R-matrices for Elliptic Calogero–Moser Models

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Abstract. The classical R-matrix structure for the n-particle Calogero–Moser models with (type IV) elliptic potentials is investigated. We show there is no momentum independent R-matrix (without spectral parameter) when $n \geqslant 4$. The assumption of momentum independence is sufficient to reproduce the dynamical R-matrices of Avan and Talon for the type I, II, III degenerations of the elliptic potential. The inclusion of a spectral parameter enables us to find R-matrices for the general elliptic potential.

1. Introduction

The Calogero–Moser model [1, 2] is perhaps the paradigm of a completely integrable system of n-particles on the line which interact via pairwise potentials $v(q_i - q_j)$. The most general form of the potential in such models [3] is the (so called type IV) elliptic potential, $v(q) = a^2 \wp(aq)$, where $\wp$ is the Weierstrass elliptic function. The various degenerations of this function yield rational (type I, $v(q) = 1/q^2$), hyperbolic (type II, $v(q) = a^2/\sinh(aq)^2$) and trigonometric (type III, $v(q) = a^2/\sin(aq)^2$) potentials. In accord with their importance these models have been studied from many different perspectives. They have a Lax pair formulation, $i\dot{L} = [L, M]$; the ansatz for this Lax pair leads to a study of functional equations [4–7]. Further, these models may be expressed as the Hamiltonian reduction [8, 9] of integrable flows on the cotangent bundle of symmetric spaces. The quantisation [10] of these models has also been of some interest. Recently, the related (type V) potential $v(q) = 1/q^2 + gq^2$ has been shown to be relevant to the collective field theory of strings [11].

The proof of the complete integrability of a system given in terms of a Lax pair involves several stages. The first, an immediate consequence of a Lax pair formulation, is the observation that the quantities $\mathrm{Tr}_E L^k$ are conserved. Here the trace is taken over the representation E of the Lie algebra $\mathfrak{g}$ to which the operator L is associated. Another stage is to show these provide enough functionally independent conserved quantities. Finally, and this is perhaps the most tedious step, one must show the quantities are in involution, i.e. $\{\mathrm{Tr}_E L^k, \mathrm{Tr}_E L^m\} = 0$. This step is model dependent. For the Calogero–Moser this stage may be achieved by arguments based on asymptotics [2], inverse scattering [12] or direct recursion [13]. Given an L-operator, an alternative approach to proving this Poisson commutativity proceeds

via the R-matrix [14–16]. An R-matrix is an $E \otimes E$ matrix satisfying

$$\{L \underset{,}{\otimes} L\} = [R, L \otimes 1] - [R^\pi, 1 \otimes L]. \tag{1}$$

An immediate consequence of the existence of an R-matrix is that

$$\{\mathrm{Tr}_E L^k, \mathrm{Tr}_E L^m\} = \mathrm{Tr}_{E \otimes E}\{L^k \underset{,}{\otimes} L^m\} = km\, \mathrm{Tr}_{E \otimes E} L^{k-1} \otimes L^{m-1}\{L \underset{,}{\otimes} L\} = 0.$$

The vanishing follows from (1) by expressing $\{L \underset{,}{\otimes} L\}$ as a commutator and using the cyclicity of the trace. An R-matrix also allows one to canonically construct the matrix M of the Lax pair [14]. Recently Avan and Talon [17] constructed the R-matrices for the Calogero-Moser models and potentials of type I, II, III and V, thus providing this alternate means of proof. The R-matrices found by Avan and Talon were dynamical: that is, they depended on the dynamical variables (in their case positions) of the model. In contrast to systems governed by purely numerical R-matrices [15], dynamical R-matrices needn't satisfy the Yang-Baxter equation and the theory of dynamical R-matrices is not well understood [18–20]. One hopes that such concrete examples of dynamical R-matrices as are provided by the Calogero–Moser models will aid in the elucidation of this theory.

Our present work investigates the R-matrix structure of the Calogero–Moser models further. The R-matrices of Avan and Talon were constructed on the basis of two assumptions, namely momentum independence and the vanishing of certain terms of the R-matrix. These assumptions were found to be consistent with the potentials of type I, II, III, V but did not allow the type IV potential. One might ask what happens if these assumptions are relaxed. We shall show that the second of Avan and Talon's assumptions actually follows from that of momentum independence (given sufficient particles). We can therefore conclude that no momentum independent R-matrix exists for the L-operators under consideration. In [21] Krichever enlarged the class of L-operators yielding type IV potentials to include a spectral parameter. The usual L-operators exactly correspond to those values of the spectral parameter for which the operators are hermitian. We show that the inclusion of the spectral parameter allows us to construct a momentum independent R-matrix.

