Then,

$$\rho_{VQ}(x) = \sup_{y \in V}\{y^T x - (y^T y)/2\} = (x^T x)/2 - \inf_{y \in V} \|y - x\|^2/2.$$

We need to show that the function

$$d(x) \equiv \inf_{y \in V} \|y - x\|^2$$

has the difference-definite property. (In Ref. 2, it is already shown that $\rho_{VQ}(x)$ is a CPQF.) Note that $d(x)$ is the square Euclidean distance from x to V. It may change its expression only if the projection of x on V changes from one face of V to another face of different dimension. The neighboring expressions of f, say f_1 and f_2, correspond to two faces of V, say F_1 and F_2, such that one (say F_1) is contained in the boundary of another (say F_2). Since F_1 is on the boundary of F_2, the square Euclidean distance from x to F_1 is not less than that from x to F_2. The corresponding expressions $d_1(x)$ and $d_2(x)$ of $d(x)$ then have the following property:

$$d_1(x) - d_2(x) \geq 0, \qquad \text{for all } x \in R^n.$$

This is only possible if $d_1(x) - d_2(x)$ has a positive semidefinite Hessian. □

Remark 2.2. Proposition 2.2 indicates that not all CPQF can be expressed as monitoring functions.

3. Separability of the Difference-Definite CPQF

Now we would like to know whether the difference-definite CPQF can be decomposed into a separable form for computational purposes. We notice that, if a CPQF is separable, then the quadratic formula on each of the polyhedra in its domain has the form $x^T Dx + q^T x + r$, where D is a diagonal $n \times n$ matrix. Of course, D, q, r may vary on different polyhedra. Such a CPQF is said to be *diagonal.* On the other hand, the diagonality of a difference-definite CPQF on two neighboring polyhedra implies by Proposition 2.1 that their common boundary should be parallel to a coordinate hyperplane unless these expressions differ only by a linear function. Hence, for a difference-definite CPQF, diagonality suggests a box structure of its domain. In this section, we show that the opposite is almost true. Namely, if all $P_i \subset \operatorname{dom} f$ are of the form $\{x \mid \epsilon_j \leq x_j \leq \delta_j, j = 1, \ldots, n\}$ (ϵ_j may be $-\infty$ and δ_j may be $+\infty$), then such $f(x)$, called the CPQF defined on boxes, must be a sum of a convex quadratic function and a separable CPQF. It should be mentioned that this type of domain structure often arises in practice (e.g., Refs. 2, 4, 6, 9) and that it is not too narrow to make this assumption in theory as one might have imagined.

Proposition 3.1. If a diagonal CPQF is defined on boxes, then this function must be separable. Namely, there exist one-dimensional convex piecewise quadratic functions $f_j(x_j)$, $j=1,\dots,n$, so that $f(x)=f_1(x)+\cdots+f_n(x_n)$.

Proof. The common boundaries of the polyhedra in dom f are parallel to coordinate hyperplanes. The boundaries, together with coordinate hyperplanes, partition dom f into boxes,

$$\text{dom } f = B_1 \cup \cdots \cup B_m.$$

$f(x)$ has a diagonal expression on each of B_i for $i=1,\dots,m$. Let us call a vertex of B_i the southwest corner of B_i if each component of the vertex is not greater than each corresponding component of other vertices of B_i. Without loss of generality, we assume that:

(A) $0 \in \text{dom } f$ and $f(0)=0$, for otherwise the same arguments below can be made for the function $f(x+x^0)-f(x^0)$, where $x^0 \in \text{dom } f$.

(B) The southwest corner of B_1 is the origin and there is a vertex of B_i, $i>1$, $(d_1,\dots,d_n)$, such that the n edges of B_i initiated from $(d_1,\dots,d_n)$,

$$\{x \in B_i \mid x_j = d_j \forall j \neq 1,\ 1 \leq j \leq n\}, \dots,$$

$$\{x \in B_i \mid x_j = d_j \forall j \neq n,\ 1 \leq j \leq n\},$$

are contained either by boxes B_k, $k<i$, or by a coordinate axis.

To achieve (B), we can order B_i's in this way. First, label boxes in R^n_+ according to the lexicographic order of their southwest corner; then, we reflect the second quadrant into R^n_+ and do the same for its boxes, then reflect the third quadrant into the second, and so on.

We now prove that

$$f(x)=f(x_1,0,\dots,0)+f(0,x_2,0,\dots,0)+\cdots+f(0,\dots,0,x_n), \tag{6}$$

for $x \in \text{dom } f$, by induction. The formula is true on box B_1 and all coordinate axes by (A) and direct verification. Now, suppose that this formula is valid for all B_k, $k<i$, and consider box B_i. By assumption (B), each edge of box B_i that goes through the vertex $(d_1,\dots,d_n)$ either belongs to some box B_k, $k<i$, or belongs to some coordinate axis, hence this formula is valid on these edges of B_i. Since f is diagonal on B_i, direct verification shows that, for any $x \in B_i$, the following formula is valid:

$$f(x)=f(x_1,d_2,\dots,d_n)+\cdots+f(d_1,\dots,d_{n-1},x_n)-(n-1)f(d_1,\dots,d_n).$$

By the validity of formula (6) on the mentioned edges, for $x \in B_i$ we have

$$\begin{aligned}
&f(x_1, d_2, \ldots, d_n) + \cdots + f(d_1, \ldots, d_{n-1}, x_n) - (n-1)f(d_1, \ldots, d_n) \\
&= [f(x_1, 0, \ldots, 0) + f(0, d_2, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, d_n)] + \cdots \\
&+ [f(d_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, d_{n-1}, 0) + f(0, \ldots, 0, x_n)] \\
&- (n-1)[f(d_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, d_n)] \\
&= f(x_1, 0, \ldots, 0) + f(0, x_2, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, x_n).
\end{aligned}$$

Thus, (6) is true in B_i. This completes the induction. □

Corollary 3.1. For any difference-definite CPQF $f(x)$ defined on boxes, if it is diagonal on one of these boxes, then it is diagonal on all the boxes; hence, $f(x)$ must be separable.