The plan of this Letter is as follows. In Section 2 we will introduce our notation, the L-operators under consideration and the equations to be solved. Section 3 looks at the simplifications resulting from the assumption of momentum independence. Section 4 then shows that for the usual L-operator no momentum independent R-matrix can be found. Upon introducing a spectral parameter in Section 5, we then exhibit a solution to the corresponding equations. We conclude with a brief discussion.

2. Preliminaries

As an alternative of the matrix entry calculations often presented, we give our calculations in terms of a basis of the underlying Lie algebra. Although we will

ultimately specialise to the gl_n case, we believe this to be both computationally and conceptually easier. It will also enable us to isolate those features peculiar to gl_n. We begin by deriving the relevant equations to be solved and introducing our notation.

Let X_μ denote a Cartan–Weyl basis for the (semi-simple) Lie algebra $\mathfrak{g}$ associated with the operator L. That is $\{X_\mu\} = \{H_i, E_\alpha\}$, where $\{H_i\}$ is a basis for the Cartan subalgebra $\mathfrak{h}$ and $\{E_\alpha\}$ is the set of step operators (labelled by the root system Φ of $\mathfrak{g}$) and

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \quad \text{and}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Phi.$$

With $[X_\mu, X_\nu] = c_{\mu\nu}^\lambda X_\lambda$ defining the structure constants of $\mathfrak{g}$, we see, for example[*], that $c_{i\alpha}^\beta = \delta_{\alpha,\beta} \alpha_i$ and $c_{\alpha-\alpha}^i = \alpha_i$. Further, we adopt the Hermiticity convention $E_\alpha^\dagger = E_{-\alpha}$ for the representations of $\mathfrak{g}$. The structure constants may then be chosen to be real and to have the symmetries: $c_{\alpha\beta}^\lambda = -c_{\beta\alpha}^\lambda = c_{\lambda-\beta}^\alpha = -c_{-\alpha-\beta}^{-\lambda}$.

With this notation at hand we express the L operator as

$$L \equiv \sum_\mu L^\mu X_\mu = p \cdot H + i \sum_{\alpha \in \Phi} w_\alpha E_\alpha,$$

where [21]

$$w_\alpha \equiv w(\alpha \cdot q; u) = \frac{\sigma(u - \alpha \cdot q)}{\sigma(u)\sigma(\alpha \cdot q)} e^{\zeta(u)\alpha \cdot q}. \tag{2}$$

Here $\sigma(x)$ and $\zeta(x) = \sigma'(x)/\sigma(x)$ are the Weierstrass sigma and zeta functions [22]. The quantity u in (2) is known as the spectral parameter and we will only make its appearance explicit when confusion might otherwise arise. It will also be convenient to use the shorthand f_α for a function on $\mathfrak{h}$ that takes the value $f(\alpha \cdot q)$ when evaluated at q. The functions w_α satisfy the addition formula

$$w_\alpha w_\beta' - w_\beta w_\alpha' = (z_\alpha - z_\beta)w_{\alpha+\beta}, \quad \text{where} \quad z_\alpha(u) = \frac{w_\alpha''(u)}{2w_\alpha(u)} = \wp_\alpha + \tfrac{1}{2}\wp(u). \tag{3}$$

Clearly $L = L^\dagger$ when $w_\alpha = -w_{-\alpha}$ and this is the case usually considered. Requiring Hermiticity restricts the spectral parameter with the result that $u \in \{\omega_1, \omega_2, -\omega_1 - \omega_2\}$, where $2\omega_{1,2}$ are the periods of the associated elliptic functions.

For the Lie algebra gl_n, the case of most interest to us, $\Phi = \{e_i - e_j, 1 \leqslant i \neq j \leqslant n\}$ with the e_i an orthonormal basis of $\mathbb{R}^n$. If e_{rs} denotes the elementary matrix with (r,s)-th entry one and zero elsewhere, then the $n \times n$ matrix representation $H_i = e_{ii}$ and $E_\alpha = e_{ij}$ when $\alpha = e_i - e_j$ gives the usual representation of L. Working with the simple algebra a_n corresponds to the center of mass frame.

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[*]Throughout, Roman indices will denote the Cartan subalgebra basis elements while the early Greek indices $\alpha, \beta, \dots$ will denote the step operators.

Let us begin unravelling (1). The left-hand side becomes

$$\{L \overset{,}{\otimes} L\} = \sum_{\mu,\nu} \{L^{\mu}, L^{\nu}\} X_{\mu} \otimes X_{\nu} = i \sum_{j,\alpha} \alpha_{j} w'_{\alpha} (H_{j} \otimes E_{\alpha} - E_{\alpha} \otimes H_{j}) \tag{4}$$

upon using $\{p_{j}, w_{\alpha}\} = \{p_{j}, \alpha \cdot q\} w'_{\alpha} = \alpha_{j} w'_{\alpha}$. Turning now to the right-hand side of (1), we have $R = R^{\mu\nu} X_{\mu} \otimes X_{\nu}$ and $R^{\pi} = R^{\nu\mu} X_{\mu} \otimes X_{\nu}$. Then

$$\begin{aligned}
[R, L &\otimes 1] - [R^{\pi}, 1 \otimes L] \\
&= R^{\mu\nu}([X_{\mu}, L] \otimes X_{\nu} - X_{\nu} \otimes [X_{\mu}, L]) \\
&= R^{\mu\nu} L^{\lambda}([X_{\mu}, X_{\lambda}] \otimes X_{\nu} - X_{\nu} \otimes [X_{\mu}, X_{\lambda}]) \\
&= (R^{\tau\nu} c^{\mu}_{\tau\lambda} L^{\lambda} - R^{\tau\mu} c^{\nu}_{\tau\lambda} L^{\lambda}) X_{\mu} \otimes X_{\nu}.
\end{aligned}$$

In terms of the Lie algebra basis, (1) then becomes the equation

$$\{L^{\mu}, L^{\nu}\} = R^{\tau\nu} c^{\mu}_{\tau\lambda} L^{\lambda} - R^{\tau\mu} c^{\nu}_{\tau\lambda} L^{\lambda}. \tag{5}$$

Observe that (5) has the structure of a matrix equation,

$$VR + R^{\dagger}V = A,$$

for the unknown matrix R in terms of the specified

$$A^{\mu\nu} = \{L^{\mu}, L^{\nu}\} \quad \text{and} \quad V^{\mu\nu} = c^{\mu}_{\nu\lambda} L^{\lambda}.$$

Equation (5) yields three different equations, depending on the range of $\{\mu, \nu\}$. For $(\mu, \nu) = (i, j)$, (i, α) and (α, β), respectively, these are

$$0 = \sum_{\alpha}(R^{\alpha j}\alpha_{i} - R^{\alpha i}\alpha_{j})w_{-\alpha}, \tag{6}$$

$$-\alpha_{i}w'_{\alpha} = i\alpha \cdot pR^{\alpha i} + \alpha \cdot R^{i}w_{\alpha} + \sum_{\beta}(\beta_{i}w_{\beta}R^{-\beta\alpha} + w_{\alpha-\beta}R^{\beta i}c^{\alpha}_{\beta\,\alpha-\beta}) \tag{7}$$

and

$$\begin{aligned}
0 = \alpha \cdot R^{\beta}w_{\alpha} &- \beta \cdot R^{\alpha}w_{\beta} + i(\alpha \cdot pR^{\alpha\beta} - \beta \cdot pR^{\beta\alpha}) + \\
&+ \sum_{\gamma}(R^{\gamma\beta}c^{\alpha}_{\gamma\alpha-\gamma}w_{\alpha-\gamma} - R^{\gamma\alpha}c^{\beta}_{\gamma\beta-\gamma}w_{\beta-\gamma})
\end{aligned} \tag{8}$$

Here we have introduced the shorthand $\beta \cdot R^{\mu} \equiv \Sigma_{i}\beta_{i}R^{i\mu}$. Equations (6)–(8) are the components of (1) in our basis.

3. Momentum Independence

We now turn to the solution of Equations (6)–(8) subject to the assumption that the R-matrix is independent of momentum. This assumption (introduced in [17]) means that

$$R^{\alpha i} = 0, \quad R^{\alpha - \alpha} + R^{-\alpha\alpha} = 0 \quad \text{and} \quad R^{\alpha\beta} = 0 \quad \text{if } \alpha \neq \pm\beta. \tag{9}$$

The first of these restrictions follows from (7) while the remainder come from (8). For example, in the matrix components of gl_{n} introduced earlier, we have $R^{\alpha i} = 0 \Leftrightarrow$

$R^{jkii} = 0$ and we thus obtain Equations (14) of [17]. At this stage, Equation (6) is satisfied identically and the variables remaining are R^{ij}, $R^{i\alpha}$, $R^{\alpha\alpha}$ and $R^{\alpha-\alpha}$. The remaining equations to be solved are

$$-\alpha_i w'_\alpha = \alpha \cdot R^i w_\alpha + \alpha_i R^{-\alpha\alpha} w_\alpha - \alpha_i R^{\alpha\alpha} w_{-\alpha} \tag{10}$$

and

$$\begin{aligned}\alpha \cdot R^\beta w_\alpha &- \beta \cdot R^\alpha w_\beta \\ &= c^\beta_{\alpha\gamma}(R^{\alpha\alpha} w_\gamma - R^{\beta\beta} w_{-\gamma}) + c^\beta_{-\alpha\gamma}(R^{-\alpha\alpha} w_\gamma + R^{-\beta\beta} w_\gamma).\end{aligned} \tag{11}$$