Proof. Assume that $\operatorname{dom} f = B_1 \cup \cdots \cup B_m$ and $f(x)$ is diagonal on B_1, where the order of B_i's satisfies the same condition as in the proof of Proposition 3.1. By repeatedly using Proposition 2.1, we imply the diagonality of f_{i+1} from f_i, $i = 1, \ldots, m-1$. Proposition 3.1 then ensures the separability. □

Corollary 3.1 says that the inseparability of a CPQF defined on boxes might be caused by a bad expression on a single box. The following result confirms this observation.

Proposition 3.2. Any difference-definite CPQF defined on boxes can be expressed as the sum of a convex quadratic function and a separable CPQF. Moreover, the quadratic function in the sum is exactly the expression on one of the boxes in dom f.

Proof. Let $B_1, \ldots, B_m$ and $f_1, \ldots, f_m$ be the boxes in dom f and the expressions associated with them. Consider the auxiliary function

$$g(x) \equiv f(x) - f_1(x) + \lambda(x_1^2 + \cdots + x_n^2),$$

where $\lambda \geq 0$ is large enough to ensure the convexity of g. Because g is a difference-definite CPQF defined on boxes and is diagonal on B_1, by Corollary 3.1, there exist one-dimensional CPQFs $g_1, \ldots, g_n$ such that $g(x) = g_1(x_1) + \cdots + g_n(x_n)$. Suppose that, for $j = 1, \ldots, n$,

$$g_j(x_j) = \begin{cases} +\infty, & \text{if } x_j < c_{j0}, \\ p_{j1}x_j^2 + q_{j1}x_j + r_{j1}, & \text{if } c_{j0} \leq x_j \leq c_{j1}, \\ \cdots & \\ p_{jk_j}x_j^2 + q_{jk_j}x_j + r_{jk_j}, & \text{if } c_{jk_j-1} \leq x_j \leq c_{jk_j}, \\ +\infty, & \text{if } x_j > c_{jk_j}. \end{cases}$$

Let

$$p_j = \min\{p_{jk} \mid k = 1, \ldots, k_j\}, \qquad \text{for } j = 1, \ldots, n.$$

Then the n-tuple $(p_1, \ldots, p_n)$ corresponds to at least one box, say B_2, so that the expression of g on B_2 is

$$G(x) = p_1 x_1^2 + \cdots + p_n x_n^2 + \cdots.$$

Here, ... denotes the nonquadratic terms. Now, we show that $f - f_2$ is convex and separable. Note that

$$f(x) - f_2(x) = g(x) - [f_2(x) - f_1(x) + \lambda(x_1^2 + \cdots + x_n^2)]$$
$$= g(x) - G(x) = \sum_{j=1}^{n} [g_j(x_j) - p_j x_j^2] - \cdots.$$

Since

$$p_{jk} \geq p_j, \qquad \text{for } k = 1, \ldots, k_j,$$

$f - f_2$ is a (separable) convex function. This completes the proof. □

Remark 3.1. Proposition 3.2 says that minimizing any difference-definite CPQF defined on boxes can be reduced to minimizing the sum of a separable CPQF and a smooth convex quadratic function. The problem of minimizing such a function $x^T Rx + q^T x + \sum_i f_i(x_i)$ can in turn be reformulated as

$$\min\left\{ y^T y + q^T x + \sum_i f_i(x_i) \,|\, y = Qx \right\}, \qquad \text{where } Q^T Q = R.$$

Thus, this type of problem is essentially a monotropic piecewise quadratic programming problem. Of course, it can also be solved by other decomposition techniques, e.g., the one recently proposed by Han (Ref. 12).

4. Conclusions

Convex piecewise quadratic functions can be divided into two classes—difference-definite and difference-indefinite ones. The expressions of a difference-definite CPQF are determined by its expression on one polyhedron plus a linear combination of $[(a^i)^T x - b_i]^2$ and $(a^i)^T x - b_i$, where $(a^i)^T x - b_i = 0$, $i = 1, \ldots, t$, are equations of the common boundaries between the neighboring polyhedra in its domain. The same is true for a difference-indefinite CPQF, but with additional terms of the form $[(a^i)^T x - b_i][(\bar{a}^i)^T x - \bar{b}_i]$, where $\bar{a}^i$ is linearly independent from a^i. The existence of such $\bar{a}^i$ makes a difference-indefinite CPQF inseparable under any nonsingular affine transformation of the variables. The difference-definite class is important for applications, because it includes the monitoring function as a special case. If, in addition, a difference-definite CPQF is defined on boxes, then it can be expressed as the sum of a convex quadratic function and a separable CPQF. Therefore, their minimization problems can be reduced to monotropic piecewise quadratic programs.

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