The first term on the right-hand side of (11) is nonvanishing only for $\gamma = \beta - \alpha \in \Phi$ while the second term is nonvanishing for $\gamma = \beta + \alpha$. We note that for the simply-laced algebras ($\alpha \cdot \alpha = 2, \forall \alpha \in \Phi$), at most one of the terms on the right-hand of (11) can be nonvanishing and we henceforth assume this to be the case. Now, by viewing the root $\gamma = \beta - \alpha$ as being also the sum $\gamma = \beta + (-\alpha)$, we obtain the two equations

$$\alpha \cdot R^\beta w_\alpha - \beta \cdot R^\alpha w_\beta = c^\beta_{\alpha\gamma}(R^{\alpha\alpha} w_\gamma - R^{\beta\beta} w_{-\gamma}) \tag{12}$$

and

$$-\alpha \cdot R^\beta w_{-\alpha} - \beta \cdot R^{-\alpha} w_\beta = c^\beta_{\alpha\gamma}(R^{\alpha-\alpha} w_\gamma + R^{-\beta\beta} w_\gamma). \tag{13}$$

We shall utilise the consistency of these equations below.

Our first observation is that $R^{ij} = \eta \delta^{ij} + P^{ij}$ for some constant η that we shall later determine and matrix P^{ij} orthogonal to the roots, $\alpha \cdot P^j = 0 \forall j$. To see this, view (10) as an equation between vectors; thus $\alpha \cdot R^i$ must be proportional to α_i. If we define the constant of proportionality by $\alpha \cdot R^i = \eta_\alpha \alpha_i$, where η_α could in principle depend on α, then by linearity

$$(\alpha + \beta) \cdot R^i = \eta_{\alpha+\beta}(\alpha_i + \beta_i) = \eta_\alpha \alpha_i + \eta_\beta \beta_i$$

and so $\eta_{\alpha+\beta} = \eta_\alpha = \eta_\beta \equiv \eta$. Now for each α we have $\Sigma_i \alpha_i R^{ij} = \eta \alpha_j$, and so $R^{ij} = \eta \delta^{ij} + P^{ij}$. The matrix P^{ij} orthogonal to the roots arises when we have $u(1)$ factors present in $\mathfrak{g}$. Thus we have

$$-w'_\alpha = \eta w_\alpha + R^{-\alpha\alpha} w_\alpha - R^{\alpha\alpha} w_{-\alpha}. \tag{14}$$

The assumption of momentum independence leads then to the two equations (11) and (14).

4. The Case $w_\alpha = -w_{-\alpha}$

Our discussion has so far made no use of the form of w_α. For the remainder of this section we will assume that w_α is an odd function, the case usually considered. As we shall see, this results in some quite strong conclusions. The next section, which deals with the inclusion of a spectral parameter, will consider the more general case. First let us show

LEMMA 4.1. $\eta = 0$.

Proof. Upon subtracting from (14) the analogous equation obtained by replacing α with $-\alpha$ and using the second equation of (9), we find

$$\eta = -\tfrac{1}{2}(R^{\alpha\alpha} + R^{-\alpha-\alpha}). \tag{15}$$

Further, upon subtracting (12) from (13), we obtain

$$\beta \cdot (R^\alpha - R^{-\alpha})w_\beta = c_{\alpha\gamma}^\beta (R^{\alpha-\alpha} + R^{-\beta\beta} - R^{\alpha\alpha} - R^{\beta\beta})w_\gamma. \tag{16}$$

The same operations applied to the analogous equations based now on $\gamma = (-\alpha) - (-\beta)$ and $-\gamma = \alpha - \beta$ yield

$$\beta \cdot (R^\alpha - R^{-\alpha})w_\beta = c_{\alpha\gamma}^\beta (R^{-\alpha-\alpha} + R^{-\beta-\beta} - R^{-\alpha\alpha} - R^{\beta-\beta})w_\gamma. \tag{17}$$

Upon comparing these last two equations and using (15) together with $R^{\alpha-\alpha} + R^{-\alpha\alpha} = 0$, we find that $\eta = 0$. $\qquad\square$

Therefore $R^{ij} = P^{ij}$; the choice $P^{ij} = 0$ corresponds to the middle two equations of Avan and Talon's second assumption [17]. We have now reduced the possible nonzero variables of the R-matrix to $R^{i\alpha}$, $R^{\alpha\alpha}$ and $R^{\alpha-\alpha}$ subject to

$$R^{\alpha-\alpha} + R^{-\alpha\alpha} = 0, \quad R^{\alpha\alpha} + R^{-\alpha-\alpha} = 0, \quad R^{\alpha\alpha} + R^{-\alpha\alpha} = -\frac{w'_\alpha}{w_\alpha} \tag{18}$$

and

$$\alpha \cdot (R^\beta - R^{-\beta})w_\alpha$$
$$= c_{\alpha\gamma}^\beta \left(-\frac{w'_\alpha}{w_\alpha} - \frac{w'_\beta}{w_\beta} + 2R^{\alpha-\alpha} \right)w_\gamma - c_{-\alpha\gamma'}^\beta \left(-\frac{w'_\alpha}{w_\alpha} + \frac{w'_\beta}{w_\beta} + 2R^{\alpha-\alpha} \right)w_{\gamma'}. \tag{19}$$

Of course, $c_{\alpha\gamma}^\beta = 0$ unless $\gamma = \beta - \alpha \in \Phi$ (in which case $\alpha \cdot \gamma \ne 0$). Our method of solving these equations proceeds as follows. Let us define the quantity $A_{\beta\gamma}$ by

DEFINITION 4.2.

$$A_{\beta\gamma} \equiv \beta \cdot (R^\gamma - R^{-\gamma})w_\gamma - c_{\gamma-\beta\beta}^\gamma z_\gamma. \tag{20}$$

Obviously this is closely related to the left hand side of (19). Our aim will be to show this quantity to be constant, from which we will be able to deduce the remaining equations of Avan and Talon's second assumption.

LEMMA 4.3. $A_{\alpha\beta} = A_{\beta\alpha} = -A_{\beta\gamma} = A_{\gamma-\alpha} = A_{-\beta\gamma}$.

Proof. First let us motivate the definition of $A_{\beta\gamma}$ and then derive its symmetries. Suppose $\gamma = \beta - \alpha \in \Phi$ so only the first term in (19) is nonvanishing. The analogous equation to (19) for $\beta = (\beta - \alpha) - (-\alpha)$ is

$$\alpha \cdot (R^\gamma - R^{-\gamma})w_\alpha = c_{-\alpha\beta}^\gamma \left(\frac{w'_\alpha}{w_\alpha} - \frac{w'_\gamma}{w_\gamma} + 2R^{-\alpha\alpha} \right)w_\beta. \tag{21}$$

Upon adding $w_\beta \times (19)$ to $w_\gamma \times (21)$ and using the symmetries of the structure constants together with the addition formula (3) for w_α, we obtain

$$\alpha \cdot (R^\beta - R^{-\beta}) w_\beta + \alpha \cdot (R^\gamma - R^{-\gamma}) w_\gamma = c_{\alpha\gamma}^\beta (z_\gamma - z_\beta).$$

(For the case at hand, $z_\gamma = z_{-\gamma}$.) After substituting $\alpha = \beta - \gamma$ in this expression and making use of the fact $\alpha \cdot (R^\alpha - R^{-\alpha}) = 0$, which follows from (19), we obtain

$$\beta \cdot (R^\gamma - R^{-\gamma}) w_\gamma - c_{-\alpha\beta}^\gamma z_\gamma = \gamma \cdot (R^\beta - R^{-\beta}) w_\beta - c_{\alpha\gamma}^\beta z_\beta. \tag{22}$$

Therefore $A_{\beta\gamma} = A_{\gamma\beta}$. Upon using $\alpha = \beta - \gamma$ the remaining symmetries are similarly shown. $\qquad\square$

Thus, to every triangle formed by three roots α, β, γ we have associated a single constant $A_{\alpha\beta}$ (up to a sign which is taken care of below).

Now suppose the root α may be expressed as a sum of two distinct pairs of roots, $\alpha = \beta - \gamma = \beta' - \gamma'$. This requires that $n \geqslant 4$. For the simply-laced case being considered, we may further assume $\alpha \cdot (\gamma - \gamma') = 0$ and that our labelling is such that $\gamma - \gamma' \in \Phi$. What then is the relation between $A_{\alpha-\gamma}$ and $A_{\alpha-\gamma'}$? Using (19) we see that

$$(\gamma - \gamma') \cdot (R^\alpha - R^{-\alpha}) = 0. \tag{23}$$

Further $c_{\beta-\gamma}^\alpha = c_{\beta'-\gamma'}^\alpha$, and so

$$A_{\alpha-\gamma} = -\gamma \cdot (R^\alpha - R^{-\alpha}) w_\alpha - c_{\beta-\gamma}^\alpha z_\alpha = -\gamma' \cdot (R^\alpha - R^{-\alpha}) w_\alpha - c_{\beta'-\gamma'}^\alpha z_\alpha = A_{\alpha-\gamma'}.$$

We have just shown that the constants $A_{\alpha-\gamma}$ associated with a triangle of roots are the same whenever they share a root α as a common edge. Now we can get from one triangle of roots to any other by intermediate root triangles. Therefore the constants $A_{\alpha-\gamma}$ depend on all the roots in the same fashion and we have shown

LEMMA 4.4. $A_{\alpha\beta} = c_{\beta-\alpha\alpha}^\beta \mathscr{A}$ for some function $\mathscr{A}$.

On combining this lemma and (19) we see that

$$\alpha \cdot (R^\beta - R^{-\beta}) w_\alpha w_\beta = c_{\gamma\alpha}^\beta \left(\frac{w'_\alpha}{w_\alpha} + \frac{w'_\beta}{w_\beta} + 2R^{-\alpha\alpha} \right) w_\gamma w_\beta = c_{\gamma\alpha}^\beta (\mathscr{A} + z_\beta) w_\alpha. \tag{24}$$

Thus $\mathscr{A}$ determines $R^{-\alpha\alpha}$ and (via (18)) $R^{\alpha\alpha}$, assuming we are given w_α and z_α. Further,

LEMMA 4.5. For $n \geqslant 4$ $\mathscr{A}$ is a constant and $R^{-\alpha\alpha}$ is a function of α only.

Proof. Once again, suppose the root α may be expressed as a sum of two distinct pairs of roots, $\alpha = \beta - \gamma = \beta' - \gamma'$. Comparing (24) with the analogous equation in β', γ' enables us to show that

$$\left(\frac{w'_\beta}{w_\beta} - \frac{z_\beta w_\alpha}{w_\gamma w_\beta} \right) - \left(\frac{w'_{\beta'}}{w_{\beta'}} - \frac{z_{\beta'} w_\alpha}{w_{\gamma'} w_{\beta'}} \right) = \left(\frac{w_\alpha}{w_\gamma w_\beta} - \frac{w_\alpha}{w_{\gamma'} w_{\beta'}} \right) \mathscr{A},$$

and so we may solve for $\mathscr{A}$ explicitly in terms of the roots shown. We have argued however that $\mathscr{A}$ depends on all of the roots in the same fashion. Therefore $\mathscr{A}$ is a constant. Having shown that $\mathscr{A}$ is a constant, let us rewrite (24) again, assuming that $\alpha = \beta - \gamma = \beta' - \gamma'$. Then,

$$(\mathscr{A} + z_\beta)\frac{w_\alpha}{w_\gamma w_\beta} - \frac{w'_\beta}{w_\beta} = 2R^{-\alpha\alpha} + \frac{w'_\alpha}{w_\alpha} = (\mathscr{A} + z_{\beta'})\frac{w_\alpha}{w_{\gamma'} w_{\beta'}} - \frac{w'_{\beta'}}{w_{\beta'}}.$$

The right-hand side of this equation is a function of β and γ only, while the left is a function of β' and γ'. Thus both are functions of α only and the remainder of the lemma follows. $\square$

The final stage of our argument consists of showing there is no constant $\mathscr{A}$ which makes $2R^{-\alpha\alpha} + w'_\alpha/w_\alpha$ a function of α only for the elliptic potentials being considered. Take, for example [22], $w_\alpha = 1/sn(\alpha \cdot x, k)$. Here

$$(\mathscr{A} + z_\beta)\frac{w_\alpha}{w_\gamma w_\beta} - \frac{w'_\beta}{w_\beta} = \left(\mathscr{A} - \frac{1 + k^2}{2} + \frac{k^2}{w_\alpha^2}\right)\frac{w_\alpha}{w_\gamma w_\beta} - \frac{w'_\alpha}{w_\alpha},$$

and we cannot both have $\mathscr{A}$ constant and this expression depending only on α unless $k = 0$. For $k = 0$, which corresponds to the type III degeneration, we find $R^{-\alpha\alpha} = -w'_\alpha/w_\alpha$, $R^{\alpha\alpha} = 0$ and

$$\alpha \cdot (R^\beta - R^{-\beta}) = c_{\gamma\alpha}^\beta w_\beta.$$

For gl_n, this has a solution $R^{i\beta} = w_\beta/2$ when $\beta \cdot e_i \neq 0$, and zero otherwise. We have thus obtained the remainder of Avan and Talon's assumptions together with their solution [17]. We have therefore shown

THEOREM 4.6. *If $n \geqslant 4$ there are no momentum independent R-matrices for the nondegenerate type IV potential and $w_\alpha = -w_{-\alpha}$.*

When $n = 2, 3$ our consistency arguments do not arise. For $n = 2$ there is only one root and $n = 3$ only one root triangle and solutions in both cases are possible.

5. Inclusion of a Spectral Parameter

Having shown there are no momentum independent R-matrices for L-operators with $w_\alpha = -w_{-\alpha}$, several possibilities remain. We may for example relax the assumption of momentum independence and solve the full equations (6–8), or we may look at a broader class of functions w_α. We will adopt the latter approach in this note and seek momentum independent R-matrices for the class of L-operators (introduced by Krichever) containing a spectral parameter. The generalisation of (1) to the situation with spectral parameter is

$$\{L(u) \overset{\otimes}{,} L(v)\} = [R(u, v), L(u) \otimes 1] - [R^\pi(u, v), 1 \otimes L(v)]. \tag{25}$$

If $R(u, v) = R^{\mu\nu}(u, v)X_\mu \otimes X_\nu$ then $R^\pi(u, v)$ is defined by $R^\pi(u, v) = R^{\nu\mu}(v, u)X_\mu \otimes X_\nu$.

We proceed in the same manner given earlier. The left-hand side of (25) is

$$\{L(u) \underset{,}{\otimes} L(v)\} = i \sum_{j,\alpha} (\alpha_j w'_\alpha(v) H_j \otimes E_\alpha - \alpha_j w'_\alpha(u) E_\alpha \otimes H_j),$$

and in terms of our basis (25) takes the form

$$\{L^\mu(u), L^\nu(v)\} = R^{\tau\nu}(u, v) c^\mu_{\tau\lambda} L^\lambda(u) - R^{\tau\mu}(v, u) c^\nu_{\tau\lambda} L^\lambda(v). \tag{26}$$

Again three equations arise, depending on the range of $\{\mu, \nu\}$. (The new possibility $(\mu, \nu) = (\alpha, i)$ yields the same equation as $(\mu, \nu) = (i, \alpha)$ with u and v interchanged.)

Once again the assumption that $R(u, v)$ is momentum independent greatly reduces the possible nonzero components of $R(u, v)$. We find

$$R^{\alpha i}(u, v) = 0, \qquad R^{\alpha - \alpha}(u, v) + R^{-\alpha\alpha}(v, u) = 0 \quad \text{and}$$
$$R^{\alpha\beta}(u, v) = 0 \quad \text{if } \alpha \neq \pm\beta. \tag{27}$$

The components to be determined are $R^{ij}(u, v)$, $R^{i\alpha}(u, v)$, $R^{\alpha\alpha}(u, v)$ and $R^{\alpha - \alpha}(u, v)$. Again we may argue that $R^{ij}(u, v) = \eta(u, v)\delta^{ij} + P^{ij}$, and we arrive at two equations

$$-w'_\alpha(v) = \eta(u, v)w_\alpha(v) + R^{-\alpha\alpha}(u, v)w_\alpha(u) - R^{\alpha\alpha}(v, u)w_{-\alpha}(u) \tag{28}$$

and

$$\alpha \cdot R^\beta(u, v)w_\alpha(u) - \beta \cdot R^\alpha(v, u)w_\beta(v)$$
$$= c^\beta_{\alpha\gamma}(R^{\alpha\alpha}(v, u)w_\gamma(v) - R^{\beta\beta}(u, v)w_{-\gamma}(u)) +$$
$$+ c^\beta_{-\alpha\gamma}(R^{-\alpha\alpha}(v, u)w_\gamma(v) + R^{-\beta\beta}(u, v)w_\gamma(u)). \tag{29}$$

These equations are the analogues of (14) and (11), respectively.

At this stage we make the ansatz

$$\eta(u, v) = \zeta(v - u) + \zeta(u) - \zeta(v), \qquad R^{\alpha\alpha}(v, u) = 0 \tag{30}$$

and

$$R^{-\alpha\alpha}(u, v) = w_\alpha(v - u)\, e^{-(\zeta(v-u) - \zeta(v) + \zeta(u))\alpha}. \tag{31}$$

For any Lie algebra, this ansatz solves (28) and reduces (29) to the equation

$$\alpha \cdot \frac{R^\beta(u, v)}{w_\beta(v)} - \beta \cdot \frac{R^\alpha(v, u)}{w_\alpha(u)} = c^\gamma_{\alpha\beta}. \tag{32}$$

The consistency conditions exploited in the last section are implicit here in the structure constants $c^\gamma_{\alpha\beta}$. Certainly, for the case of gl_n we may solve (32) by setting

$$R^{i\alpha}(u, v) = \tfrac{1}{2}w_\alpha(v), \quad \text{whenever } \alpha \cdot e_i \neq 0 \tag{33}$$

and zero otherwise. The remainder of this section will be devoted to proving some of these assertions.

First let us show that (28) is satisfied. Now

$$R^{-\alpha\alpha}(u, v) \frac{w_\alpha(u)}{w_\alpha(v)} = \frac{\sigma(u - \alpha)\sigma(v)\sigma(v - u - \alpha)}{\sigma(u)\sigma(v - \alpha)\sigma(\alpha)\sigma(v - u)}$$

$$= \zeta(-u) + \zeta(v - \alpha) + \zeta(\alpha) - \zeta(v - u).$$

The exponential factors in our ansatz for $R^{-\alpha\alpha}(u, v)$ have been chosen so that there is cancellation leaving only the σ factors in the middle term. The final equality makes use of the identity [22]

$$\frac{\sigma(x + y)\sigma(y + z)\sigma(z + x)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x + y + z)} = \zeta(x) + \zeta(y) + \zeta(z) - \zeta(x + y + z).$$

Finally, upon making use of

$$w'_\alpha(v)/w_\alpha(v) = \zeta(v) - \zeta(\alpha) - \zeta(v - \alpha),$$

we find (28) holds for our choice of η.

As for (29), first observe that

$$R^{-\alpha\alpha}(v, u)w_{\alpha+\beta}(v) + R^{-\beta\beta}(u, v)w_{\alpha+\beta}(u)$$

$$= \left[-\frac{\sigma(u - v - \alpha)\sigma(v - \alpha - \beta)}{\sigma(\alpha)\sigma(v)} + \frac{\sigma(v - u - \beta)\sigma(u - \alpha - \beta)}{\sigma(\beta)\sigma(u)} \right] \times$$

$$\times \frac{e^{\zeta(u)\alpha + \zeta(v)\beta}}{\sigma(\alpha + \beta)\sigma(v - u)}$$

$$= w_\alpha(u)w_\beta(v).$$

To obtain the final equality, we have employed the 'three-term equation' of Weierstrass [22, §20·53],

$$\sigma(x - y)\sigma(x + y)\sigma(z - t)\sigma(z + t) + \sigma(y - z)\sigma(y + z)\sigma(x - t)\sigma(x + t) +$$
$$+ \sigma(z - x)\sigma(z + x)\sigma(y - t)\sigma(y + t) = 0.$$

This observation means that (29) reduces to (32).

6. Discussion

This paper has further investigated the R-matrix structure of the Calogero–Moser models under the assumption of momentum independence. We have shown that for the usual L-operator $(L = L^\dagger \Leftrightarrow w_\alpha = -w_{-\alpha})$ and nondegenerate type IV potential, no momentum independent R-matrix exists whenever $n \geqslant 4$. Indeed our analysis showed that momentum independence actually gives the R-matrices of [17] for the type I–III potentials when the otherwise arbitrary projection operator P^{ij} is chosen to vanish. For $n = 2$ and 3, solutions may however be found for the type IV potential.

By enlarging the class of L-operators under consideration to the family considered by Krichever, we were able to construct an appropriate spectral parameter-dependent R-matrix. This was given by (30), (31), (33). We have worked throughout in terms of a basis of the Lie algebra. This has the merit of reducing the problem to the two equations (28), (29) and highlighting the implicit consistency conditions. The ansatz presented by (30), (31) is independent of the Lie algebra to which L is associated. While we can certainly find consistent solutions to the resulting equation (32) in the case of gl_n we have not fully examined this equation in the general setting.

We must conclude by mentioning the very recent, related work of Sklyanin [23] which also constructs R-matrices for the type IV Calogero–Moser model with spectral parameter. At first glance our solutions are different and we have delayed the written presentation of this work in order to clarify this point. Certainly Sklyanin's approach is very different from our own. We may easily verify that Sklyanin's ansatz satisfies our (30), (31) and so provides a solution. The differences between the solutions has its origin in our respective presentation of the L-operators. Sklyanin in fact works with a conjugate of Krichever's L-operator, $UL(u)U^{-1}$ where $U_{ij} = e^{\zeta(u)x_i}\delta_{ij}$. Once this is observed, our solutions are in fact in agreement, our works providing independent proofs of this fact.

Acknowledgements

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